On the Relation Between Pressure and Wind, with Particular Reference to a Vortex.[1]

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Abstract

It is shown that if the boundary conditions to solve the vorticity and balance equations are prescribed independently, then the primitive equations of motion are not satisfied.

Given the wind field of a symmetrical vortex embedded in a resting atmosphere, analytical expressions for the pressure and wind tendency are derived. A numerical example is worked out and the results are used to obtain information regarding the effect of the boundary conditions and of the variation of the Coriolis parameter in a forecast with a barotropic model.

The barotropic non-divergent flow is defined by the following equations:

$$\frac{dv}{dt} + fu = -\frac{\partial \phi}{\partial y} \tag{1}$$

$$\frac{du}{dt} - fv = -\frac{\partial \phi}{\partial x} \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

where $d/dt \equiv \partial/\partial t + u\partial/\partial x + v\partial/\partial y$, u and v are the components along the x- and y-axes, respectively of the horizontal wind, the x-axis points to the east and the y-axis to the north, f is the Coriolis parameter; and $\phi = p/\varrho$, where ϱ is the density and p the pressure.

Let $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$, then (3) is automatically satisfied and we only have to deal with equations (1) and (2).

By differentiating (1) with respect to x and (2) with respect to y and subtracting we obtain the well-known vorticity equation that contains only the stream function as dependent variable:

$$\nabla^2 \frac{\partial \psi}{\partial t} = J(\nabla^2\psi,\ \psi) - \beta \frac{\partial \psi}{\partial x} \tag{4}$$

Differentiating (1) with respect to y and (2) with respect to x and adding we get

$$2[\psi_{xy}^2 - \psi_{xx}\,\psi_{yy}] - f\nabla^2\psi - \nabla\psi \cdot \nabla f + \nabla^2\phi = 0 \tag{5}$$

which gives a relation between wind and pressure. (The pressure field is given by $\phi\varrho$, where ϱ is constant; we shall, however, refer to ϕ as the pressure.) This equation has been used by BOLIN (1955, 1956) and CHARNEY (1955) in an attempt to improve numerical forecasting by removing the geostrophic assumption.

[1] This research was started at the International Meteorological Institute in Stockholm, while the author was there. Part of the material is based on an unpublished manuscript, in which the late Prof. C.-G. Rossby was co-author.

Given the pressure field ϕ in a region A enclosed in a boundary C where we prescribe ψ, if we assume, for the sake of simplicity, that equation (5) is of the elliptic type, we can obtain the stream function ψ by solving this equation (BOLIN, 1956, p. 2).

When one substitutes in the primitive equations of motion the given pressure field and the ψ computed using (5), they determine, except for an arbitrary constant, a unique solution for $\partial\psi/\partial t$ in the whole region A and its boundary C.

It follows therefore that we cannot choose arbitrary boundary conditions to solve the vorticity equation, because when solving it to determine $\partial\psi/\partial t$ we must obtain the same solution as by solving the primitive equations. In other words, by prescribing ϕ in A and ψ and $\partial\psi/\partial t$ in C arbitrarily independent to solve (4) and (5) we obtain a solution that does not necessarily satisfy the primitive equations.

Similarly, if we prescribe in the region A the stream function, we cannot prescribe both the tendency and the pressure at the boundary in order to solve equations (4) and (5).

The above consideration shows that the problems of finding the pressure-wind relation and formulating the best boundary conditions are closely connected; and that these two problems in numerical forecasting are not independent and need to be investigated from a unified viewpoint.

In the present stage of numerical forecasting, the arbitrariness of the boundary conditions cannot be avoided, and the following questions can be formulated in connection with the above considerations:

1. In what type of problems is it essential to use consistent boundary conditions to solve (4) and (5)?

2. How can one choose the boundary conditions to solve equations (4) and (5) in order that the solution be also the solution of the primitive equations of motion?

The answers to these two questions are not given here, but they will be the subject of further research.

In the remaining part of this paper we shall deal with one example in which an analytical solution to equations (4) and (5) can be found which, because of its simplicity, lends itself to comparative studies and discussions.

We introduce a polar coordinate system (r, Θ) and prescribe at $t=0$ the stream field ψ as a function of r at $r \leq r_0$ and $\psi=0$ at $r \geq 0$. We assume that ψ and its three first derivatives are continuous differentiable functions. This stream function represents a vortex.

