# Barotropic Waves in Straight Parallel Flow with Curved Velocity Profile

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## Abstract

The paper contains a study of the dynamics of two-dimensional straight parallel flow with a continuously curved profile. The differential equation for the stream-function of a neutral wave disturbance is used as an analytical tool to investigate the properties and the evolution of waves having a certain initial configuration of the flow. The behaviour of the waves is explained by using an extension of the method applied by HOLMBOE (1953) in his study of the Kelvin and Rayleigh waves. In order to do that, the problem has been approached from a point of view which deviates somewhat from the customary treatment. The method is first illustrated by a model with a hyperbolic tangent profile (*Th*-flow), which is shown to have a very simple analytical solution for the stationary wave $L_s$. The form of the solution makes it possible to interpret it for other wave lengths ($L \gtrless L_s$) as a forced stationary wave which is the resultant of a non stationary physical wave and a Rayleigh wave induced by the proper infinite sliding vorticities at the central level. When this foreign field is removed from the solution, the remaining field, which now is a physically possible initial configuration of the flow, will propagate with different speed from level to level. As the tilt develops the wave will begin to amplify if $L > L_s$, and it will be damped if $L < L_s$. So the stationary wave represents the transition from short stability waves to long instability waves as had been anticipated by Fjørtoft (1). The method is then extended to more general types of profiles. Some general results on the conditions for the existence of stationary waves, which were proved or anticipated by RAYLEIGH (1880), FJØRTOFT (1950) and HÖILAND (1951), are given a unified treatment. It is finally shown that, if a system has stationary waves, they represent the transition from damped to amplified waves. If the number of stationary waves is even, both the shortest and the longest waves are damped waves, and the intermediate intervals of the spectrum are alternate bands of amplified and damped Iwaves. If the number of stationary waves is odd, the longest waves in the spectrum are amplified.

### Introduction

The dynamics of two-dimensional, straight parallel flow in barotropic models with linear profiles has recently been studied in detail by HOLMBOE (1953). His physical analysis of the problem, with the aid of the fundamental theorems of circulation and vorticity has brought the theory to a degree of clarity which had never been attained before. The present paper is an extension of the theory to the case of flows having a curved velocity profile.

The models considered by Holmboe are systems with continuous velocity profiles having a number of kinks and straight line segments between them (multiple layer Couette flow). The simplest of these models is the double layer Couette flow: two unbounded layers with different values of the vorticity, separated by an interface where the velocity is continuous, so that the vorticity has a discontinuity equal to the difference between the shear of the two layers. When the interface is given a sinusoidal deformation, the resulting field of motion will generate a neutral wave (a wave without change in shape and intensity) moving towards the concave side of the kink in the Couette flow profile: the Rayleigh wave. In the treatment of this model given in HOLMBOE (1953), the field of motion is separated into two components, one of which is the double-layer Couette flow with the plane undeformed interface; the second component is irrotational everywhere except in the space between the undeformed and deformed position of the interface. All the properties of the Rayleigh wave are thus derived in terms of the interface deformation and the jump in the vorticity at the interface. In the case of three-layer Couette flow the same method was applied, by considering the total field (for arbitrary deformations of the two interfaces) as the resultant of three component fields: the Couette flow with plane (undeformed) interfaces, and two periodic fields which are defined by the vorticities in the areas between the two positions of the upper and lower interfaces respectively. The evolution of the field can, therefore, be described by the interaction of the Rayleigh waves which are produced at the two interfaces.

The models we are going to study in this paper are of a more complex nature, due to the fact that the profiles to be considered have a continuous variation of vorticity. The analytical treatment of these systems gives rise to mathematical difficulties, but the physical interpretation of the problem and the qualitative description of the field can be carried through without effort. The whole treatmen is restricted to the cases in which the profile of the mean flow has inflection points. In all other cases the method cannot be applied. The solutions obtained through the use of the neutral wave differential equation are much too restricted to describe any physically possible disturbance in these more general cases. Therefore, to solve these problems, the stream function of the periodic field must be treated as an arbitrary function of time, and the vorticity equation has to be solved by other methods.

**Acknowledgement**

Without the kind assistance of Professor J. HOLMBOE, this paper would never have been written. The author is particularly indebted to him for the physical interpretation of the "forced" stationary wave.

1. — *Kinematic.* We shall consider only homogeneous, incompressible (non-divergent), inviscid fluids. The types of motion to be investigated belong to a rather restricted class. The system is unbounded and the trajectory of any parcel is entirely contained in a plane which will be called its *plane of motion.* The planes of motion of all parcels define a set of parallel planes such that there is no variation of the motion in the direction normal to them. We can describe the motion at any time either by giving the velocity vector at each point in the plane of motion, or by giving the isopleths of the vorticity. In accordance with the field theorem, in an unbounded field, one can be obtained from the other, so that both representions are equivalent.[1]) We shall make use of the vorticity representation for reasons that will become immediately apparent. In the models we are going to deal with all vorticity isopleths are assumed to be, at some prescribed time, sine curves oscillating around straight lines whose direction, being unique for the whole plane, will be called *the direction of the mean flow.*

The motion will be described with reference to a right-handed system of mutually perpendicular unit vectors (**i, j, k**) such that **j** is perpendicular to the plane of motion and **i** is along the direction of the mean flow. The lines $Z$ = const. will be referred to as *levels* at the *height* $z$.

