Hydraulic Jump in a Fluid System of Two Layers

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Abstract

The number of conjugate states for the flow of a fluid system of two layers is investigated by means of the momentum principle. The uniqueness of the conjugate state is proved for the cases in which the modified Froude number for either layer is predominantly large. Specific experimental results for three special cases demonstrate the uniqueness of the state downstream from the hydraulic jump, and show that, for a first approximation, the simple analysis provides a means for determining the downstream depths, with smaller errors for lower jumps.

Introduction

Although fluid motion in a stratified system was investigated more than a century ago by STOKES (1847), and not long afterwards by HELMHOLTZ (1868), WEBB (1884), GREENHILL (1887) and LOVE (1891), it is only during recent years that it has attracted the serious attention of oceanographers, meteorologists, and hydraulicians. As a result of the revived interest of the geophysicists, the subject is at present enjoying a period of intensive investigation. Among the recent contributors may be mentioned ROSSBY (1951), CRAYA (1951), KUELEGAN (1953), STOMMEL and FARMER (1952), TEPPER (1952), LONG (1953), and BENTON (1953).

In spite of the encouraging advancements of recent years, much is still to be learned about the subject and many questions at issue are yet to be settled. Among these the most important is the one concerning the determination of the state downstream from a hydraulic jump for a completely specified state upstream. It is with a view toward answering this question that an analysis based on simple assumptions was made for the case of two-layer flow and the relevant experiments were performed. The results obtained will be presented in the following sections of this paper.

Analysis

General Considerations

The system under study consists of two superposed layers of fluids flowing over a plane bottom. The upper surface is assumed to be free. For simplicity, the bottom is taken to be horizontal.

As indicated in Fig. 1, the density, the discharge per unit width, and the depths upstream and downstream from the hydraulic jump are respectively denoted by ϱ_1, q_1, h_1, and h_1' for the upper fluid, and by ϱ_2, q_2, h_2, and h_2' for the lower fluid. The gravitational acceleration is denoted by g. For fixed values of the densities, of the discharges, and of g, and for given upstream depths (h_1, h_2), the dynamically possible depths (h_1', h_2') downstream from the jump are said to be conjugate to (h_1, h_2), and the upstream and downstream states are conjugate by definition. A critical state is then defined as one which is conjugate

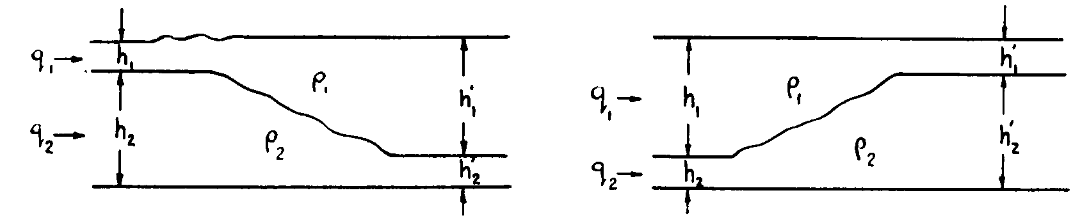

Fig. 1. Definition sketches (a) Normal jump (b) Inverted jump.

to itself, corresponding to an infinitesimal hydraulic jump. The number of conjugate states and the determination thereof will be the chief concern of the present analysis.

If a jump occurs in a fluid system of two layers, there is usually rather violent mixing at the interface. The interfacial shear is generally larger than the shear at the solid boundary, and cannot be neglected without introducing sizable error except under favorable conditions. Furthermore, due to the motion of the fluids the pressure distribution is no longer hydrostatic. As is well known, the distribution of the dynamic part of the pressure at the interface depends directly on the existence and location of the point of separation and therefore indirectly on the distribution of the interfacial shear. Since so little is known about the interfacial shear, it is evident that a rigorous theory cannot be achieved at present. However, if one is content with a first approximation, an *a priori* analysis can be constructed by neglecting the interfacial shear and assuming hydrostatic distribution for the pressure. This analysis will at least yield some conclusions of a qualitative nature which may be expected to hold even if interfacial shear is to be taken into account. Furthermore, for low jumps its conclusions may even be quantitatively correct. Thus, at the present stage of its development, the subject may well benefit from such an analysis.