Assuming $f = f_0 + \beta y$ where f_0 and β are constants, we can compute the tendency field and the pressure field from the Poisson equations (4) and (5) (ADEM, 1956, p. 365). We get:

$$\left(\frac{\partial\psi}{\partial t}\right)_{t=0} = -\frac{\beta x}{r^2}\int_0^r \psi r dr + xC_1 \qquad (6)$$

$$\phi = f\psi + \int_0^r \frac{1}{r}\left(\frac{d\psi}{dr}\right)^2 dr - \frac{\beta y}{r^2}\int_0^r \psi r dr + yC_2 + C_3 \qquad (7)$$

where C_1, C_2 and C_3 are arbitrary constants to be determined from the boundary conditions. We have retained only the arbitrary harmonic solutions pertinent to our discussion, and all others are zero. If we prescribe that the normal velocity is zero at the boundary $r = R_0$, then $(\partial\psi/\partial t)_{t=0} = 0$ at $r = R_0$, and we get

$$C_1 = \frac{\beta}{R_0^2}\int_0^{R_0} \psi\, r dr$$

On the other hand substituting (6) and (7) in (1) and (2) we see that our solution satisfies the primitive equations if $C_1 = -C_2$. Therefore the specification of $\partial\psi/\partial t$ at $r=R_0$ is enough to determine the arbitrary constant C_2 in (7). In other words, as stated above, one cannot prescribe both the pressure and the tendency at the boundary. If one prescribes the boundary conditions to solve the vorticity equation (4) one has no freedom to choose boundary conditions for the balance equation (5) and vice-versa. Finally the arbitrary constant C_3 is irrelevant and can be taken as zero.

Substituting the values of the arbitrary constants in (7) we obtain

$$\phi = f\psi + \int_0^r \frac{1}{r}\left(\frac{d\psi}{dr}\right)^2 dr - \frac{\beta \sin \Theta}{r}\int_0^r \psi r dr -$$

$$- \frac{\beta r \sin \Theta}{R_0^2}\int_0^{r_0} \psi r dr \tag{8}$$

This formula gives the pressure field of a vortex of radius $r = R_0$ enclosed in a concentric circular region of radius $r = R_0$. We have only assumed that the normal velocity is zero at $r = R_0$.

In this formula the term

$$\int_0^r \frac{1}{r}\left(\frac{d\psi}{dr}\right)^2 dr$$

is due to the centripetal acceleration of the vortex and corresponds to the term $2\,(\psi_{xx}\psi_{yy} - \psi_{xy}{}^2)$ in equation (5).

The term

$$\beta \sin \Theta\left[r\psi - \frac{1}{r}\int_0^r \psi r dr - \frac{r}{R_0^2}\int_0^{r_0} \psi r dr\right]$$

corresponds to the variation of the Coriolis parameter.

It is interesting to note that at $r \geqq r_0$ where $\psi = 0$ (wind field zero) a pressure field, due entirely to the variation of the Coriolis parameter, exists and is given by

$$\phi = -\beta \sin \Theta\left[\frac{1}{r} + \frac{r}{R_0^2}\right]\int_0^{r_0} \psi r dr$$

From the primitive equation (1) we can compute, by multiplying by ϱ and integrating over a circle of radius r, the resultant net force on a horizontal slice of unit height and with an area equal to the circle of radius r:

$$R = \varrho \int\int \frac{dv}{dt} dA = -\beta\varrho\pi\left[\int_0^r \psi r dr -\right.$$

$$\left. - \frac{r^2}{R_0^2}\int_0^{r_0} \psi r dr\right] \tag{9}$$

In this integration, the contribution from integrating the term $-fu$ is

$$F = -\varrho \int\int f u dA = \beta\varrho\pi[r^2\psi - 2\int_0^r \psi r dr] \tag{10}$$

while that from integrating $-\partial\phi/\partial y$ is

$$P = -\varrho \int\int \frac{\partial \phi}{\partial y} dA =$$

$$= \beta\varrho\pi\left[\int_0^r r\psi dr - r^2\psi + \frac{r^2}{R_0^2}\int_0^{r_0} \psi r dr\right] \tag{11}$$

This last term represents the resultant pressure force on the material line of radius r and could be obtained directly from (8) by integrating at such line.

When $r = r_0$

$$(R)_{r=r_0} = -\beta\varrho\pi\left[1 - \frac{r_0^2}{R_0^2}\right]\int_0^{r_0} \psi r dr \tag{12}$$

and if $R_0 \gg r_0$ we get the relations

$$(R)_{r=r_0} = \frac{1}{2}(F)_{r=r_0} = -(P)_{r=r_0}$$

showing that the net resultant force on the vortex is reduced by one half due to the pressure resultant. In a preliminary computation, ROSSBY (1948) neglected the resultant pressure force, but as pointed out by him afterwards (ROSSBY, 1949) and as shown by these computations the net force is decreased by one half if the resultant pressure force is included.

The net resultant is directed northwards in the case of a cyclone and southwards in the case of an anticyclone.

It is of interest to point out that the last term of the second member of (8), which represents the pressure field due to the boundary effect, tends to zero when the boundary $R_0 \to \infty$. However, as shown by (11), the integrated pressure resultant at $r = R_0$ does not go to zero and accounts for the necessary balance of forces implied by the primitive equations.

Let $\psi = \psi_0\ [1 - (r/r_0)^2]^4$ when $r \leq r_0$ and $\psi = 0$ when $r \geq r_0$.

This stream function represents a cyclone if $\psi_0 < 0$ and an anticyclone if $\psi_0 > 0$.

