On the Use of Space-Smoothing in Physical Weather Forecasting

By RAGNAR FJØRTOFT, University of Copenhagen

(Manuscript received April 15, 1955)

Abstract

In section 1—2 a certain space-smoothing operation is defined and its usefulness in solving elliptic equations is demonstrated in the case of a Poisson equation. It leads to solutions in a closed form which possess the numerical simplicity of the ordinary iteration methods, but is converging more rapidly. The reverse operation of unsmoothing is also defined as far as it can be done, and it is mentioned that the combined processes of smoothing and unsmoothing are convenient tools for obtaining a spectral analysis of horizontal scalar fields, and also to remove systematical errors which are made when derivatives are taken as finite differences.

In sections 3—6 the application of smoothing is shown in the barotropic forecasting problem. At first a general theorem is proved concerning trajectories of two-dimensional non-divergent flow. It states that if the streamfunction ψ_1 of such a flow can be decomposed into two components ψ_2, α of which α is individually conserved in the ψ_1-motion, then the displacements of the fluid particles up to any time can be found by at first displacing in the *stationary* flow $\alpha_{t=0}$ = const and then adding from the resulting positions the displacements in the flow with the streamfunction ψ_2. The theorem is first applied to barotropic flow. In this case the first stationary field to displace in is the deviation between the actual and smoothed flow, while the second field to displace in is the smoothed flow.

The space-smoothing is next applied to an equation expressing the individual conservation of a quantity s in a two-dimensional non-divergent flow. The Reynolds term belonging to the smoothed equation is studied and found to depend essentially upon the deformation properties of the velocity field. The role of deformation for the net spectral flow of energy in the s-field is studied. The smoothing is in particular applied to the vorticity equation to show how this possibly can be utilized in the integration problem.

In sections 7—11 the baroclinic case is considered. In sections 7—8 is shown the fundamental role of deformation for the interchange of potential and kinetic energy. It is found that in the advective model there is direct proportionality between the change in total kinetic energy and total thermal wind energy, and also a direct proportionality between the change in kinetic energy for the vertically mean motion and the thermal wind energy.

In section 9 is discussed the possible importance of non-linear interference for the understanding of the creation and local distribution of disturbances in the atmosphere.

The integration problem is discussed in sections 10—12. At first an extension of the barotropic displacement rule is given for the vertically mean motion. The trajectory problem for levels other than the mean level is touched in section 12, and a simple non-advective model discussed shortly in section 11.

Introduction

In an article in Tellus (FJØRTOFT, 1952) it was shown that a certain operation of space-smoothing of the horizontal streamfield could be utilized in connection with the barotropic forecasting problem. On the one hand it led to rapidly converging solutions of the elliptic equation appearing in the problem. On the other hand the smoothing was of such a character that vorticity would be conserved in the smoothed motion as well. This reduced considerably the number of iterations which are necessary for a forecast over a certain period, provided the numerical problem was treated in a Lagrangian sense. Since the appearance of the mentioned article the further work with the method has continued along the following directions: First, the method has been improved for the barotropic case, and extended to simple baroclinic models. Secondly, it has been tested to a number of 50 cases of which some also have been treated on numerical computers. By comparison the results hitherto seem to be practically equally good.

The application of the developed integration methods will be brought in a second part of this paper because all the additional problems which arise as soon as theory is applied to practical cases deserve a separate treatment. Instead, however, is included in this part some results of more general theoretical interest which are closely related to the integration problem.

On a certain space-smoothing operation

1. Definition of the smoothing operator

Let β be any function of position on a spherical surface, ∇_h the spherical nabla-operator, and β_q the eigenvalues of the eigen-valueproblem

$$\nabla_h^2 \beta_q + a_q \beta_q = 0 \tag{1}$$

$$\beta_q = \text{unique on a spherical surface} \tag{2}$$

The a_q's are, as is known, given by

$$a_q = \frac{q(q+1)}{R^2}, \quad q = 1, 2, \ldots \tag{3}$$

where R is the radius of the sphere. It will be supposed that β satisfies the conditions for an expansion after the eigensolutions of (1), (2):

$$\beta = \sum_{q=1}^{\infty} \beta_q \tag{4}$$

We now define as $\beta^{(r)}$ an operation performed on β which is given by

$$\beta^{(r)} = \beta + \frac{\nabla_h^2 \beta}{a_r} \tag{5}$$

Inserting here from (4) and using (1) we arrive at

$$\beta^{(r)} = \sum_{q=1}^{\infty} \left(1 - \frac{a_q}{a_r}\right) \beta_q \tag{6}$$

Hence it is seen that the operation which has been introduced smooths out entirely the r-component of β and damps all β_q for which $a_q < a_r$, e.i. according to (3) for $q < r$. Therefore if (4) is approximated by a finite series

$$\beta = \sum_{q=1}^{N} \beta_q \tag{7}$$

it is seen that $\beta^{(N)}$ will imply a smoothing of all components of β, since then

$$\beta^{(N)} = \sum_{q=1}^{N} \left(1 - \frac{a_q}{a_N}\right) \beta_q \tag{8}$$

where now $0 \leq 1 - \frac{a_q}{a_N} < 1$.

For the following arguments it will be supposed that β is given or approximated by a finite series (7). It should be mentioned that in the practical applications to be treated in the second part of this paper, the differential operation $^{(r)}$ will be replaced by a corresponding finite difference operation $^{-r}$ which satisfies the following conditions

$$\bar{\beta}_q^r = \text{or} \simeq \beta_q^{(r)} \text{ when } q \lesseqgtr r \tag{9}$$

and

$$|\bar{\beta}_q^r| \simeq 0 \text{ when } q > r. \tag{10}$$

In the following we shall make such use of the operation $^{(r)}$ which makes it effective for components β_q with $q \leq r$ only. Whether we use (9) or (5) will therefore not change the results we arrive at essentially. However, for the theoretical reasoning in this paper it will be simpler to use the operation$^{(r)}$ because of the simple way in which the eigenvalues a_q enter into the coefficients of the expansion of $\beta^{(r)}$.

A repeated smoothing will now imply the relationship

$$\beta^{(r),(s)\dots(t)} = \sum_{q=1}^{N} \left(1 - \frac{a_q}{a_r}\right)\left(1 - \frac{a_q}{a_s}\right)\dots$$

$$\left(1 - \frac{a_q}{a_t}\right)\beta_q \tag{11}$$

and thus result in a complete damping out of the components β_q with the indexes $q = r, s, \dots, t$.

The identity (5) may also be written

$$\beta = -\frac{\nabla_h^2 \beta}{a_r} + \beta^{(r)} \tag{12}$$

If we write a similar identity for $\beta^{(r)}$ we obtain

$$\beta^{(r)} = -\frac{\nabla_h^2 \beta^r}{a_s} + \beta^{(r)(s)} \tag{13}$$

where s may be taken equal to or different from r. Combining (12) and (13) we obtain

$$\beta = -\frac{\nabla_h^2 \beta}{a_r} - \frac{\nabla_h^2 \beta^{(r)}}{a_s} + \beta^{(r)(s)}$$

Iterating this procedure we arrive at

$$\beta = -\left[\frac{\nabla_h^2 \beta}{a_r} + \frac{\nabla_h^2 \beta^{(r)}}{a_s} + \dots + \frac{\nabla_h^2 \beta^{(r),(s)\dots(t)}}{a_u}\right] +$$

$$+ \beta^{(r)(s)\dots(t)(u)} \tag{14}$$

We observe that

$$\nabla_h^2 \beta^{(r)(s)\dots(t)} = (\nabla_h^2 \beta)^{(r)(s)\dots(t)} \tag{15}$$

Let us now first take $r, s, \dots, t, u, \dots = N$. We then have

$$\beta^{(r)(s)\dots(t)(u)\dots} \equiv \beta^{(N)^p} =$$

$$= \sum_{q=1}^{N} \left(1 - \frac{a_q}{a_N}\right)^p \beta_q \rightarrow 0$$

when $p \rightarrow \infty$. In this case therefore, using (15), we may write (14) in the limit as

$$\beta = -\frac{1}{a_N} \lim_{p \to \infty} \left[\nabla_h^2 \beta + (\nabla_h^2 \beta)^{(N)} + \dots +\right.$$

$$\left. + (\nabla_h^2 \beta)^{(N)^p} + \dots\right] \tag{16}$$

In the second case we choose successively $r = N, s = N - 1, \dots, t = 2, u = 1$. Since now

$$\beta^{(N)(N-1)\dots(2)(1)} = \sum_{q=1}^{N} \left(1 - \frac{a_q}{a_N}\right)$$

$$\left(1 - \frac{a_q}{a_{N-1}}\right)\dots\left(1 - \frac{a_q}{a_2}\right)\left(1 - \frac{a_q}{a_1}\right)\beta_q,$$