Neglecting the shear, assuming hydrostatic distribution of pressure, and taking the mean head over the jump section as $\frac{1}{2}(h_1 + h_1')$, one has, by applying the momentum principle to the lower layer,

$$\varrho_2 q_2^2 \left(\frac{1}{h_2'} - \frac{1}{h_2}\right) = h_2 h_1 \varrho_1 g + \frac{1}{2} h_2^2 \varrho_2 g + \frac{1}{2}(h_1 + h_1') \cdot (h_2' - h_2)\,\varrho_1 g - h_2' h_1' \varrho_1 g - \frac{1}{2} h_2'^2 \varrho_2 g \qquad (1)$$

This can be rewritten as

$$2a_2 (h_2 - h_2') = h_2 h_2' (h_2 + h_2') [r(h_1 - h_1') + (h_2 - h_2')] \qquad (2)$$

in which

$$a_2 = \frac{q_2^2}{g}, \quad r = \frac{\varrho_1}{\varrho_2}$$

Similarly, one obtains, for the upper layer,

$$2a_1 (h_1 - h_2') = h_1 h_1' (h_1 + h_1') [(h_1 - h_1') + (h_2 - h_2')] \qquad (3)$$

in which

$$a_1 = \frac{q_1^2}{g}$$

Given h_1 and h_2, one has to solve Eqs. (2) and (3) simultaneously for the conjugate depths h_1' and h_2'. Since these equations are all of the third degree in h_1' and h_2', there are in general nine solutions. One solution is obviously $h_1' = h_1$ and $h_2' = h_2$. Thus at first sight there may be as many as eight other solutions representing eight states conjugate to the given one. However, to have a physical meaning, the solutions must be real and positive. Therefore, the number of conjugate states is decided by that of the positive solutions of Eqs. (2) and (3), aside from the obvious one corresponding to the given state.

It may be noted that since the critical state is conjugate to itself by definition, the differences $h_1' - h_1$ and $h_2' - h_2$ are infinitesimal, and may be replaced by the differentials dh_1 and dh_2. Equations (2) and (3) then become the differential equations

$$\frac{dh_2}{dh_1} = \frac{rh_2^3}{a_2 - h_2^3} \qquad (4)$$

$$\frac{dh_1}{dh_2} = \frac{h_4^3}{a_1 - h_1^3} \qquad (5)$$

from which it immediately follows that

$$h_1^3 = \frac{a_1(a_2 - h_2^3)}{a_2 - (1 - r)h_2^3} \qquad (6)$$

in which h_1 and h_2 are now the critical depths. This equation was already given by BENTON (1953) in an elegant manner from a somewhat different approach. Thus the present definition of a critical state is equivalent to that of Benton.

To investigate the number of conjugate states, one starts by tracing the graphs of Eqs. (3) and (2), using h_1' and h_2' as the variables. For this purpose these equations are rewritten as:

$$h_2' = \frac{2a_1(h_1' - h_1)}{h_1 h_1'(h_1 + h_1')} + h_2 - (h_1' - h_1) \qquad (3\,\mathrm{a})$$

$$h_1' = \frac{2a_2(h_2' - h_2)}{r h_2 h_2'(h_2 + h_2')} + h_1 - \frac{h_2' - h_2}{r} \qquad (2\,\mathrm{a})$$

From Eq. (3 a), it can be seen immediately that h_2' has a discontinuity at $h_1' = -h_1$ and $h_1' = 0$ in such a way that $h_2' = -\infty$ for $h_1' = -h_1 - 0$ and for $h_1' = +0$, and $h_2' = +\infty$ for $h_1' = -h_1 + 0$ and for $h_1' = -0$. Furthermore, for large values of $|h_1'|$, h_2' behaves like $-h_1'$.