The maximum velocity $v_{max} = (d\psi/dr)_{max}$ is at $(r/r_0)^2 = 1/7$ and ψ_0 is given as a function of the radius and of the maximum velocity by

$$\psi_0 = -0.525\ r_0\ (d\psi/dr)_{max}$$

Let us consider a cyclone with maximum velocity $v_{max.} = 30$ m/s and $r_0 = 1{,}000$ km. Let $f_0 = 0.9 \times 10^{-4}$ sec^{-1}, $\beta = (1.7) \times 10^{-13}$ cm^{-1} sec^{-1}. The results of the computation using this data to evaluate the different terms in expression (8) for the pressure field are shown in Table 1. Column I gives the values of $f_0\psi$ for different values of r/r_0. Columns II and III give the values of the terms due to the variation of the Coriolis parameter for $\sin\Theta = 1$. Column IV gives the values of the term due to the centripetal acceleration of the cyclone.

To get the actual quantities from Table 1, one has to multiply the values of the table by $(\psi_0)\ 10^{-4}$ and the results will then be given in cm^2 sec^{-1}.

For an anticyclone the sign of Columns I, II and III has to be changed, but the sign of IV remains unchanged, showing that for a cyclone the terms I and IV have to be added

Table 1. Values of the different terms of the pressure field in a cyclone.

The stream function is $\psi = \psi_0\ [1 - (r/r_0)^2]^4$. $v_{max} = 30$ m/s, $r_0 = 1{,}000$ km, $f_0 = 0.9 \times 10^{-4}$ sec^{-1} and $\beta = (1.7) \times 10^{-13}$ cm^{-1} sec^{-1}.

	I	II	III	IV
r/r_0	$f_0\psi$	$r\beta\psi$	$-\beta/r\int_0^r rdr$	$\int_{r_0}^{r} \frac{1}{r}\left(\frac{d\psi}{dr}\right)^2 dr$
0	—.900	0	0	—.720
.2	—.764	—.029	.016	—.541
.4	—.448	—.034	.025	—.212
.6	—.151	—.017	.025	—.032
.8	—.015	—.002	.021	—.006
1.0	0	0	.017	0
1.2	0	0	.014	0
1.4	0	0	.012	0
1.6	0	0	.011	0
1.8	0	0	.010	0
2.0	0	0	.009	0

and that for an anticyclone they have to be subtracted. This fact shows that a negligible pressure field can be associated with a strong wind field in the case of an anticyclone, and that given a cyclone and an anticyclone of the same wind intensity and at the same latitude, the pressure field associated with the cyclone is always stronger than the one corresponding to the anticyclone (in the northern hemisphere).

It is of interest to see how the values in this table vary when one varies v_{max}, r_0, f_0 and β. Column I is proportional to f_0, Columns II and III are proportional to r_0 and β, and Column IV is proportional to v_{max}/r_0.

The following conclusions regarding the importance of the β-term in the balance equation (5) are worth mentioning:

The β-term of equation (5) is locally small compared with the other terms. However, its integrated effect can be important.

The β-term increases proportionally with the dimensions of the disturbances to be forecasted; and, therefore, if it is of significance, its effects should appear mainly in forecasting large scale motions.

The tendency of the stream field is given by

$$\left(\frac{\partial\psi}{\partial t}\right)_{t=0} = -\frac{\beta x}{r^2}\int_0^r \psi r dr + \frac{\beta x}{R_0^2}\int_0^{r_0} \psi r dr \qquad (13)$$

If $R_0 \gg r_0$, we see by comparison with (8) that except for a shift of 90° it is the same field as the term of the pressure field responsible for the pressure resultant on the vortex. The tendency outside the vortex for $r \ll R_0$ is proportional to the net resultant force on the vortex, as is shown by comparison of (13) with (12).

From the analytical expressions for the pressure and wind tendency given by equations (8) and (13) we see that the boundary effect is due entirely to the variation of the Coriolis parameter and that if we vary in our example the boundary $r = R_0$ its effect is proportional to $1/R_0^2$. On the other side, for a fixed boundary $r = R_0$ the boundary influence at a given point is proportional to the distance between the point and the center of the vortex.

REFERENCES

ADEM, J., 1956: A series solution for the barotropic vorticity equation... Tellus **8**, pp. 364—371.

BOLIN, B., 1955: Numerical forecasting with the barotropic model. Tellus **7**, pp. 27—49.

BOLIN, B., 1956: An improved barotropic model... Tellus **8**, pp. 1—15.

CHARNEY, J., 1955: The use of the primitive equations of motion in numerical prediction. Tellus **7**, pp. 22—26.

ROSSBY, C.-G., 1948: On displacements and intensity changes of atmospheric vortices. J. Marine Research **7**, pp. 175—187.

ROSSBY, C.-G., 1949: On a mechanism for the release of potential energy in the atmosphere. J. Meteor. **6**, pp. 178.