(14) can by reference to (15) be written

$$\beta = -\left[\frac{\nabla_h^2 \beta}{a_N} + \frac{(\nabla_h^2 \beta)^{(N)}}{a_{N-1}} + \dots +\right.$$

$$\left. + \frac{(\nabla_h^2 \beta)^{(N)(N-1)\dots(2)}}{a_1}\right] \tag{17}$$

The solution of a Poisson equation

$$\nabla_h^2 \beta = F \tag{18}$$

with the boundary conditions (2) and correct for the N first components can now be obtained by replacing $\nabla_h^2 \beta$ with the known function F in either (16) or (17). In the former case the form of the solution corresponds completely to the one of successive iterations used in Richardsons method (19). In (17), however, the solution is given in a closed form while at the same time maintaining the computational simplicity which characterizes the method of successive iterations. The main advantage, however, in using (17) for numerical purposes is explained by the fact that taking a few terms in (17) gives a much closer approximation to the solution than taking the same number of terms in (16).

2. The "reverse" operation of unsmoothing

Suppose now that β is an unknown function satisfying a relation

$$\beta^{(N)} = f \tag{19}$$

where f is known. While obviously β_N cannot be found because $\beta_N^{(N)}$ must be zero, the other components can be determined by a reverse process of unsmoothing. For instance is

$$\beta = \beta_N + \lim_{p\to\infty}\left[\beta^{(N)} - \frac{\nabla_h^2 \beta^{(N)}}{a_N} + \frac{\nabla_h^4 \beta^{(N)}}{a_N^2} + \dots + (-1)^p \frac{\nabla_h^{2p}\beta^{(N)}}{a_N^p} + \dots\right] \qquad (20)$$

as can be verified by substituting $\beta = \sum_{q=1}^{N} \beta_q$ on both sides. It can be shown that there exists also a similar relationship between β, β_N, and $\beta^{(N)}$ in a closed form. The combined process of smoothing and unsmoothing will prove very useful in the numerical work of obtaining a spectral analysis of a field β as will be demonstrated in an article to appear later. It has also been used to remove systematical errors arising when space derivatives are obtained by taking finite differences.

The barotropic forecasting problem

3. Formulation of the integration problem

In the so-called barotropic case velocity $\mathbf{v}$ is supposed to be horizontal and non-divergent

$$\mathbf{v} = -\nabla_h \psi \times \mathbf{k} \qquad (21)$$

and the motion governed by the equation

$$\frac{D}{dt}\nabla_h \times \mathbf{v}\cdot\mathbf{k} = 0. \qquad (22)$$

For simplicity the motion will be considered in an absolute coordinate system. $\frac{D}{dt}$ is defined from

$$\frac{D}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_h$$

Eq. (22) then simply expresses the individual conservation of the vertical component of absolute vorticity. Using (21) we can also write (22) as

$$\frac{D}{dt}\nabla_h^2\psi = 0. \qquad (23)$$

Suppose now that ψ and hence $\nabla_h^2\psi$ is known initially. Suppose further that on the basis of this information the trajectories of the fluid particles, or rather their displacements up to a time t_1 are known with some accuracy. In accordance with (23) we will then be able to find the distribution of $\nabla_h^2\psi$ at time t_1 with a corresponding accuracy by simply letting the fluid particles carry along the vorticities they possess initially. If $F(\mathbf{r})$ denotes this distribution of $\nabla_h^2\psi$ at time $t = t_1$, $\mathbf{r}$ being the position vector for points in the considered horizontal surface, a solution of the Poisson equation

$$\nabla_h^2\psi^\star = F(\mathbf{r})$$

will yield the field $\psi_{t=t_1}$ with some accuracy. By n iterations of the operations above we will obtain a forecast for the period $n\,t_1$.

According to the integration scheme outlined above the integration problem consists of two essentially distinct parts. The second part has already been dealt with in some detail in section 1. In the following section the first part will be considered more closely.

4. The trajectory or displacement problem for barotropic flow

If $\mathbf{p}$ denotes the displacement of a particle up to a time t and $\mathbf{p}_1'$ the corresponding displacement in the stationary flow $\mathbf{v} = \mathbf{v}_{t=0}$, we obviously have

$$\lim_{t\to 0}\frac{\mathbf{p}'}{t} = \lim_{t\to 0}\frac{\mathbf{p}}{t}$$

For $t \le t_1$ we can therefore, if t_1 is taken sufficiently small consider $\mathbf{p}'$ as a sufficiently good approximation to $\mathbf{p}$, meaning thereby that $\mathbf{p}' - \mathbf{p}$ then will be small of higher order than $\mathbf{p}$ itself. In other words, the approximation

$$\mathbf{v} = \mathbf{v}_{t=0};\ t \le t_1 \qquad (24)$$

will be sufficiently accurate as a basis to find $\mathbf{p}$, if only t_1 is taken sufficiently small.

Applied to the atmosphere we probably have to take t_1 not > 2 h. if we shall avoid serious errors in a 24 h. forecast, thus making it necessary with a total of 12 iterations for a forecast over this period. In the following a method will be given by means of which t_1 can be increased considerably. The method is based upon a certain theorem which will find repeated use in this paper. For that reason it will be most convenient to give the proof of it in general terms. Let then ψ_1 symbolize a streamfunction defining a velocity $\mathbf{v}_1$ in some motion by

$$\mathbf{v}_1 = -\nabla_h\psi_1 \times \mathbf{k},$$

while $\frac{D_1}{dt}$ is defined from: $\frac{D_1}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla_h$. Suppose further that it is possible to write

$$\psi_1 = \psi_2 + \alpha \tag{25}$$

$$\frac{D_1 \alpha}{dt} = 0. \tag{26}$$

With $\mathbf{v}_\alpha$, $\mathbf{v}_2$, $\frac{D_2}{dt}$ defined by

$$\mathbf{v}_\alpha = -\nabla_h \alpha \times \mathbf{k}$$

$$\mathbf{v}_2 = -\nabla_h \psi_2 \times \mathbf{k}$$

$$\frac{D_2}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_2 \cdot \nabla_h$$

we obtain

$$\frac{D_1 \alpha}{dt} = \frac{\partial \alpha}{\partial t} + (\mathbf{v}_2 + \mathbf{v}_\alpha) \cdot \nabla_h \alpha = \frac{D_2 \alpha}{dt} + \mathbf{v}_\alpha \cdot \nabla_h \alpha$$

Since now $\mathbf{v}_\alpha \cdot \nabla_h \alpha = 0$ we obtain by comparing with (26):

$$\frac{D_2 \alpha}{dt} = 0. \tag{27}$$

Hence, α which according to (26) is individually conserved in our motion, is also so in the fictitious motion with the velocities $\mathbf{v}_2$.