For any given level $z$, there will be a sinusoidal vorticity isopleth, whose value we shall designate by $Q = Q(z)$, with the inflection points on the line $z$ = const. Let A be any point at this level, and B the point of the Q-isopleth on the vertical through A (fig. 1). The differenc between the vorticities at B and A is given by a Taylor's series as a function of the distance $\zeta$ and the derivatives of the vorticity at either A or B. We shall assume the amplitude of the Q-isopleth to be so small that only the first term in the Taylor's expansion need to be considered in order to have the desired approximation. The vorticity at any point on the level $z$ can therefore be separated into two terms, one of which (Q) is the mean vorticity

[1] There are a few obvious exceptions to this rule: (i) A field of translation may always be added without changing the vorticity field; (ii) in unbounded one layer Couette flow the *direction* of the flow is not specified by the constant vorticity of the flow.

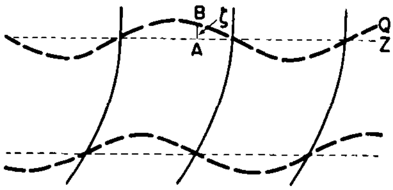


Fig. 1.

at that level, and the other ($q$) is the departure from the mean value. We have, then, for each level $z$

$$Q + q = Q - Q'\zeta$$

or

$$q = -Q'\zeta \tag{1.1}$$

In accordance with the field theorem, the total velocity vector **V** can be considered as the sum of a *mean flow* **U**, whose vorticity is **Q**, and a departure from it, **v**, whose vorticity is **q**. Consequently, if we write

$$\mathbf{V} = U\mathbf{i} + \mathbf{v} \tag{1.2}$$

we have

$$Q\mathbf{j} = \nabla U \times \mathbf{i} = U'\mathbf{j} \tag{1.3}$$

and

$$q\mathbf{j} = \nabla \times \mathbf{v}$$

We can represent the periodic flow either in terms of its rectangular components or of its stream-function

$$\mathbf{v} = u\mathbf{i} + w\mathbf{k} = \mathbf{j} \times \nabla \psi, \tag{1.4}$$

which gives

$$\nabla \times \mathbf{v} = q\mathbf{j} = \nabla^2 \psi \mathbf{j} \tag{1.5}$$

2. — *Dynamics.* The physical properties of the fluids, listed at the beginning of the previous section, have been chosen to give the simplest possible dynamics. It is well known that any motion of a fluid with these properties can be entirely described by means of a unique dynamic principle expressing the conservation of the vorticity of the fluid parcels along the path.

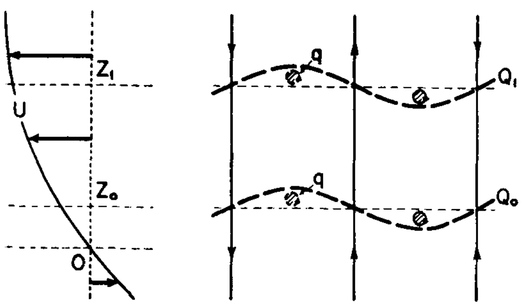


Fig. 2.

In fig. 2 we have assumed the existence of a motion of the described type with profile of the mean flow $U = U(z)$ represented at the left. At the right are shown two sinusoidal vorticity isopleths at some arbitrary levels. They are considered to be in phase at all levels. The circles with the arrows indicate the sense of the periodic vorticity $q$. If we focus our attention on the level $z_0$, the velocity field associated only with the periodic vorticities $q_0$ at that level has the configuration of a Rayleigh wave (see HOLMBOE 1953, p. 15). If there were no other velocity fields except the mean flow and the periodic flow of this Rayleigh wave, the parcels would move rhythmically up and down with the same periodicity. The $Q_0$-isopleth will, therefore, change with time in such a way that it will appear as a wave moving towards the right, with reference to the mean flow at the level $z_0$, with a phase speed proportional to the value of the vertical velocities at the inflection points (which in turn are entirely defined by $q_0$). However, the velocity field associated with the periodic vorticity $q_1$, at any other level $z_1$, will have the same characteristics as the previous one. It will therefore produce at the level $z_0$ a transversal velocity component whose value will be the intensity at the level $z_1$ multiplied by the factor $e^{-k|z_0-z_1|}$, where $k = \frac{2}{L}$ is the wave number and $L$ is the wave length of the sinusoidal vorticity isopleth. The total intensity of the transversal velocity field at $z_0$, which determines the motion of the $Q_0$-isopleth, will be given, consequently, by the contributions of the periodic vorticities from all levels.

The previous analysis makes evident that each layer of the fluid contributes with a different "weight" to the total motion of the Q-waves at each level. The expression (1.1) shows clearly that it is the curvature of the profile of the mean flow that, for a given distribution of the Q-isopleth deformation, determines the intensity of the periodic vorticity field at that level. We can therefore speak of any level as being more or less "active" than another one according to the values of the curvature of the mean profile.

The integrated effect of all periodic vorticities within the fluid will change from level to level, and we shall have the Q-waves moving with different speeds at different levels. The crest and trough lines, initially vertical, will in general become tilted with time. The result of such a process has been analyzed, from a general point of view, by FJØRTOFT (1950) (see also HOLMBOE 1953 p. 9—10), showing its relation to the amplification or damping of the waves. We shall return later on to this question.