Now if h_2' in Eq. (3 a) is differentiated twice and the result equated to zero, the following equation is obtained:

$$h_1'^3 - 3h_1'^2 - 3h_1'h_1^2 - h_1^3 = 0$$

the only real root of which is

$$h_1' = h_1(2^{1/3} - 1)^{-1} \qquad (7)$$

corresponding to a point of inflection of the graph of Eq. (3), located to the right of the h_2'-axis. Differentiating Eq. (3 a) once with respect to h_1' and setting the result to zero, one has

$$2a_1(h_1^2 + 2h_1h_1' - h_1'^2) - h_1h_1'^2(h_1 + h_1')^2 = 0 \qquad (8)$$

the roots of which correspond to the maxima and minima of h_2'. The left side of Eq. (8) is definitely negative for $h_1' < -h_1$, so that there can be no maxima or minima for such values of h_1'. Since $h_2' = +\infty$ for $h_1' = -0$ and $h_1' = -h_1 + 0$, and since there is no point of inflection for negative values of h_1', there is only one minimum of h_2' in the range $-h_1 < h_1' < 0$. For $h_1' = +0$ and $h_1' = +\infty$, Eq. (3 a) shows that $h_2' = -\infty$. With the only point of inflection given by Eq. (7), there can be only one maximum for positive values of h_1'. This can occur only at a value of h_1' smaller than that given by Eq. (7). Furthermore, $h_2' = -\infty$ for $h_1' = -h_1 - 0$, and the function h_2' is monotonic for $h_1' < -h_1$. From the foregoing considerations, the graph of Eq. (3) or (3 a)—with branches I, II, and III—can be traced, and is shown in Fig. 2 in dimensionless terms.

Similarly the graph of Eq. (2) or (2 a) can be traced, as shown in Fig. 2 with branches IV, V, and VI. It is evident that I intersects IV and VI, VI intersects III, and the lower branch of VI intersects the left branch of III. The intersections will be denoted by A, B, C, and F. Furthermore, it should be noted that the left branch of III and the lower branch of VI have no point of inflection. These facts will prove useful in the investigation of the number of intersections of III and VI, which of all possible intersections, alone have positive coordinates and thus determine the number of conjugate states.

If one writes

$$\Delta h_2 = h_2' - h_2^3,\ \Delta h_1 = h_1' - h_1 \qquad (9)$$

and equates the expressions for $\Delta h_2/\Delta h_1$ obtained from Eqs. (2) and (3), one has

$$\frac{2F_1}{x(x+1)} - 1 = r\left(\frac{2F_2}{2F_2 - y(y+1)} - 1\right) \qquad (10)$$

in which

$$F_1 = a_1/h_1^3,\ F_2 = a_2/h_2^3,\ x = h_1'/h_1,\ y = h_2'/h_2 \qquad (11)$$

By a translation of coordinates Eq. (10) can be reduced to the symmetrical form

$$\left[\xi^2 - \left(\frac{1}{4} + 2F_1'^2\right)\right]\left[\eta^2 - \left(\frac{1}{4} + 2F_2'^2\right)\right] = \frac{4rF_1^2F_2^2}{(1-r)^2} \qquad (12)$$