We consider now a particle which at time $t = 0$ is at some arbitrary point A on some arbitrary level line $\alpha_{t=0} = \alpha_1$ (fig. 1). In accordance with (26) this particle moves with the level line and hence must at time $t = T$ be at some point C at $\alpha_{t=T} = \alpha_1$. In the fictitious motion with the velocities $\mathbf{v}_2$ the "particle" which at time $t = T$ also arrives at C must in accordance with (27) have originated from some point, say B, at the level line $\alpha_{t=0} = \alpha_1$. We can now find the point B in a simple way, using only the *initial* conditions, by the following arguments: We consider all particles which at time $t = 0$ are on the line $\alpha_{t=0} = \alpha_1$. The set of trajectories of these particles in the *fictitious* motion are next considered. If these are labelled in some unique fashion by a set $\{l\}$ this will introduce a function $f(\mathbf{r})$ such that $f(\mathbf{r}) = \text{const} = l$ will represent the equation for the trajectory numbered l. Particularly we shall label the

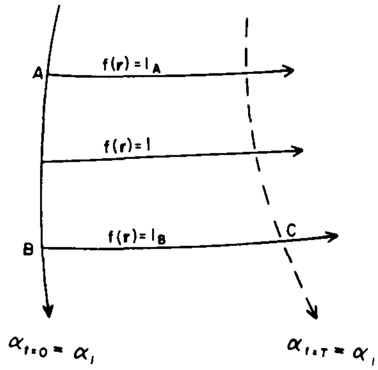

Fig. 1. Illustration to the text in section 4.

trajectories through A and B with l_A and l_B, respectively. Now, according to definition

$$\frac{\partial f}{\partial t} = 0 \tag{28}$$

Hence $\frac{D_1 f}{dt} = \mathbf{v}_1 \cdot \nabla_h f$. Substituting here $\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_\alpha$ we obtain

$$\frac{D_1 f}{dt} = \mathbf{v}_2 \cdot \nabla_h f + \mathbf{v}_\alpha \cdot \nabla_h \tag{29}$$

Suppose now that in particular $\left(\frac{D_1 f}{dt}\right)_A$ denotes the individual derivative for the particle which initially is at A. In this case, repeating the arguments from earlier, $\mathbf{v}_2$ must be the velocity of a "particle" in the fictitious motion which is initially at $\alpha_{t=0} = \alpha_1$, and hence must be parallel to the trajectory $f(\mathbf{r}) = l$ which passes through the considered point. Hence $\mathbf{v}_2 \cdot \nabla_h f = 0$, and accordingly (29) can be written when applied to the particle in question

$$\left(\frac{D_1 f}{dt}\right)_A = \mathbf{v}_\alpha \cdot \nabla_h f,$$

or if we recall that $\mathbf{v}_\alpha \cdot \nabla_h f = \nabla_h \alpha \times \nabla_h f \cdot \mathbf{k}$

$$\left(\frac{D_1 f}{dt}\right)_A = \nabla_h \alpha \times \nabla_h f \cdot \mathbf{k} \tag{30}$$

If next $\mathbf{v}_3$ is defined as the stationary velocity $\mathbf{v}_3 = -\nabla_h \alpha_{t=0} \times \mathbf{k}$, and $\frac{D_3}{dt}$ is the correspond-

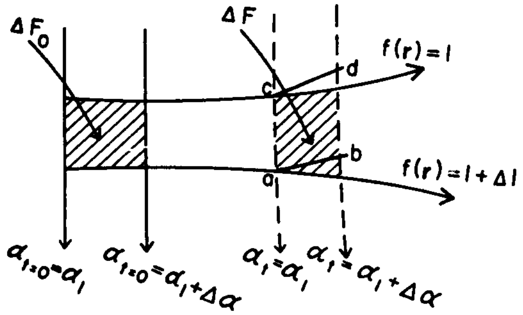

Fig. 2. Illustration for the text below.

ing "individual" derivative: $\frac{D_3}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_3 \cdot \nabla_h$, we obtain by reference to (28)

$$\left(\frac{D_3 f}{dt}\right)_A = \mathbf{v}_3 \cdot \nabla_h f$$

or

$$\left(\frac{D_3 f}{dt}\right)_A = \nabla_h \alpha_{t=0} \times \nabla_h f \cdot \mathbf{k} \tag{31}$$

We have as is known

$$\nabla_h \alpha \times \nabla_h f \cdot \mathbf{k} = \lim_{\triangle\alpha,\, \triangle l \to 0} \frac{\triangle\alpha \triangle l}{\triangle F} \tag{32}$$

and

$$\nabla_h \alpha_{t=0} \times \nabla_h f \cdot \mathbf{k} = \lim_{\triangle\alpha,\, \triangle l \to 0} \frac{\triangle\alpha \triangle l}{\triangle F_0} \tag{33}$$

Here $\triangle F$ is the area between the level lines $\alpha_t = \alpha_1$, $\alpha_t = \alpha_1 + \triangle\alpha$, some fictitious trajectory $f(\mathbf{r}) = l$, and the neighbour one $f(\mathbf{r}) = l + \triangle l$, (fig. 2), while $\triangle F_0$ correspondingly is the area between the level lines $\alpha_{t=0} = \alpha_1$, $\alpha_{t=0} = \alpha_1 + \triangle\alpha$, some fictitious trajectory $f(\mathbf{r}) = l'$, and the neighbour one $f(\mathbf{r}) = l' + \triangle l$. Let now $\triangle F_1$ denote the area of the surface element $abcd$ into which $\triangle F_0$ transforms in the $\mathbf{v}_2$-motion. Because of (27) and the definition of $f(\mathbf{r})$ the corners a, b, c, d must be situated as the diagram shows. Since $\nabla_h \cdot \mathbf{v}_2 = 0$, $\triangle F_1 = \triangle F$. On the other hand if in particular $l' = l$, from continuity reasons $\triangle F / \triangle F_1 \to 1$ when $\triangle\alpha \to 0$. Accordingly also $\triangle F_0 / \triangle F \to 1$ when $\triangle\alpha \to 0$. This in connection with (33), (32), (31), and (30) gives

$$\left(\frac{D_1 f}{dt}\right)_A = \left(\frac{D_3 f}{dt}\right)_A. \tag{34}$$

provided f assumes the same value on both sides. Since now both particles, namely the one of the real motion and the one of the stationary field $\alpha_{t=0}$, initially have the same value l_A for $f(\mathbf{r})$, we obtain by integration of (34) that also at any later time the two particles assume identical values of f. Particularly, the particle moving in the stationary field will arrive at B on the trajectory $f(\mathbf{r}) = l_B$ at the same time as the particle in the real motion arrives at C on the same fictitious trajectory. To summarize we have therefore obtained the following result:

(35) If a flow is determined from a streamfunction ψ_1 which satisfies the conditions

$$\psi_1 = \psi_2 + \alpha$$

$$\frac{D_1 \alpha}{dt} = 0$$

we can find the displacements of the fluid particles up to any time t by displacing at first in the stationary velocity field $-\nabla_h \alpha_{t=0} \times \mathbf{k}$ and then adding from the resulting positions the corresponding displacements in the fictitious motion with the velocities $\mathbf{v}_2 = -\nabla_h \psi_2 \times \mathbf{k}$.

The result that trajectories can be added in this way seems surprising at first sight. For usually trajectories can not be added by virtue of the highly non-linear operations involved in finding them. When we nevertheless can do so in the present case it must be due to the rather special conditions expressed in (25), (26). However, it should be borne in mind that the order in which we let the displacements take place is not arbitrary. This can easily be seen by for instance taking a particle initially in a stagnation point of the field $\alpha_{t=0}$. Then the first displacement is zero, and the net displacement is accordingly determined from the ψ_2-flow alone. If we had started with this displacement the next displacement in the $\alpha_{t=0}$-field would, however, in general not have been zero.

In this section we shall make a first use of the above theorem in an application to barotropic flow. We then write down the identity

(12) applied to the streamfunction ψ, taking $r = N$, together with eq. (23) after this has been divided by $-a_N$. This gives

$$\psi = \psi^{(N)} - \frac{\nabla_h^2 \psi}{a_N} \tag{36}$$

$$\frac{D}{dt}\left(-\frac{\nabla_h^2 \psi}{a_N}\right) = 0 \tag{37}$$

This system is a special case of (25), (26), as it is seen if we let $\psi_1 \to \psi$ and $\alpha \to -\frac{\nabla_h^2 \psi}{a_N}$. We obtain therefore the following rule:

$$\left.\begin{array}{l}\text{The displacements in barotropic flow up to any time may be found by at first}\\ \text{displacing in the stationary velocity field with the streamfunction } -\left(\frac{\nabla_h^2\psi}{a_N}\right)_{t=0},\\ \text{and then from the resulting positions adding the corresponding displacements}\\ \text{in the smoothed motion with the streamfunction } \psi^{(N)}.\end{array}\right\} \tag{38}$$

We now put as an approximation

$$\mathbf{v}^{(N)} = \mathbf{v}^{(N)}_{t=0}; \quad t \leq t_1' \tag{39}$$

Since there is no upper limit to the period over which we can find the displacements in the stationary field, the upper limit to the period over which we can find the displacements of the particles according to the combined displacement rule above, will be given by t_1' in (39). Since $\mathbf{v}^{(N)}$ represents a smoothed flow in space, it must be less timevariable than the velocities $\mathbf{v}$ themselves. Accordingly we may take t_1' in (39) larger than t_1 in (24). Precisely how large we can take t_1' must depend upon the spectral distribution in space of ψ, partly because this among other things will determine the number N of the component at which we will break the expansion after eigensolutions ψ_q in (7). When applied to the flow near the 500 mb surface experience has shown that the 2 h. for t_1 can be increased approximately to 24 h. for t_1'.