3. — *The stationary wave.* If the wave motion we have considered in the previous sections is such that the streamlines coincide with the vorticity isopleths, the field of motion will not change the configuration of the vorticity field. We can then resort again to the field theorem and conclude that there will be no local change of the velocity at any point. The wave is said to be *stationary*. The Q-isopleth deformation also represents, in this case, the vertical displacements of the fluid parcels, and is related to the stream function $\psi$ by the expression

$$U\zeta = -\psi \tag{3.1}$$

which leads to the well known *differential equation of the stationary wave:*

$$\psi'' = \left(k^2 + \frac{U''}{U}\right)\psi \tag{3.2}$$

If we now consider a frame moving with the constant speed $C$ with reference to the one used above, (3.1) and (3.2) become respectively

$$(U - C)\zeta = -\psi, \tag{3.3}$$

and

$$\psi'' = \left(k^2 + \frac{U''}{U-C}\right)\psi. \tag{3.4}$$

Equation (3.3) is called the *neutral wave condition* since it expresses the relationship between the values of the stream-function and the displacements of the parcels in a wave propagated *without change of shape and amplitude* with the phase velocity $C$.

Before attempting any general discussion of equation (3.2) or (3.4) and its conditions of solubility, we shall turn now, in the following sections, to a detailed consideration of a particular case.

4. — *Flow with hyperbolic tangent profile (Th-flow).* We shall consider a model having a profile of the mean motion which in many ways gives the simplest analytical results and allows, at the same time, a clear physical analysis of the underlying mechanism which governs the evolution of the waves. The mean speed varies from level to level proportionally to the hyperbolic tangent of the height: (the fluid is assumed to be unbounded.)

$$U = -\mathrm{th}z \tag{4.1}$$

The mean vorticity has the value —1 at the origin and 0 at infinity. At the level $z = 2$ it has decreased already to 0.1. It follows that in any wave motion with this mean flow, the most active part, in the sense defined in section 2, will be concentrated in two relatively shallow layers symmetrically located with reference to the central level.

5. — *The stationary wave in Th-flow.* For reasons that will become apparent in section 10, we shall look only for waves at rest with reference to the fluid at the level of the inflection point of the mean profile. The periodic velocity field will then have a stream-function satisfying the differential equation

$$\psi'' = (k^2 - 2\,\mathrm{sch}^2)\psi, \tag{5.1}$$

with the general solution

$$\psi = [\varepsilon_1 (k + \mathrm{th}z)\,e^{-kz} + \varepsilon_2(k - \mathrm{th}z)\,e^{kz}]\cos kx \tag{5.2}$$

To avoid infinite values of the field at infinity, we take the first term as the solution for $z > z_0$, and the second term as the solution for $z < z_0$, where $z_0$ is any level with a finite value of the height.

At the level $z_0$, the expressions for both $\psi$ and $\psi'$ must satisfy the requirement of continuity, which gives

$$k^2 = 1,$$

regardless of the level $z_0$ which we select. Therefore, the only waves stationary with reference to the central level correspond to a wave length $L = 2\pi$. The simplest case is given by $z_0 = 0$, since it will have complete symmetry with reference to the central level.

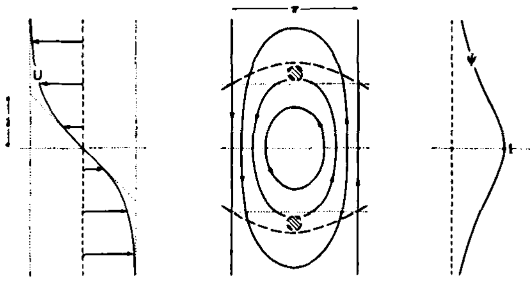


Fig. 3. The stationary wave in Th-flow.

We shall describe in detail this particular case, which has the stream-function

$$\psi = (1 + |U|)\, e^{-|z|} \cos kx, \qquad (5.3)$$

where the amplitude factor $\varepsilon$ has been omitted.[1]

In fig. 3 we have drawn the curve (5.3), at point of abscissa $n\pi$, showing the variation of the periodic field with height. Two typical vorticity isopleths above and below the central level are shown in the center of the diagram. The field of periodic vorticity $q$ is represented by the circles with arrows. The Q wave at any level, if driven only by the local periodic vorticities at that level, would move, with respect to the mean flow, towards the concave side of the profile. The effect of the periodic vorticities at all other levels is to increase the speed of the wave relative to the mean flow until its relative phase speed reaches a value equal and opposite to the corresponding velocity of the mean flow. The wave so obtained has, therefore, the character of a stationary $b$-wave as defined by HOLMBOE (1953).

6. — *The forced stationary wave in Th-flow.* In order to find the stationary wave it was necessary to adjust the general solution of the differential equation for the stream-function so as to satisfy the boundary conditions. However, by giving up one of the restrictions, namely the continuity of $\psi'$, it is possible to get valuable information about the behavior of periodic motions with different wave lengths.

If $\psi'$ is discontinuous at some level, it simply means that the fluid slides on itself. If the fluid is homogeneous, no possible motion within the fluid itself is able to set up that type of discontinuity. For this reason, all solutions containing them have been eliminated from the outset. A closer scrutiny of the meaning of these solutions shows, however, that they have a deeper significance. The solution

$$\psi = (k + |U|)\, e^{-k|z|} \cos kx, \qquad (6.1)$$

when $k \neq 1$ represents a periodic motion that cannot be produced by displacements alone. We shall show, however, *that the addition of a periodic sliding vorticity of appropriate strength, at the level $z = 0$, to a field produced by displacements, will suffice to get the desired stationary wave.* By subtracting from (6.1) the field of sliding vorticities we can find the behavior of the remaining field for values of $k \neq 1$.