in which F_1' and F_2' are the modified Froude numbers defined by

$$F_1'^2 = F_1^2(1-r)^{-1},\ F_2'^2 = F_2^2(1-r)^{-1} \qquad (13)$$

and

$$\xi = x + \frac{1}{2},\ \eta = y + \frac{1}{2}$$

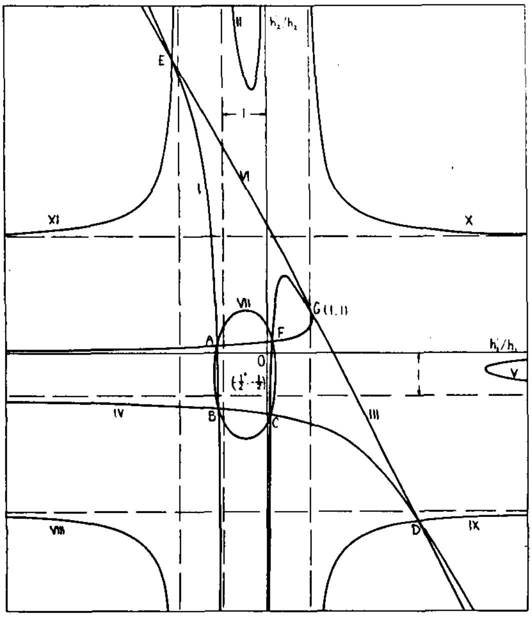

Fig. 2. Graphs of the momentum equations and of Eq. (10) or (12)

The solutions of Eqs. (2) and (3), except the one corresponding to the given state, are those of Eqs. (2) and (10), or those of Eqs. (3) and (10).

The graph of Eq. (10) can be easily traced, and is shown in Fig. 2—branches VII to XI. It is symmetrical with respect to the lines $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$, and consists of one closed convex piece around the center $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and four branches of a hyperbolic shape with asymptotes

$$x = -\frac{1}{2} \pm \left(\frac{1}{4} + 2F_1'^2\right)^{1/2}$$

$$y = -\frac{1}{2} \pm \left(\frac{1}{4} + 2F_2'^2\right)^{1/2} \qquad (14)$$

It is immediately clear that IX must intersect III at a point D and XI must intersect VI at a point E. Since there are already five intersections A, B, C, D, and E which are not those of III and VI, these two branches can intersect in four points at most. That is, for a given state, there can be at most three conjugate states.

Since there is no point of inflection on I, IV, the left branch of III, and the lower branch of VI, these can intersect VII (which is convex) only at A, B, C, and F, as shown in Fig. 2. It is then obvious that III, VI, and VII cannot intersect simultaneously at any point other than F. For otherwise the point, being on III, would lie to the right of F, and being on both VI and VII, would have an ordinate at once greater and smaller than that of F. Thus, if X does not meet III and VI, there can be only two points at which III intersects VI, namely, G (1, 1) and F, G representing the given state and F representing the unique conjugate state.

A few *sufficient* conditions for the uniqueness of the conjugate state can be easily obtained. From Eqs. (8) and (3a) it is evident that the maximum ordinate on III depends on a_1, h_1, and h_2 only. But from Eq. (14) the horizontal asymptote of X can be so far above the x-axis for predominantly large values of F_2' that X and III, and consequently III and VI, do not intersect. Similarly if F_1' is predominantly large X and VII and consequently III and VII do not meet. Thus, under these conditions, the conjugate state is unique. It should be noted that whenever these conditions are satisfied, the conjugate depths are either both greater or both smaller than the corresponding depths of the given state—in other words, any jump that occurs cannot be primarily internal in character.

The results concerning the finiteness of the number of conjugate states in general and the uniqueness of the conjugate state under special conditions are in direct contrast to Benton's claim (Benton, 1953) that, given a complete description of the upstream state, infinitely many downstream states are possible. It seems improbable that the present conclusions would be invalidated if shear were to be taken into account, since the process involved would still —perhaps *a fortiori*—have a deterministic nature.

If the conjugate state is unique, whether a hydraulic jump can occur is decided, as usual, from energy considerations. Otherwise several situations may present themselves. If the energy flux for the given state is less than that of each of the three conjugate states, a jump cannot occur, whereas if it is greater than that for one of the conjugate states only, a jump can occur and the downstream state is unique if it occurs. In all other situations a jump can occur but, if it occurs, the downstream state cannot be uniquely determined by momentum and energy considerations alone, and which of the two or three possible conjugate states will be realized after a jump depends primarily on the controls downstream.