We may now take advantage of this in any forecasting problem of the following nature: Let s denote a quantity which is individually conserved in barotropic motion:

$$\left.\begin{array}{l}\dfrac{Ds}{dt} = 0\\[2ex] \dfrac{D\nabla_h^2\psi}{dt} = 0\end{array}\right\} \tag{40}$$

If now s and ψ are given initially the displacement rule above will determine the distribution of s after the period t_1'. If particularly $s = \nabla_h^2\psi$, the first displacement of the level lines $(\nabla_h^2\psi)_{t=0} =$ const in the stationary field with the streamfunction $-\left(\frac{\nabla_h^2\psi}{a_N}\right)_{t=0}$ yields zero changes. In this case it is therefore sufficient with a displacement only in the smoothed flow. This is simply another expression of (27), which, when $\alpha \to -\frac{\nabla_h^2\psi}{a_N}$, $\frac{D_2}{dt} \to \frac{D^N}{dt}$, after multiplication by $-a_N$ transforms to

$$\frac{D^N}{dt}\nabla_h^2\psi = 0 \tag{41}$$

$\frac{D^N}{dt}$ being defined by

$$\frac{D^N}{dt} = \frac{\partial}{\partial t} + \mathbf{v}^{(N)} \cdot \nabla_h \tag{42}$$

For any other conservative quantity s, however, the level lines $s_{t=0} =$ const must generally at first be displaced in the stationary field with the streamfunction $-\left(\frac{\nabla_h^2\psi}{a_N}\right)_{t=0}$.

5. A few studies of the dynamical equations of the smoothed flow

The operation of space smoothing$^{(N)}$ was introduced in section 1 at first merely as a method of solving a Poisson equation. In the section above this operation of smoothing has also got dynamical interest in the sense that it can be utilized with considerable advantage in the problem of finding trajectories of fluid particles in barotropic motion. The question then naturally arises to what extent the dynamics of such a smoothed motion is determined in terms of the smoothed motion itself. This is essentially a question of the relative importance of certain additional Reynold terms which appear in the dynamical equation as soon as they are being smoothed. Some inquiry into the nature and interpretation of the Reynold term is the chief object of this section.

Suppose now that we have

$$\frac{Ds}{dt} = H \tag{43}$$

where H is a source for s which is a known function of space and time. For the following argumentation it does not matter if we put $H = 0$, giving

$$\frac{Ds}{dt} = 0 \tag{40}$$

or alternatively

$$\frac{\partial s}{\partial t} = -\mathbf{v}\cdot\nabla_h s. \tag{44}$$

Applying here the smoothing $^{(N)} = 1 + \frac{\nabla_h^2}{a_N}$ on both sides we get after some computation

$$\frac{D^N s^{(N)}}{dt} = (R_s)_N \tag{45}$$

where

$$(R_s)_N = (\mathbf{v} - \mathbf{v}^{(N)})\cdot\nabla_h (s - s^{(N)}) + \frac{2}{a_N} \Lambda(\psi, s) \tag{46}$$

with

$$\Lambda(\psi, s) = \frac{\partial^2\psi}{\partial x\partial y}\left(\frac{\partial^2 s}{\partial y^2} - \frac{\partial^2 s}{\partial x^2}\right) + \frac{\partial^2 s}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right) \tag{47}$$

In (47) the derivatives with respect to x, y refer to a locally cartesian coordinate system in the considered horizontal surface. It should be noted that $\Lambda(\psi, s)$ depends entirely upon the mutual deformation properties of the vector fields $-\nabla_h\psi\times\mathbf{k}$ and $-\nabla_h s\times\mathbf{k}$. The interpretation of $\Lambda(\psi, s)$ is seen from the following arguments: Recalling that $\frac{D}{dt} = \frac{D^N}{dt} + (\mathbf{v} - \mathbf{v}^{(N)})\cdot\nabla_h$, (45) can also be written

$$\frac{Ds^{(N)}}{dt} = (\mathbf{v} - \mathbf{v}^{(N)})\cdot\nabla_h s + \frac{2}{a_N}\Lambda(\psi, s) \tag{48}$$

Multiplying here both sides by s and integrating over the total spherical surface F we obtain by reference to (40):

$$-\frac{d}{dt}\int_F s(s - s^{(N)})\,dF = \int_F s(\mathbf{v} - \mathbf{v}^{(N)})\cdot\nabla_h s\,dF + \int_F \frac{2s}{a_N}\Lambda(\psi, s)\,dF \tag{49}$$

However, since $\nabla_h\cdot(\mathbf{v} - \mathbf{v}^{(N)}) = 0$, we obtain

$$\int_F s(\mathbf{v} - \mathbf{v}^{(N)})\cdot\nabla_h s\,dF = \int_F \nabla_h\cdot\frac{s^2}{2}(\mathbf{v} - \mathbf{v}^{(N)})\,dF$$

and hence by Gauss's theorem

$$\int_F s(\mathbf{v} - \mathbf{v}^{(N)})\cdot\nabla_h s\,dF = 0 \tag{50}$$

Then, after multiplication by a_N in (49), recalling that in accordance with (5) $\nabla_h^2 s = -a_N(s - s^{(N)})$, we obtain

$$\frac{d}{dt}\int_F -s\nabla_h^2 s\,dF = -\int_F 2s\Lambda(\psi, s)\,dF \tag{51}$$

From (40), however, we obtain simultaneously

$$\frac{d}{dt}\int_F s^2 dF = 0 \tag{52}$$

If we introduce the expansions $s = \sum_{q=1}^{\infty} s_q$ and $\nabla_h^2 s = -\sum_{q=1}^{\infty} a_q s_q$ into (51), (52) and recall the orthogonality properties of the s_q's,

$$\int_F s_q s_r dF = 0,\ q \neq r,$$

we obtain

$$\sum_{q=1}^{\infty}\frac{d}{dt}\int_F a_q s_q^2 dF = -\int_F 2s\Lambda(\psi, s)\,dF$$

$$\sum_{q=1}^{\infty}\frac{d}{dt}\int_F s_q^2 dF = 0$$

Using these equations together with the condition $a_{q+1} > a_q$ we obtain that there will be a net flow of s_q^2 to the higher or lower frequency end of the spectrum according as $-\int_F s\Lambda(\psi, s)\, dF \gtrless 0$. The term $\Lambda(\psi, s)$, which depends entirely upon the mutual deformation properties of the vector fields $\mathbf{v}$ and $-\nabla_h s \times \mathbf{k}$, is therefore solely responsible for a contingent net flow of amplitude squared, s_q^2, in the one or other direction of the spectrum.

If H in (43) were not zero, such net flows of s_q^2 could of course also take place with $\Lambda(\psi, s) \equiv 0$. However, this kind of spectral changes would not depend directly upon the properties of the velocity field.

The interpretation of $\Lambda(\psi, s)$ may be formulated in a somewhat different manner by using the following arguments: We have

$$\frac{d}{dt}\int_F - s\nabla_h^2 s dF = \frac{d}{dt}\int_F (\nabla_h s)^2 dF$$

so that by comparison with (51):

$$\frac{d}{dt}\int_F (\nabla_h s)^2 dF = -\int_F 2\, s\Lambda(\psi, s)\, dF \qquad (53)$$

Further, if $\frac{Ds}{dt} = 0$, then, since $\nabla_h \cdot \mathbf{v} = 0$, the area between adjacent level lines $s = s_1$, and $s = s_1 + \triangle s$ must be conserved during the motion. Then, evidently $\frac{d}{dt}\int_F (\nabla_h s)^2 dF \gtrless 0$ according as the lengths of the level lines $s = \text{const}$ are on an average either increased or decreased. We may therefore say that $\Lambda(\psi, s)$ may be interpreted as in an integrated sense to be responsible for whether on an average the lengths of the level lines $s = \text{const}$ will increase or decrease, if $\frac{Ds}{dt} = 0$, or at least tend to do so if $\frac{Ds}{dt} = H \neq 0$. Whether there is locally an equally clear interpretation of $\Lambda(\psi, s)$ is not as yet clear to the author.