The solution (6.1) which gives a continuous field of $\psi$, but has at the level $z = 0$ the jump in $\psi'$ whose magnitude is

$$\Delta\psi' = 2\,(1 - k^2) \cos kx, \qquad (6.2)$$

is a stationary wave produced by the addition of two fields: (i) a continuous field, which we shall call $\psi_b$, whose vorticity arises from the displacements of the fluid parcels, in agreement with the law $q = -U''\zeta$; (ii) a Rayleigh field $\psi_R$, irrotational everywhere except at $z = 0$ where it has the sliding vorticity (6.2), i.e.

$$\psi_R = (k - k^{-1}) e^{-k|z|} \cos kx. \qquad (6.3)$$

When (6.3) is subtracted from (6.1) we get

$$\psi_b = (k^{-1} + |U|)\, e^{-k|z|} \cos kx. \qquad (6.4)$$

The stationary wave (6.1) whose analytical representation is now

$$\psi = \psi_b + \psi_R,$$

will be called a *forced stationary wave*, to signify that it is produced by some foreign field which is artificially introduced into the system. The removal of the Rayleigh field $\psi_R$ will change the field of motion in such a way that the Q-waves no longer remain stationary, but the knowledge of the form of the remaining field

[1] The amplitude factor $\varepsilon$ is still arbitrary, subject only to the restriction imposed by the assumption of small amplitude of the waves. It can be seen that near the origin the field of vertical velocity approaches the value $-\varepsilon$, whereas the mean flow itself goes to zero. The theory breaks down, then, at this level, since in equation (3.1) it was assumed, in order to obtain (3.2), that the periodic field is negligible as compared to the mean speed. It can be shown, however, that equation (3.2) is valid at any level.

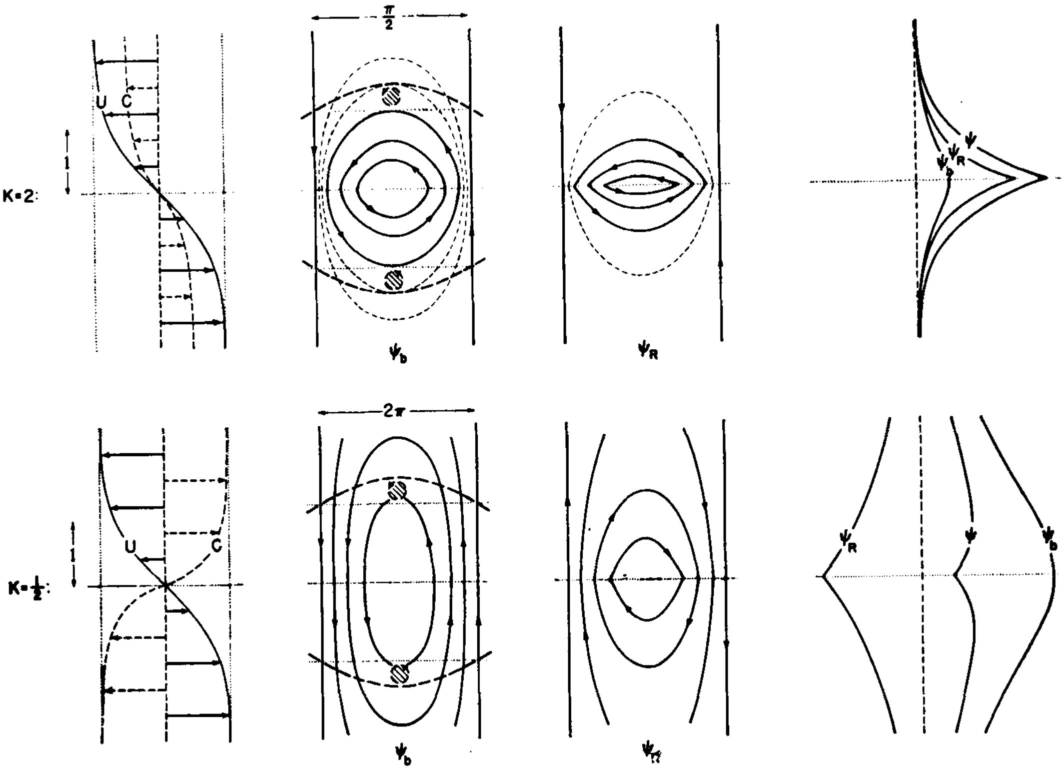


Fig. 4. The b-wave in Th-flow.

will allow us to get information about the subsequent evolution of the field. This field is analogous to the *b*-wave in skew-symmetric triple layer Couette flow (see fig. 10 a in HOLMBOE 1953). We shall therefore call it the *b*-wave in *Th*-flow.

7. — *The b-wave in Th-flow.* In fig. 4 is shown an initial state of *Th*-flow with forced stationary waves for two values of the wave length. In order to fix ideas the values $k = 2$ and $k = {}^1/_2$ have been selected as typifying "short" and "long" waves, respectively.

A direct inspection of fig. 4 shows that the field of motion associated with the periodic vorticities of *short waves needs the support of additional sliding vorticities of the same sense,* i.e., such that they provide a field of motion which speeds up the Q-waves at all levels. It is obvious that the removal of the sliding vorticities will have the effect of slowing down the Q-waves at all levels, producing a relative displacement of the waves such that the crest and trough lines, originally vertical, become tilted with the same sense as the shear of the mean flow. In accordance with the Reynolds-Fjörtoft's energy transfer principle that type of motion will produce a *damping* of the wave. This result can also be directly obtained by a direct analysis of the field of motion of the tilted wave. In fact, if we focus our attention on a Q-wave at any level in the upper layer we see that it is damped by the periodic fields associated with all Q-waves below that level, and amplified by the periodic field associated with all Q-waves above. On the other hand, if we consider a Q-wave at any level in the lower layer, we see that it is damped by the field of motion associated with all Q-waves above its level, and amplified by the periodic motion associated with the Q-waves below. The net result is, then, a damping of all Q-waves. The mechanism is identical with the interaction of the two Rayleigh waves in triple layer Couette flow (see for instance fig. 10a in HOLMBOE 1953).