Since the validity of the momentum equations and the uniqueness of the downstream state are to be tested for three cases of primarily internal jumps, specific results from Eqs. (2) and (3) will be given here for these cases. For the case in which the downstream velocities are the same for both layers, one has

$$\frac{q_1}{q_2} = \frac{h_1'}{h_2'} = \lambda$$

and Eq. (2) becomes

$$(1 + r\lambda)\left(\frac{h_2'}{h_2}\right)^3 + r\left(-\frac{h_1}{h_2} + \lambda\right)\left(\frac{h_2'}{h_2}\right) -$$

$$-\left(1 + r\frac{h_1}{h_2} + 2F_2^2\right)\left(\frac{h_2'}{h_2}\right) + 2F_2^2 = 0 \qquad (15)$$

This equation has two positive roots: one is $h_2' = h_2$, corresponding to the given state, and the other corresponds to the conjugate state. If the upper layer is at rest, $a_1 = 0$, and from Eqs. (2) and (3) one obtains

$$\frac{h_2'}{h_2} = \frac{1}{2}\left(\sqrt{1+8F_2'^2} - 1\right) \qquad (16)$$

Similarly, if the lower layer is at rest, one has

$$\frac{h_1'}{h_1} = \frac{1}{2}\left(\sqrt{1+8F_1'^2} - 1\right) \qquad (17)$$

One notes in Eq. (16) that if h_2' is greater than h_2 by a finite amount, F_2' is definitely greater than 1. Physically, this means that the velocity q_2/h_2 must be greater than the celerity of the fastest long waves of infinitesimal amplitude in order to hold the finite jump in place, i.e., finite disturbances progress faster than infinitesimal ones. In this light, Eq. (16) can be considered as a formula giving the celerity of progression of disturbances of finite amplitudes. In fact, the greater the ratio $\Delta h_2/h_2$, the greater the celerity. The same remarks can be made in connection with Eq. (17).

It may be noted also that, if $a_1 = 0$, from Eq. (3) it follows that the free surface is level, whereas if $a_2 = 0$ but $a_1 \neq 0$, Eq. (2) gives $r\Delta h_1 = -\Delta h_2$, and the free surface is not level. Of course, if $a_2 = 0$ and r is very near 1, the change in the free-surface elevation is imperceptible. The truth of the foregoing remarks has been verified in the experiments performed to establish the validity of Eqs. (15) to (17). These experiments will be presented in the following section.

Experimental results

To test the validity of the momentum equations and the uniqueness of the downstream state if a jump occurs, experiments were carried out for three cases[1]. In the first case, the upper fluid was at rest, and the jump of the lower fluid was upward (normal jump). In the second case, the lower fluid was at rest, and the jump of the upper fluid was downward (inverted jump). In the third case, both fluids were in motion, but in such a way that the downstream velocities were the same for both layers. The limitation on the downstream condition for the third case was imposed by an experimental artifice which had to be adopted for the sake of expediency. Experimentally it would be rather difficult and expensive to enable the discharges to vary independently and to realize a stationary jump with two moving layers. Instead of attempting to have a stationary jump, one tried to obtain a surge by filling a channel sealed at the downstream end with the lighter fluid, and discharging the heavier fluid into it at the bottom. As the lower fluid reached the end of the channel, it rose in height to form a surge which moved upstream with a celerity depending on the actual discharge of the lower fluid and the upstream depths. On taking velocities relative to the surge, the situation of two moving layers was achieved, but with the restriction that the downstream velocities were necessarily the same for both layers.

Water was used for the moving fluid in het first two cases (stationary jumps), and for het

[1] The experiments for Cases 1 and 2 were performed by the second author for his M. S. Thesis at the Iowa Institute of Hydraulic Research. The first author is responsible for the rest of the material contained in this paper.