If we put $s = \nabla^2\psi$, eq. (45) becomes

$$\frac{D^N}{dt}\nabla_h^2\psi^{(N)} \approx (\mathbf{v} - \mathbf{v}^{(N)}) \cdot \nabla_h\nabla_h^2(\psi - \psi^{(N)}) + \frac{2}{a_N}\Lambda(\psi, \nabla_h^2\psi) \qquad (54)$$

We will discuss this equation from a particular point of view in the next section.

6. Further inquiry into the displacement problem for barotropic motion

While the Reynolds term in eq. (54) may be a very important one in characterizing important features of the motion, its magnitude on the other hand may be such that it can be neglected in connection with the displacement problem for a period t_1'' which may possibly be greater than the period t_1' over which $\mathbf{v}^{(N)}$ was put constant. For the period t_1'' we may now write down the approximation

$$\frac{D^N}{dt}\nabla_h^2\psi^{(N)} = 0; \quad t \leq t_1''. \qquad (55)$$

It is readily seen that this equation can be used in connection with the displacement problem. For writing down the system

$$\psi^{(N)} = \psi^{(N)(N-p)} - \frac{\nabla_h^2\psi^{(N)}}{a_{N-p}} \qquad (56)$$

$$\frac{D^N}{dt}\left(-\frac{\nabla_h^2\psi^{(N)}}{a_{N-p}}\right) = 0, \qquad (57)$$

the latter having been obtained by division by $-a_{N-p}$ in (55), we obtain by identifying ψ_1 with ψ^N, and α with $-\frac{\nabla_h^2\psi^{(N)}}{a_N}$ in eqs. (25), (26):

Under the condition (55) the displacements in the smoothed motion with the velocities $\mathbf{v}^{(N)}$ may be found up to a time $t \leqq t_1''$ by at first displacing in the stationary flow with the streamfunction $-\left(\frac{\nabla_h^2\psi^{(N)}}{a_N}\right)_{t=0}$ and then adding from the resulting positions the corresponding displacements in the doubly smoothed flow with the velocities $\mathbf{v}^{(N)(N-p)}$. (58)

Putting now as an approximation

$$\mathbf{v}^{(N)(N-p)} \approx \mathbf{v}_{t=0}^{(N)(N-p)}; \quad t \leq t_1''' \qquad (59)$$

it will be understood that we can take the t_1''' in (59) larger than the t_1' in (39), because $\mathbf{v}^{(N)(N-p)}$ as a doubly smoothed flow must be less time-variable than $\mathbf{v}^{(N)}$. Let T denote the larger of t_1'' and t_1'''. The slight experience we have had hitherto with forecasting by the aid of this triple displacement rule seems to indicate that we can choose at least $\dfrac{a_N}{a_{N-p}} = 2$, and that in this case with the proper choice of N, T may be taken as large as about 48 h.

A discussion of the spectral distribution of ψ which is most unfavourable to a neglect of the Reynolds term in eq. (54) has been carried out and will be presented in a separate paper.

We can now combine the displacement rule (58) above with the earlier rule (38), giving a triple displacement rule:

$$\left.\begin{array}{l}\text{To find the displacements of fluid particles in barotropic motion up to a}\\ \text{time } t \leq T, \text{ we may at first displace in the stationary field } -\left(\dfrac{\nabla_h^2\psi}{a_N}\right)_{t=0}, \text{ then}\\ \text{from the resulting positions add the displacements in the stationary field}\\ -\left(\dfrac{\nabla_h^2\psi^{(N)}}{a_{N-p}}\right)_{t=0}, \text{ and then finally add the displacements in the doubly smoothed}\\ \text{flow } \psi^{(N)(N-p)}, \text{ which may be taken stationary and equal to } \psi^{(N)(N-p)}_{t=0} \text{ when } t \leq T.\end{array}\right\} \quad (60)$$

Obviously we could continue these arguments and find the displacements in the doubly smoothed flow by neglecting the Reynolds term $R_{N,N-p}$ belonging to a doubly smoothed vorticity equation,

$$\frac{D^{N,N-p}}{dt}\nabla_h^2\psi^{(N)(N-p)} = R_{N,N-p}.$$

However, since we have as yet no practical experience in the use of the doubly smoothed vorticity equation we shall abstain from a further discussion of this at the present moment.

Before proceeding to the baroclinic motions we shall mention an other aspect of the results arrived at with respect to displacements of fluid particles in barotropic motion. Let then $\mathbf{p}^\star$ denote the approximate displacements up to any time which would be found in accordance with the methods developed above, as compared to the true displacements $\mathbf{p}$. Provided

$$\mathbf{p}^\star \simeq \mathbf{p};\ t \leq T \qquad (61)$$

the distribution of $\nabla_h^2\psi$ at time t and thereby by integration ψ and $\mathbf{v}$, will be found with a corresponding accuracy by simply displacing $\nabla_h^2\psi_{t=0}$ the vector distances $\mathbf{p}^\star$. However, the integration of the Poisson equation becomes altogether unnecessary if we also have

$$\frac{\partial \mathbf{p}^\star}{\partial t} \simeq \frac{\partial \mathbf{p}}{\partial t};\ t \leq T'. \qquad (62)$$

Now (62) does not necessarily follow from (61). On the other hand (61) could not be true if (62) was unsystematically untrue. The unsystematical errors in (62) are relatively easy to get rid of. Provided this has been accomplished we should therefore expect it possible to find velocity $\mathbf{v}$ approximately, directly from

$$\mathbf{v} = \frac{\partial \mathbf{p}^\star}{\partial t} \simeq \frac{\mathbf{p}^\star(t+\triangle t) - \mathbf{p}^\star(t-\triangle t)}{2\triangle t};\ t \leq T' \qquad (63)$$

Here probably, T' has to be taken somewhat smaller than T in (61).

Experience has shown that T' at least can be taken as large as 24 hrs. when we use the triple displacement rule (60).

The baroclinic case

7. The equation for vertical velocity

In the present case we shall not any longer have the restriction that velocity $\mathbf{v}_3$ shall be horizontal and non-divergent. We may express this by writing

$$\mathbf{v}_3 = \mathbf{v} + \mathbf{v}_\alpha$$

where

$$\mathbf{v} = -\nabla_h\psi \times \mathbf{k}$$

$$\mathbf{v}_\alpha = -\nabla_h\alpha + w\mathbf{k}$$

We shall a priori exclude sound phenomena. Since we in this paper are mainly concerned with the matter from a theoretical point of view we shall do this in the simplest way by assuming

$$\nabla \cdot \mathbf{v}_3 = 0 \qquad (64)$$

The boundary conditions to be considered will be taken as

$$w = 0 \text{ for } z = -h,\ z = h \qquad (65)$$

where h is the depth of the atmosphere.

We shall next make use of certain equilibrium conditions, essentially in the manner demonstrated by CHARNEY (1948) and ELIASSEN (1948). One of the essential consequences of the method of Charney and Eliassen is the fact that the component $\mathbf{v}_\alpha$ of the velocity field cannot be chosen arbitrary, but is in fact instead determined from a knowledge of the pressure distribution and the boundary conditions alone. In this paper we shall derive the equations for the components of $\mathbf{v}_\alpha$ by at first eliminating $\frac{\partial \ln\vartheta}{\partial t}$ and $\frac{\partial\psi}{\partial t}$ between three equations. The first of these expresses the relation which must exist between ψ and potential temperature ϑ if the $\mathbf{v}_\alpha$-component of the velocity field shall be in equilibrium:

$$\frac{d}{dt}\int_\tau \frac{1}{2}\mathbf{v}_\alpha^2 d\tau \equiv 0 \tag{66}$$

Here it is integrated over the total volume τ of the atmosphere. Considering the coriolis parameter $2\Omega_z$ as a virtual constant under the gradient $\nabla_h \cdot 2\Omega_z \frac{\partial\psi}{\partial z}$,

$$\nabla_h \cdot 2\Omega_z \frac{\partial\psi}{\partial z} \simeq 2\Omega_z \nabla_h \frac{\partial\psi}{\partial z} \tag{67}$$

and considering also the convective acceleration $\mathbf{v}\cdot\nabla_h\mathbf{v}$ negligible in comparison with $2\Omega_z\mathbf{k}\times\mathbf{v}$ when differentiated with respect to height z,

$$\left|\frac{\partial}{\partial z}(\mathbf{v}\cdot\nabla_h\mathbf{v})\right| \ll \left|\frac{\partial 2\Omega_z\mathbf{k}\times\mathbf{v}}{\partial z}\right|, \tag{68}$$

it has been shown elsewhere (unpublished), that the necessary and sufficient condition for (66) is:

$$2\Omega_z\frac{\partial\psi}{\partial z} + g\ln\vartheta = k(z,t) \tag{69}$$

where k is independent of the horizontal coordinates.