The *long waves* have a component field $\psi_b$ associated with the $q$-vorticities, such that it must be *opposed by additional sliding vorticities of opposite sense to become stationary*. The Rayleigh field necessary for the forced stationary wave has the effect of slowing down the Q-waves at each level. The removal of this field will produce, therefore, a speeding up of the waves at all levels. The same type of reasoning applied above shows that the net result is an *amplification* of the wave, in agreement with the energy transfer theorem.

The same result can be obtained analytically by calculating the instantaneous speed of the Q-waves at each level after the removal of the Rayleigh field. The transversal displacements,

$$\zeta = -\frac{\psi}{U}$$

in the forced stationary wave represent the deformation of the Q-isopleths. Although the Q-waves initially propagate with different speeds $C(z)$ at different levels, the fact that there is no tilt at the initial time will secure that the amplitudes of all the Q-waves are initially stationary, so we may use the neutral wave condition (3.5) *at this moment*, i.e.

$$\zeta = -\frac{\psi_b}{U - C(z)}$$

From the two expressions for $\zeta$ we get

$$C(z) = U\frac{\psi_R}{\psi} \qquad (7.1)$$

When the values (6.1) and (6.3) are introduced for $\psi$ and $\psi_R$, we get

$$C = U\frac{k^2 - 1}{k(k + |U|)}$$

which is represented in fig. 4 for the values $k = 2$ and $k = 1/2$.

It can be seen that $C$ has the same sign as $U$ when $k > 1$ (short waves), and opposite sign when $k < 1$ (long waves). At great distances from the central level, $C$ approaches asymptotically the value

$$C_b = \frac{1 - k}{k} \qquad (7.2)$$

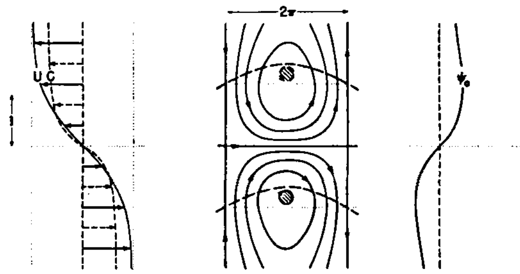


Fig. 5. The a-wave in Th-flow.

8. — *The a-wave in Th-flow.* The solutions we have analyzed so far lead only to a very special type of motion, namely one with initial vorticity isopleths 180° out of phase above and below the central level. Let us now examine the evolution of the field if at a given instant the vorticity isopleths are in phase throughout the field. In fig. 5 we have represented schematically the problem we are considering. We have taken the values of $\zeta$ and $k$ so that the layer $z > 0$ is identical with fig. 4. If we focus our attention on the Q-wave at any one level, we can immediately see that the propagation which would be caused by its own periodic vorticities acting alone is supported by the fields associated with all the vorticities in the same layer, but is opposed by the fields associated with the vorticities in the other layer. The motion as a whole has thus the characteristics of an *a-wave* in triple layer Couette flow as defined by Holmboe (1953). A comparison with fig. 4 shows at once that the *a*-wave has also at each level a relative motion towards the concave side of the profile. The speed of propagation will be, however, less than the corresponding speed of the *b*-wave.

To compute the instantaneous speed of an *a*-wave, we have to know first the isolated effect of each of the two layers. We shall consider an auxiliary model in which the mean motion is

$$U = -thz \qquad \text{for } z > 0,$$

$$U = -z \qquad \text{for } z < 0.$$

The solution for the stationary wave is given by

$$\psi = (k + U)e^{-kz}\cos kx, \ \text{for } z > 0,$$

$$\psi = ke^{kz}\cos kx, \ \text{for } z < 0,$$

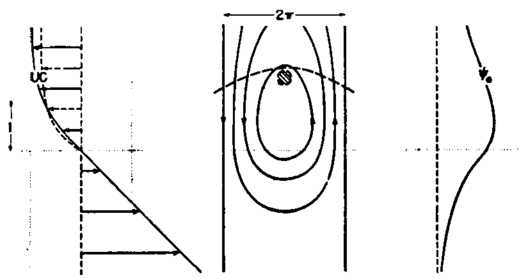

Fig. 6. The $\psi_0$-wave in Th-flow.

where the constants have been so adjusted as to give bounded and continuous values of $\psi$ at all levels. The continuity of $\psi'$ requires

$$\psi'(o) = -1 - k^2 = k^2$$

The value $k = \frac{1}{2}\sqrt{2}$ gives, then, the stationary wave of the system. For all other values of $k$ we shall have only forced stationary waves with the sliding vorticities at the central level

$$\Delta\psi' = (1 - 2k^2) \cos kx,$$

corresponding to the Rayleigh field

$$\psi_R = \frac{2k-1}{2k} e^{-k|z|} \cos kx.$$

When this field is subtracted from the total field, the remaining field,

$$\left.\begin{aligned} \psi_0 &= \left(\frac{1}{2k} - U\right) e^{-kz} \cos kx, \text{ for } z > 0, \\ \psi_0 &= \frac{1}{2k} e^{kz} \cos kx, \text{ for } z < 0, \end{aligned}\right\} \quad (8.1)$$

is the instantaneous field associated with the vorticities $q = -U''\zeta$, where $\zeta = -\frac{\psi}{U}$.