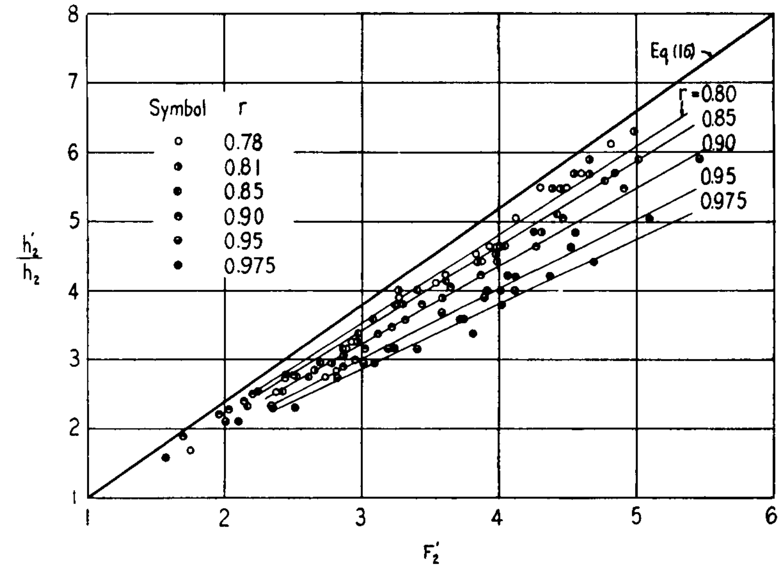

Fig. 3. Experimental results for the normal jump

lower fluid in the third case (surge). The other fluid used was either stanisol (an oil prepared by the Standard Oil Company, with specific gravity 0.777), or a mixture of stanisol and carbon tetrachloride (specific gravity 1.59). Since stanisol has a kinematic viscosity of about 3.4×10^{-5} ft²/sec, is not highly inflammable, and mixes well with carbon tetrachloride to form a homogeneous mixture, it was found to be quite suitable to use. For easy visualization of the interface, the oil or oil mixture was always dyed red.

The lucite flume used was 4 feet long, 6 inches wide, and 8 inches deep. For stationary jumps the depths were controlled by one upstream gate and one downstream gate. The upstream gate was designed to ensure parallel flow of the water into the channel, and the downstream gate was designed to reduce entrainment of oil. The discharge of water was controlled by a valve upstream and measured by a triangular weir downstream. A supply of oil was maintained to compensate for the loss due to entrainment. For the moving jump the total depth was controlled by horizontal slots on the side of the channel near the upstream end. As the water entered the sealed channel, the oil would spill over the lowest open slot, so that an approximately constant total depth was maintained at the upstream end. The depth of the water at entrance was not controlled, but was found to be about half an inch. With the total depth at the entrance controlled by the slots, different combinations of (h_1, h_2) could be achieved. The discharge was again controlled by an upstream valve, and was measured through the celerity of the moving jump and the water depths upstream and downstream from the jump. All depths were measured either visually or photographically.

Before the experimental results are presented, it may be noted that the variables h_2' and h_2' depend on h_1, h_2, ϱ_1, ϱ_2, g, q_1, q_2, and the dynamic viscocities μ_1 and μ_2. If h_2' is taken as the dependent variable, a dimensional analysis shows that

$$\frac{h_2'}{h_2} = \left(\frac{h_1}{h_2}, \frac{q_1}{q_2}, \frac{\mu_1}{\mu_2}, r, F_2, R_2\right) \tag{18}$$

in which F_2 is the Froude number for the lower layer and $R_2 = q_2 \varrho_2/\mu_2$ is the Reynolds number for the same layer. Equation (18) can be used to correlate the data obtained for Case 3 and Case 1 (for which $q_1/q_2 = 0$). For Case 2, a similar analysis yields

$$\frac{h_1'}{h_1} = F_1\left(\frac{h_1}{h_2}, \frac{\mu_1}{\mu_2}, r, F_1, R_1\right) \tag{19}$$

in which F_1 and R_1 are respectively the Froude number and the Reynolds number for the upper layer. From Eq. (2) it follows that h_2'/h_2 does not depend on μ_1/μ_2 and R_2 according to the simple theory. Furthermore,