The second of the three equations is the thermodynamic equation

$$\frac{\partial\ln\vartheta}{\partial t} = -\mathbf{v}\cdot\nabla_h\ln\vartheta - \mathbf{v}_\alpha\cdot\nabla\ln\vartheta + Q \tag{70}$$

where Q is proportional to the supply of heat per unit time and mass. Using (67) and including a horizontally constant function in the definition of ψ, elimination of $\ln\vartheta$ between (69), (70) gives:

$$\frac{\partial}{\partial t}\frac{\partial\psi}{\partial z} = -\mathbf{v}\cdot\nabla_h\frac{\partial\psi}{\partial z} - \frac{g}{2\Omega_z}\mathbf{v}_\alpha\cdot\nabla\ln\vartheta + \frac{gQ}{2\Omega_z} \tag{71}$$

The third of the equations is the equation for the vertical component of absolute vorticity $\nabla_h\times\mathbf{v}\cdot\mathbf{k} = \nabla_h^2\psi$,

$$\frac{\partial}{\partial t}\nabla_h^2\psi = (\nabla_h^2\psi)\frac{\partial w}{\partial z} - \mathbf{v}_\alpha\cdot\nabla\nabla_h^2\psi + (\nabla\times\mathbf{v})_h\cdot \cdot\nabla_h w + \nabla_h\times\mathbf{F}\cdot\mathbf{k} - \mathbf{v}\cdot\nabla_h\nabla_h^2\psi. \tag{72}$$

$\mathbf{F}$ denotes a horizontal frictional force. In the vorticity equation the only non-linear term in $\mathbf{v}_\alpha$ together with the solenoid term have been neglected as presumably small terms. Eliminating $\frac{\partial\psi}{\partial t}$ between eqs. (71), (72) we then obtain

$$L(\mathbf{v}_\alpha) = F \tag{73}$$

where $L(\mathbf{v}_\alpha)$ is a certain linear differential expression of at most second order in the components of $\mathbf{v}_\alpha$, and F a non-homogeneous term which is given by

$$F = \frac{\partial}{\partial z}(\mathbf{v}\cdot\nabla_h\nabla_h^2\psi) - \nabla_h^2\left(\mathbf{v}\cdot\nabla_h\frac{\partial\psi}{\partial z}\right) + \frac{g}{2\Omega_z}\nabla_h^2 Q - \frac{\partial}{\partial z}\nabla_h\times\mathbf{F}\cdot\mathbf{k}$$

The two other equations for the components of $\mathbf{v}_\alpha$ are obtained from the continuity equation and the condition that according to definition $\nabla_h\times\mathbf{v}_\alpha = 0$. This gives respectively

$$\frac{\partial u_\alpha}{\partial x} + \frac{\partial v_\alpha}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{74}$$

$$\frac{\partial v_\alpha}{\partial x} - \frac{\partial u_\alpha}{\partial y} = 0. \tag{75}$$

The condition that the system (73), (74), (75) above is elliptic can be shown to be given by

$$\left(\frac{\partial\mathbf{v}}{\partial z}\right)^2 < \frac{g\nabla_h^2\psi}{2\Omega_z}\cdot\frac{\partial\ln\vartheta}{\partial z} \tag{76}$$

This condition is usually satisfied in the atmosphere. With the boundary conditions (65) we must therefore obtain $w \equiv 0$ when $F \equiv 0$. The coefficients in (73), at least in the terms which are the most dominating, are relatively permanent in character as compared to the non-homogeneous term F. The variable conditions for F will therefore be the main factor in creating the variable conditions for the systems of vertical velocities we can have in the atmosphere. We shall now look a little more closely at this non-homogeneous term. Effectuating the differentiation with respect to z it turns out that we can write F on the form

$$F = a_N \left[2(\mathbf{v} - \mathbf{v}^{(N)}) \cdot \nabla_h \frac{\partial \psi}{\partial z} + \frac{2}{a_N} A\left(\psi, \frac{\partial \psi}{\partial z}\right) \right] + \frac{g}{2\Omega_z} \nabla_h^2 Q - \nabla_h \times \frac{\partial \mathbf{F}}{\partial z} \cdot \mathbf{k} \quad (77)$$

where $^{(N)}$ is defined as in (6) and $A\left(\psi, \frac{\partial \psi}{\partial z}\right)$ is the function $A(\psi, s)$ defined in (47) when we put $\frac{\partial \psi}{\partial z}$ for s. It is an interesting fact that here besides heating and friction, and an advection term $(\mathbf{v} - \mathbf{v}^{(N)}) \cdot \nabla_h \frac{\partial \psi}{\partial z}$, now a term $A\left(\psi, \frac{\partial \psi}{\partial z}\right)$ appears which in accordance with (6) only depends upon the mutual deformation properties of $\mathbf{v}$ and $-\nabla_h \frac{\partial \psi}{\partial z} \times \mathbf{k}$. The relative importance of this and the advection term will be investigated more closely in the following section.

8. The importance of deformation investigated for a simplified baroclinic model

By a partial elimination of u_α, v_α between (73), (74), (75) we obtain the following equation for vertical velocity:

$$(\nabla_h^2 \psi) \frac{\partial^2 w}{\partial z^2} + 2 \frac{\partial u}{\partial z} \frac{\partial^2 w}{\partial y \partial z} - 2 \frac{\partial v}{\partial z} \frac{\partial^2 w}{\partial x \partial z} + \left(g \frac{\partial \ln \vartheta}{\partial z} \Big/ 2\Omega_z \right) \nabla_h^2 w = F \quad (78)$$

The terms containing the first derivatives of u_α, v_α, w are small if the relative variation in space of the coefficients in (78) are negligible in comparison with the corresponding variations in the first derivatives of $\mathbf{v}_\alpha$, and have been neglected. It is understood that for the components of vertical velocity which have sufficiently large horizontal scales, eq. (78) can under the boundary condition (65) be approximated by

$$2\Omega_z \frac{\partial^2 w}{\partial z^2} = F \quad (79)$$

When we make the corresponding simplifications in the vorticity equation (72) and in the eq. (71) we obtain

$$\frac{\partial}{\partial t} \nabla_h^2 \psi = -\mathbf{v} \cdot \nabla_h \nabla_h^2 \psi + 2\Omega_z \frac{\partial w}{\partial z} + \nabla_h \times \mathbf{F} \cdot \mathbf{k} \quad (80)$$

and

$$\frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} = -\mathbf{v} \cdot \nabla_h \frac{\partial \psi}{\partial z} + \frac{g}{2\Omega_z} Q \quad (81)$$

For the following arguments it does not matter whether we ignore friction and heating, or not. For simplicity we shall therefore put both zero: $\mathbf{F} = 0$; $Q = 0$. Eq. (81) can then also be written

$$\frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} = -\mathbf{v} \cdot \nabla_h \frac{\partial \psi}{\partial z} \quad (82)$$

Let us next assume that

$$\frac{\partial^2 \psi}{\partial z^2} = 0. \quad (83)$$

We can then write

$$\psi = \hat{\psi} + \left(\frac{\partial \psi}{\partial z}\right) z; \quad \mathbf{v} = \hat{\mathbf{v}} + \left(\frac{\partial \mathbf{v}}{\partial z}\right) z, \quad (84)$$

where

$$\hat{\psi} = \psi_{z=0} = \frac{1}{2h} \int_{-h}^{h} \psi dz. \quad (85)$$

Substituting $\mathbf{v} = \hat{\mathbf{v}} + z \frac{\partial \mathbf{v}}{\partial z}$ in (82) and observing that $\frac{\partial \mathbf{v}}{\partial z} \cdot \nabla_h \frac{\partial \psi}{\partial z} = 0$, we obtain

$$\frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} + \hat{\mathbf{v}} \cdot \nabla_h \frac{\partial \psi}{\partial z} = 0. \quad (86)$$