It is obvious that the same instantaneous field of motion would be obtained by considering a *Th*-flow throughout, but with a field of $\zeta$ which at that moment is identically zero in the lower layer and has the value of the $b$-wave in the upper layer. The instantaneous phase speed of the $\psi_0$-wave at each level is found to be

$$C(z) = \frac{U\left(k - \frac{1}{2}k - 1\right)}{k - U} \quad (8.2)$$

which is drawn in fig. 6.

At great distances from the central level, $C$ approaches asymptotically the value

$$C_0 = \frac{\frac{1}{2}k^{-1} - k}{k+1}. \quad (8.3)$$

We have therefore been able to isolate the effect of each layer. A simple inspection of fig. 4, fig. 5 and fig. 6 shows that

$$\psi_a + \psi_b = 2\psi_0,$$

or

$$\psi_a = 2\psi_0 - \psi_b = -Ue^{-k|z|} \cos kx, \quad (8.4)$$

where (6.4) has been introduced for $\psi_b$, and (8.1), (8.2) for $\psi_0$. Similarly

$$C_a = \frac{-k}{k+1}.$$

Therefore, the instantaneous phase speed of the $a$-wave at each level is, for all values of $k$, less than the corresponding speed of the mean flow. Consequently, all the $a$-waves will develop a tilt in the same sense as the shear of the mean flow.

9. — *Comparison of Th-flow with skew-symmetric three-layer Couette flow with shear in the central layer.* The *Th*-flow has a close resemblance to skew-symmetric three layer Couette flow with no shear in the outer layer analyzed by HOLMBOE (1953, section 11). In fig. 7 both types of flow are shown for comparison. The dispersion diagram for the Couette flow system has been reproduced from fig. 11 a in HOLMBOE (1953). (Only the stationary $b$-wave appears in the diagram.) The spectrum is divided into two regions, one of short stable waves ($L < L_b$) and the other of long unstable waves ($L > L_b$). The phase speeds in this diagram are the phase speeds of the upper and lower component waves with reference to the fluid in the respective outer layers. The corresponding dispersion diagram for the *Th*-flow at the initial moment would represent the instantaneous phase speeds of the wave—relative to the fluid—at great distances from the central level. To find these values we have subtracted from (7.2), (8.3) and (8.5) the speed of the mean flow at infinity (—1).

The two dispersion diagrams show a striking similarity, especially for large values of the

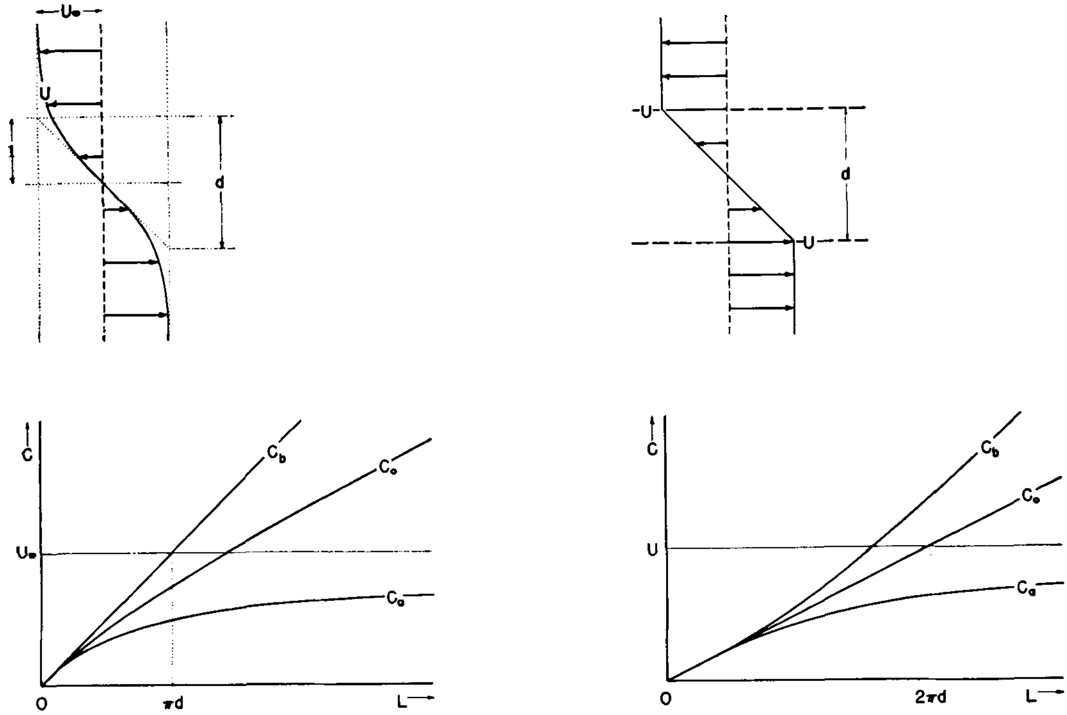


Fig. 7. Dispersion diagrams for Th-flow and three layer Couette flow.

wave lengths. This similarity suggests rather strongly a close analogy between the dynamic behavior of the two systems. It is reasonable to expect that the symmetric waves in *Th*-flow will behave similarly to such waves in Couette flow also for some time beyond the initial moment. The significant analytical difference between the two systems, however, is that in the *Th*-flow the waves will be subject to internal deformations which cannot be represented by fundamental modes.

10. — *Conditions for stationary waves in flow with arbitrary curved profiles.* The technique which has been used for *Th*-flow can now be extended to more general profiles. In order to simplify the analysis, let the fluid be bounded by two parallel walls which intercept the $z$-axis at the points of ordinates $o$ and $h$. The neutral waves in this system must be represented by a stream-function which (i) is continuous, (ii) has continuous derivatives at all levels, (iii) vanishes at both boundaries. It will follow that the function $U(z)$ must, in turn, satisfy very special conditions in order that such solutions can exist.