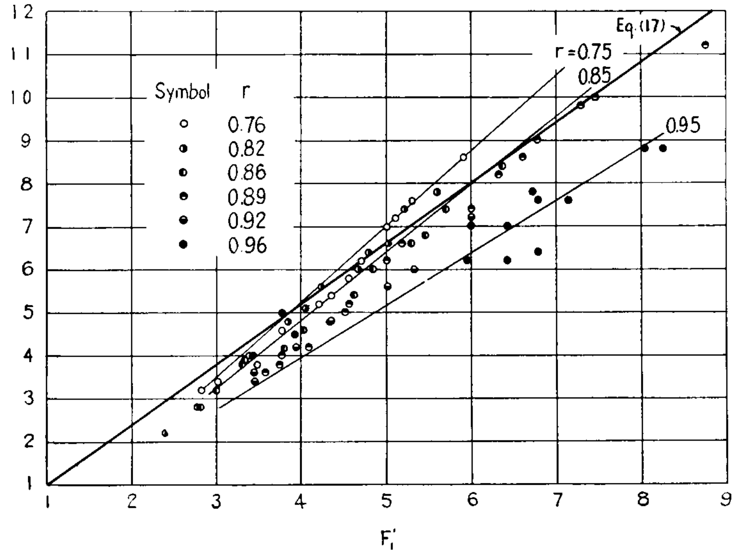

Fig. 4. Experimental results for the inverted jump

Eqs. (16) and (17) indicate that in Cases 1 and 2, respectively, the ratios h'_2/h_2 and h'_1/h_1 depends only on F'_2 and F'_1. In other words, according to the simple theory, the parameters h_1/h_2, μ_1/μ_2, and R_2 or R_1 have no effect on h'_2/h_2 or h'_2/h_1, the effect of r and F_2 are embodied in F'_2 in Case 1, and those of r and F_1 are embodied in F'_1 in Case 2. To what extent the theoretical predictions are valid will be shown by the experimental results.

The results for the normal jump are shown in Fig. 3, from which it can be seen that the ratio h'_2/h_2 depends not only on the modified Froude number F'_2, but also on the density ratio r. But F'_2 and r seem to determine h'_2/h_2 uniquely in the range of experimentation. The points all lie below the theoretical curve, with increasing deviation for increasing r and F'_2. The results show that for increasing modified Froude number and density ratio, the effect of shear (chiefly interfacial shear) becomes increasingly important. The free surface was found to be level as expected.

Results for the inverted jump (Fig. 4) show similar trend, except that some points lie above the theoretical curve. The greater scatter is partly due to the difficulty of ascertaining the upstream depth h_1, since an undular jump occurred at the free surface. The free surface was observed to be definitely higher after the jump.

Figure 5 shows the result obtained for Case 3, the velocities being taken relative to the surge. The abscissa is the theoretical value of h'_2/h_2 computed from Eq. (15), and the ordinate is the experimental value of the same ratio. Here it can be seen that except for very high jumps the theoretical prediction is very well verified experimentally. Since h_1^3 was always large as compared with a_1, the free surface could be expected to be almost level according to Eq. (3). As can be seen from Plate 1, this expectation was well verified.

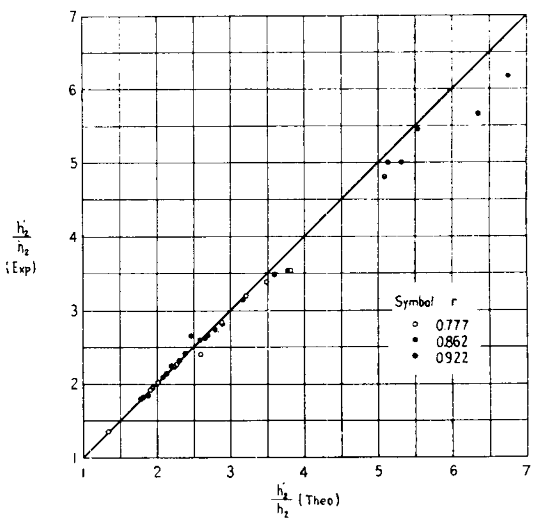

Fig. 5. Comparison of experimental results for the surge with theoretical predictions according to Eq. (15)