Differentiating (86) with respect to height and using (83) we obtain

$$\frac{\partial}{\partial t}\frac{\partial^2 \psi}{\partial z^2}=0,$$

showing that (83), when once satisfied will also be so at any later time. Substituting from (84) in the terms on the r.h.s. of (79) and recalling that

$$\Lambda\left(z\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial z}\right)=0$$

in accordance with the definition (47), we obtain

$$2\Omega_z\frac{\partial^2 w}{\partial z^2}=a_N\left[2\left(\hat{\mathbf{v}}-\hat{\mathbf{v}}^{(N)}\right)\cdot\nabla_h\frac{\partial \psi}{\partial z}+\right.$$

$$\left.+\frac{2}{a_N}\Lambda\left(\hat{\psi},\frac{\partial \psi}{\partial z}\right)\right]+a_N z\left(\frac{\partial \mathbf{v}}{\partial z}-\frac{\partial \mathbf{v}^{(N)}}{\partial z}\right)\cdot\nabla_h\frac{\partial \psi}{\partial z} \tag{87}$$

With the boundary conditions (65) the solution of (87) becomes

$$2\,\Omega_z w=a_N\left[2\left(\hat{\mathbf{v}}-\hat{\mathbf{v}}^{(N)}\right)\cdot\nabla_h\frac{\partial \psi}{\partial z}+\right.$$

$$\left.+\frac{2}{a_N}\Lambda\left(\hat{\psi},\frac{\partial \psi}{\partial z}\right)\right]\frac{1}{2}(z^2-h^2)+a_N\left(\frac{\partial \mathbf{v}}{\partial z}-\frac{\partial \mathbf{v}^{(N)}}{\partial z}\right)\cdot$$

$$\cdot\nabla_h\frac{\partial \psi}{\partial z}\,\frac{1}{6}(z^2-h^2)\,z \tag{88}$$

This solution will now be used in order to find an expression for the increase in total "volume" kinetic energy. Ignoring friction we may in accordance with a result obtained by Fjørtoft (6) write

$$\frac{d}{dt}\int_\tau\frac{1}{2}\,\mathbf{v}_3^2 d\tau=\int_\tau g\ln\vartheta w d\tau$$

Substituting here for $\ln\vartheta$ from the balance condition (69) obtained from (66) we also obtain

$$\frac{d}{dt}\int_\tau\frac{1}{2}\,\mathbf{v}_3^2 d\tau=\frac{d}{dt}\int_\tau\frac{1}{2}\,\mathbf{v}^2 d\tau=\int_\tau\frac{\partial \psi}{\partial z}\,2\,\Omega_z w d\tau,$$

since because of the continuity equation (64) and the boundary conditions (65)

$$\int_\tau k\,(z,t)w d\tau=0$$

Substituting for $2\Omega_z w$ the solution (88), the antisymmetrical part of the solution falls out under the integration and we are left with the equation

$$\frac{d}{dt}\int_\tau\frac{1}{2}\mathbf{v}^2 d\tau=\left[-\int_\tau 2\frac{\partial \psi}{\partial z}\Lambda\left(\hat{\psi},\frac{\partial \psi}{\partial z}\right)\partial\tau-\right.$$

$$\left.-\int_\tau\frac{\partial \psi}{\partial z}\left(\hat{\mathbf{v}}-\hat{\mathbf{v}}^{(N)}\right)\cdot\nabla_h\frac{\partial \psi}{\partial z}d\tau\right]\frac{h^2}{3}$$

However, using (50) this equation reduces to

$$\frac{d}{dt}\int_\tau\frac{1}{2}\,\mathbf{v}^2 d\tau=-\frac{2}{3}\int_\tau\psi_T\Lambda\left(\hat{\psi},\psi_T\right)d\tau \tag{89}$$

having for convenience introduced the quantity ψ_T defined from

$$\psi_T=h\frac{\partial \psi}{\partial z}. \tag{90}$$

Recalling now the definition of $\Lambda(\psi,s)$, we have obtained the result that changes in the total kinetic energy in our model depend entirely upon the mutual deformation properties of the velocity field $\hat{\mathbf{v}}=-\nabla_h\hat{\psi}\times\mathbf{k}$ and the "thermal" wind field $\mathbf{v}_T=-\nabla_h\psi_T\times\mathbf{k}$.

After multiplication by h eq. (86) can be written

$$\frac{D\psi_T}{dt}=0;\ \frac{D}{dt}=\frac{\partial}{\partial t}+\hat{\mathbf{v}}\cdot\nabla_h. \tag{91}$$

In agreement with (53) we now get

$$\frac{d}{dt}\int_\tau\frac{1}{2}\,\mathbf{v}_T^2 d\tau=-\int_\tau\psi_T\Lambda\left(\hat{\psi},\psi_T\right)d\tau. \tag{92}$$

Eq. (89) can therefore also be written

$$\frac{d}{dt}\int_\tau\frac{1}{2}\,\mathbf{v}^2 d\tau=\frac{2}{3}\frac{d}{dt}\int_\tau\frac{1}{2}\,\mathbf{v}_T^2 d\tau. \tag{93}$$

There is accordingly in the absence of friction and heating direct proportionality between the increase in total kinetic energy and total energy in the thermal wind field. We may now also take over the other results of section 5, thus giving:

(a) The total thermal wind energy and hence also the total kinetic energy increases or decreases in the absence of friction and heating according as the lengths of the isolines $\psi_T =$ $=$ const, essentially the horizontal isotherms, are on an average either increased or decreased.

(b) The total thermal wind energy, and hence also the total kinetic energy, increases or decreases in the absence of friction and heating according as there is a net flow of amplitude $(\psi_T)_q^2$ to either lower or higher wavelengths.

Substituting from (84) $\mathbf{v} = \hat{\mathbf{v}} + z \dfrac{\partial \mathbf{v}}{\partial z}$ in $\int_\tau \frac{1}{2} \mathbf{v}^2 d\tau$ we obtain

$$\frac{d}{dt}\int_\tau \frac{1}{2} \mathbf{v}^2 d\tau = \frac{d}{dt}\int_\tau \frac{1}{2}\hat{\mathbf{v}} d\tau + \frac{1}{3}\frac{d}{dt}\int_\tau \frac{1}{2}\mathbf{v}_T^2 d\tau \quad (94)$$

By elimination of $\dfrac{d}{dt}\int_\tau \frac{1}{2} \mathbf{v}^2 d\tau$ between (93), (94) we get

$$\frac{d}{dt}\int_\tau \frac{1}{2}\hat{\mathbf{v}}^2 d\tau = \frac{1}{3}\frac{d}{dt}\int_\tau \frac{1}{2}\mathbf{v}_T^2 d\tau. \quad (95)$$

All what was said above for the total kinetic energy is therefore also valid for the kinetic energy of the vertically mean flow.

According to (b) above an increase in the total kinetic energy necessarily is connected with a decrease in thermal wind kinetic energy on the larger scales. Whether actually the total kinetic energy also shall decrease on the same scales will, however, according to (94) also depend upon in which direction and with what magnitude the kinetic energy of the vertically mean flow changes on these scales. The results above do, however, only give a clue to an understanding of the conditions for the changes in the *total* kinetic energy, and the *total* kinetic energy of the vertically mean flow, and not of the nature of the changes in their spectral distribution.

9. On the possible importance of non-linear interference for the understanding of the creation and geographical distribution of disturbances

Let now $\hat{\psi}^{(N)\cdots(N-p)}$ and $\psi_T^{(N)\cdots(N-p)}$ denote repeatedly smoothed fields. We do then know that neither of these fields contain components with wavenumbers above $N-p-1$. The function $A(\psi^{(N)\cdots(N-p)}, \psi_T^{(N)\cdots(N-p)})$ however, may by interference contain "energy" on wavenumbers which are excluded in the smoothed fields. The distribution of the bulk of energy in these higher components must essentially depend upon the mutual deformation properties of the smoothed $\hat{\mathbf{v}}$ – and $\mathbf{v}_T$-fields. The author of this article has some experience in forecasting on the basis of initially smoothed fields. It has repeatedly turned out as a result of the displacements of the ψ_T-isolines, that the field ψ_T, though smoothed initially, ends up with "energy" on smaller scales, the bulk of which has a typical distribution relative to the smoothed fields. These smaller scale phenomena which thus enter the picture, are generally also verified. While we will return to this in the practical examples to be brought in the second part of this paper, we shall here study the same phenomenon by investigating the function $A(\psi, \psi_T)$ for the normal maps of November. Using the ordinary geostrophic approximation $\psi = \dfrac{gZ}{2\Omega_z}$, where Z denotes the height of a pressure surface we may write

$$A(\hat{\psi}, \psi_T) = k_1 \cdot A(Z_0, Z_7 - Z_0)$$

and

$$\int_\tau \psi_T A(\hat{\psi}, \psi_T) d_\tau =$$

$$= k_2 \int_F (Z_7 - Z_0)\, A(Z_0, Z_7 - Z_0)\, dF$$

where k_1, k_2 are certain positive proportionality factors. Here, Z_0, Z_7 denote the normal heights of the 1,000 mb and 700 mb surfaces, respectively, smoothed so as to give zero energy on wavelengths below approximately 5,000 km. In fig. 3 the function $A(Z_0, Z_7 - Z_0)$ is illustrated together with the normal, smoothed thickness field $Z_7 - Z_0$. It is clearly seen that the integral above will obtain a negative value when F is taken as the area represented on the map. According to the results of the preceding section we can therefore conclude that the normal fields of essentially pressure and temperature are such as to give a flow of "energy" in the horizontal temperature field towards the high-frequency end of the spectrum, thereby increasing the total kinetic energy.