Let us consider a function defined by

$$\alpha = (U\psi' - U'\psi)\,\frac{\psi}{U} \qquad (10.1)$$

whose derivative is

$$\alpha' = k^2\psi^2 + \left(\frac{U\psi' - U'\psi}{U}\right)^2 \qquad (10.2)$$

From its definition (10.1) it follows that *the zeros of* $\alpha$ *coincide with the zeros of* $\psi$. This is so even if $\psi$ and $U$ vanish simultaneously at a level $z = z_0$, since in that case we should have

$$\lim_{z\to z_0} \alpha = -\lim_{z\to z_0} \frac{U'}{U}\,\psi^2 = \frac{\lim 2\psi\psi'}{\lim\left(1 - \dfrac{U''U}{U'^2}\right)} = 0.$$

On the other hand, in view of (10.2), $\alpha'$ is positive at all levels, and therefore $\alpha$ must have a discontinuity between each pair of zeros of $\psi$. This discontinuity can only be provided by the vanishing of $U$ at some level where $\psi \neq 0$. This is the well known Rayleigh's theorem.

The same property of the function $\alpha$ shows

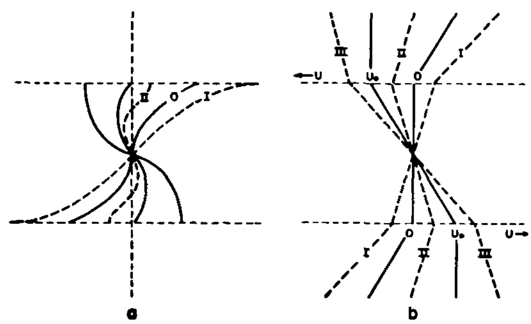


Fig. 8.

that $U$ must have a zero between any two zeros of $\psi$. Therefore the number of zeros of $\psi$ between the boundaries must be at most one unit less than the number of zeros of $U$. It follows that a profile with only one zero cannot have a stationary wave with a nodal plane.

If the level $U = 0$ is not a nodal plane ($\psi \neq 0$), $U''$ must also vanish. To prove this it is enough to consider the function $\psi\psi'$ whose derivative, in view of (3.2), is given by

$$(\psi\psi')' = \left(k^2 + \frac{U''}{U}\right)\psi^2 + \psi'^2.$$

If $U = 0$ with $U''$ being zero, the above expression and all the higher order derivatives of it become infinite. This implies that the function $\psi\psi'$ is discontinuous, which violates the conditions for the stationary wave. The zeros of $U$ must be, therefore, inflection points of the profile.

In equation (3.4) the expression in parenthesis cannot be positive for all $z$. For in that case $\psi''$ would have the same sign as $\psi$ at all levels, which is incompatible with the vanishing of $\psi$ at both boundaries. It follows that the curvature and the speed of the basic flow cannot have the same sign at all levels.

If we apply these considerations to the case of skew-symmetric flow (i.e. a flow with a profile which is skew-symmetric with reference to the level half way between the boundaries) we can see that all the properties we have derived here are in complete agreement with what was found by HOLMBOE (1953) for three-layer Couette flow systems. In fig. 8 b we have reproduced fig. 11 b of HOLMBOE (1953). The Couette flow profiles of type I can be considered as limiting cases of curved velocity profiles having the property $\frac{U''}{U} = 0$ at all levels. No stationary wave is possible in this case. The Couette flow profiles of types II and III can be considered as limiting cases of curved velocity profiles having the property $\frac{U''}{U} = 0$ at all levels. Couette flow profiles of type III, with only one zero of $U$, have only one stationary wave. Couette flow profiles of type II, with three zeros of $U$, have two stationary waves, one of them with a nodal plane at the central level.

11. — *The sliding vorticities in the forced stationary waves.* In the treatment of *Th*-flow we were able to show that stationary wave (for $k = 1$ in *Th*-flow) represents a transition from short damped waves to long amplified waves. We can now generalize this result.

If we consider again equation (3.2), the presence of an inflection point at the level where $U = 0$ implies the existence of two independent solutions, $\psi_1$ and $\psi_2$, which—together with their derivatives—are continuous and bounded for all values of $z$. The only admissible solution is, however, that which vanishes at both boundaries. If there is such a solution it means that the linear combination

$$\psi = A\psi_1 + B\psi_2,$$

when suitable values are introduced for $A$ and $B$, satisfies the conditions

$$\psi(0) = \psi(h) = 0, \qquad (11.1)$$

where 0 and $h$ are the ordinates of the boundaries. This type of solution exists, if at all, only for very special values of $k$ which are called the eigenvalues of the problem. In this section we shall designate them by $k_s$, and the corresponding solutions (eigenfunctions) by $\psi_s$, i. e.:

$$\psi_s'' - k_s^2\psi_s - \frac{U''}{U}\psi_s = 0. \qquad (11.2)$$

For other values of $k$, any solution which vanishes at 0 will not cut again the $z$-axis at $z = h$. The Sturm theory for the differential equations of the form (3.2) shows how this intersection moves along $z$ when we let $k$ either increase or decrease from its eigenvalue $k_s$ (fig. 9). The eigenvalues $k_s$ are, therefore, "isolated". By means of the Sturm's theorem