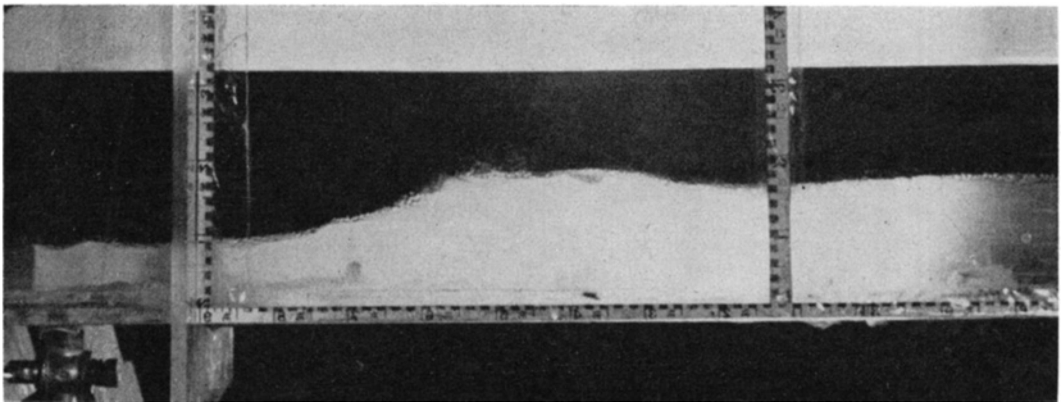

Plate 1. Photograph showing interfacial surge and the almost horizontal free surface between air and oil.

From the same plate it can be seen that the appearance of the surge was fairly smooth, and no violent interfacial mixing existed.

Because of the lack of violent interfacial mixing in Case 3 the neglecting of interfacial shear in the analysis is more nearly justified, and the agreement between theoretical and experimental results is consequently better than in Cases 1 and 2, in which rather violent mixing usually occurred at the interface. In this connection it may also be mentioned that because of the simplicity of the experimental technique employed in Case 3, the experimental limitations encountered in that case are much less severe than in the cases of steady jumps. Why the data for Cases 1 and 2 show a dispersion according to the density ratio whereas those for Case 3 does not, cannot be convincingly explained. This does not, however, prevent one from making the points in regard to the several questions at issue.

From the experiments it is evident that, for the special cases investigated, to a given upstream state there corresponds only one downstream state. Aside from this, it seems that the analysis previously presented provides a means for determining h_2'/h_2 or h_1'/h_1, at least to a first approximation, the absolute errors being small for small theoretical values of $\Delta h_2/h_2$ or $\Delta h_1/h_1$—which correspond to small values of F_2'—1 or F_1'—1 in Case 1 or 2.

Conclusions

From the foregoing investigation of the hydraulic jump in a fluid system of two layers, it can be concluded that:

1. According to a simple analysis based on the momentum principle and the neglecting of shear, there can be at most three states conjugate to a given one. These states can be determined from Eqs. (2) and (3). If the modified Froude number of either layer is predominantly large, there is only one conjugate state. If the conjugate state is unique, and if a jump can occur from the consideration of energy and does occur, the state downstream from the jump is completely determined. If there are more than one conjugate states, merely from momentum and energy considerations it may not be possible to determine uniquely the state downstream from a hydraulic jump.

2. Specific results of experiments performed on primarily internal hydraulic jumps for three special cases, shown in Figs. 3 to 5, demonstrate the uniqueness of the downstream state in these cases, which is expected from the analysis. Furthermore, these results show that the simple analysis provides means for determining the downstream depths to a first approximation, the absolute errors in h_1'/h_1 or h_2'/h_2 being small for small theoretical values of $\Delta h_1/h_1$ or $\Delta h_2/h_2$.

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