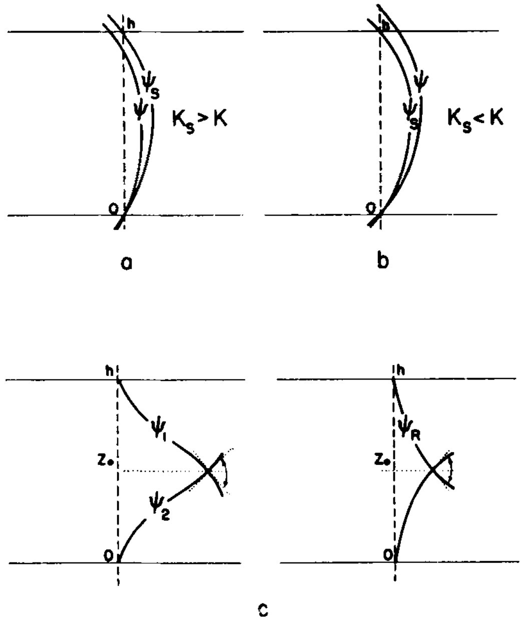


Fig. 9.

it is easy to show that the symmetric flow of type II referred to at the end of the previous section has in fact two stationary waves.

A modified form of the Sturm's theorem will now be applied in order to find how the systems behave when the wavelength of the periodic motion is such that $k \neq k_s$. As it was stated above, equation (3.2) has, in the absence of singularities, two independent solutions, $\psi_1$ and $\psi_2$, which are analytic for all $z$. These solutions can always be chosen in such a way that $\psi_1$ vanishes at $z = 0$ and $\psi_2$ vanishes at $z = h$. We can get, therefore, a continuous solution (full line in left part of fig. 9 c) which has the values

$$\psi = \begin{cases} \psi_1, & 0 \leq z \leq z_0 \\ \psi_2, & z_0 \leq z \leq h \end{cases},$$

and satisfies the boundary conditions, but has a jump in the derivative at the level $z_0$. As we did in section 6, we can again consider this jump as produced by a Rayleigh field which is Laplacian everywhere except at the level $z_0$. This field can always be represented by the expression

$$\psi_R = \sigma\phi_R, \qquad (11.3)$$

where $\phi_R$ is a combination of expotentials. When $\psi_R$ is subtracted from the total field $\psi$, we get a function $\psi_b$ which satisfies the boundary conditions and has continuous derivatives at all levels, and it is defined by the relation

$$\psi = \psi_b + \psi_R. \qquad (11.4)$$

The actual value of $\psi_b$ can be directly obtained when we replace (11.4) for $\psi$ in (3.2). We can easily find that it must satisfy the non-homogeneous differential equation

$$\psi_b'' - \left(k^2 + \frac{U''}{U}\right)\psi_b = \sigma \frac{U''}{U}\phi_R. \qquad (11.5)$$

The solution of this equation satisfying (11.1) can always be found, by means of the Green's function defined by the homogeneous equation, provided that $k \neq k_s$.

We shall consider first a profile having only one stationary wave of wave number $k_s$ with a stream function of the periodic field satisfying (11.2) and the boundary conditions (11.1). For any other wave number $k \neq k_s$ we can always find a $b$-wave whose stream-funcion $\psi_b$ satisfies (11.5) and (11.1), i.e. such that it becomes stationary by the addition of the Rayleigh field in (11.4). If we now multiply (11.2) by $\psi_b$ and (11.5) by $\psi_s$, and subtract, we get

$$(\psi_b\psi_s' - \psi_s\psi_b')' + (k^2 - k_s^2)\,\psi_b\psi_s = -\sigma\frac{U''}{U}\phi_R.$$

or, after integration between the boundaries

$$(k^2 - k_s^2)\int_0^h \psi_b\psi_s dz = -\sigma\int_0^h \frac{U''}{U}\psi_s\phi_R dz.$$

The integral on the left is always positive (due to our convention in the selection of the cell), while the integral on the right, for the types of profiles we are considering $(U''/U \leq 0)$ is always negative, so we can write

$$(k^2 - k_s^2)\left|\int_0^h \psi_b\psi_s dz\right| = \sigma\left|\int_0^h \frac{U''}{U}\psi_s\phi_R dz\right|.$$

We see, then, that waves with wave numbers $k > k_s$ would require a Rayleigh field of the same sign to make them stationary, whereas for $k < k_s$ the required Rayleigh field must have opposite sign. In other words, short $b$-waves need the help of a periodic field of the same sense in order to propagate with the relative phase speed required for the stationary condition, which in turn means that the $b$-field alone would propagate with less speed. The crest and trough lines, initially vertical, will develop tilt in the same sense as the shear of the mean flow. The same argument applied to long $b$-waves shows that they will develop tilt in opposite sense to the mean flow. The character of the stationary wave as a transition between short damped waves and long amplified waves is thereby proved.

If the system under consideration has *two stationary waves*, the same reasoning shows that, starting with very short waves and decreasing the value of $k$, every time we cross an eigenvalue of $k$ the Rayleigh field which is necessary to keep the wave stationary changes its sign. The long waves (for $k$ smaller than the smallest eigenvalue) and the short waves (for $k$ larger than the largest eigenvalue) will behave in the same way, since both need the help of a Rayleigh field of the same sign to become stationary (damped waves). The intermediate range will be of amplified waves, since they must be opposed by a Rayleigh field of different sign in order to remain stationary. We have therefore obtained a generalization of the results obtained by HOLMBOE (1953) concerning skew-symmetric three-layer Couette flow with opposite shear in the central and outer layers.

The last result can be easily generalized to the case of profiles with any number of zeros. If the number of eigenvalues is odd, the shortest and longest waves will be damped waves. If the number of eigenvalues is even, the shortest waves are damped and the longest waves will be amplified. The intermediate ranges of the wavelengths are divided in regions which alternatively will correspond to damped and to amplified waves.

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