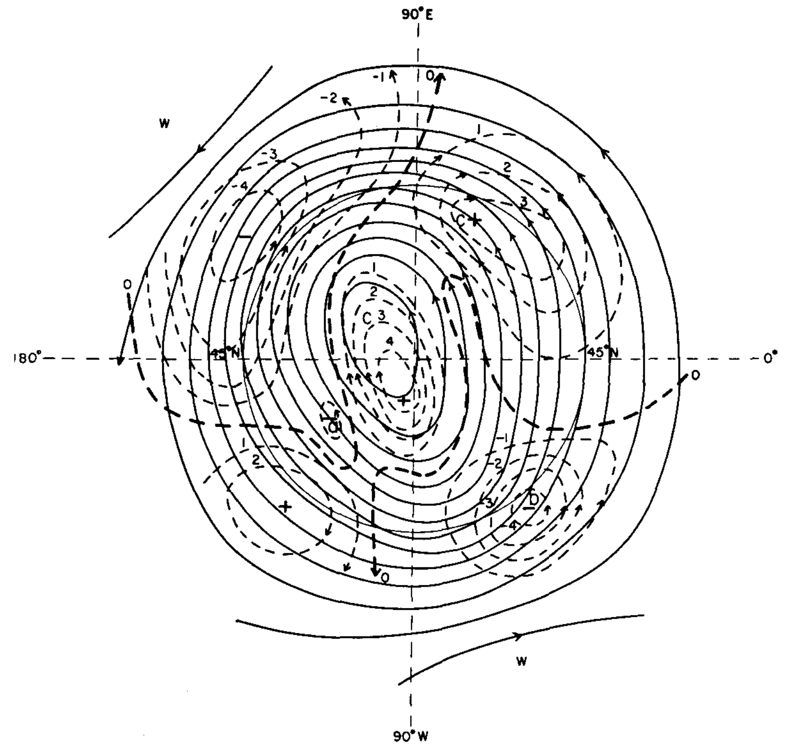

Fig. 3. Spacely smoothed normal 700—1000 mb thickness field (full lines) together with the field of Λ (Z_0, $Z_7 - Z_0$) (broken lines).

In fig. 4 we have drawn the component Λ' of $\Lambda(Z_0, Z_7 - Z_0)$ which only have energy on the scales which were discluded in the fields Z_0 and $Z_7 - Z_0$ themselves.

Much remarkably the field of Λ' consists of three pairs of centers centered around localities which coincide rather closely with the three main centers of mean diurnal variability of the height of the 500 mb surface found by Nyberg (1949) for the single month of November 1947.

The above investigations indicate that the phenomena of non-linear interference in the above sense may have a bearing on our understanding of the creation and local distribution of disturbances and is now being taken up for systematical study.

10. The integration problem

When we treated the integration problem for barotropic motion we developed methods which clearly represented an advantage when we undertook to find the displacements of the fluid particles. When we are now turning to the integration problem for baroclinic flows our main problem will be to investigate how much of these advantages, if anything, can be maintained. The following is not a complete treatment of this problem. The general baroclinic problem is so difficult that it is best to start with simple models. It is thought however, that even these will reveal the most fundamental features of baroclinic flows, and that the manner in which we are going to treat the integration problem may be rather typical for more general models.

It is an explainable fact that the barotropic model applies best to the vertically mean flow represented approximately by the horizontal flow between the 500 mb and 600 mb surfaces, while the flows in greater distances above and below certainly are very little barotropic. It is therefore natural to direct the attention at first to the vertically mean flow where the results obtained for barotropic flow must be expected to be least altered.

Maintaining the assumption (83) and averaging the vorticity equation (80) in the vertical, we obtain

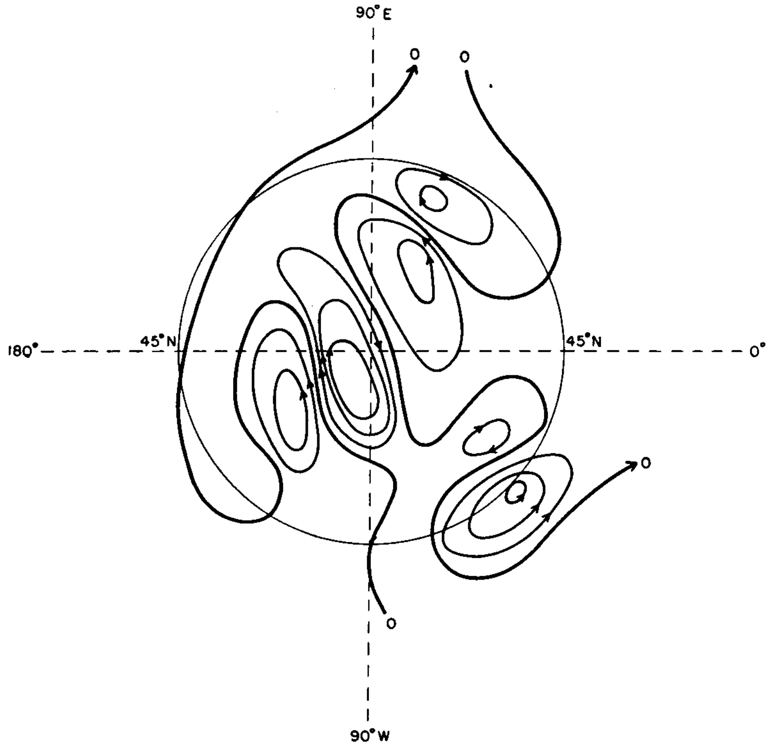

Fig. 4. The field of Λ' defined in section 9.

$$\frac{\partial}{\partial t}\nabla_h^2\hat{\psi} + \hat{\mathbf{v}}\cdot\nabla_h\cdot\nabla_h^2\hat{\psi} = H \qquad (96)$$

where

$$H = -\frac{1}{3}\mathbf{v}_T\cdot\nabla_h\nabla_h^2\psi_T. \qquad (97)$$

For an illustration of the integration methods we shall at first make the advective assumption (91):

$$\frac{D\psi_T}{dt} = 0. \qquad (91)$$

If, now $\mathbf{r}_0$ denotes the position vectors at time $t = 0$ for particles moving with the velocities $\hat{\mathbf{v}}$, we may use $\mathbf{r}_0$ to define the Lagrangian particle variables in contrast to $\mathbf{r}$ which has been used to define the coordinates of position on the horizontal surfaces. Eq. (96) may then be written in Lagrangian coordinates as

$$\frac{\partial}{\partial t}\nabla_h^2\psi(\mathbf{r}_0, t) = H(\mathbf{r}_0, t) \qquad (98)$$

and in Eulerian variables as

$$\frac{\partial}{\partial t}\nabla_h^2\psi(\mathbf{r}, t) = -\hat{\mathbf{v}}\cdot\nabla_h\nabla_h^2\hat{\psi} + H(\mathbf{r}, t) \equiv I(\mathbf{r}, t). \qquad (99)$$

Let us assume that

$$\left|\frac{\partial}{\partial t}H(\mathbf{r}_0, t)\right| \ll \left|\frac{\partial I(\mathbf{r}, t)}{\partial t}\right|. \qquad (100)$$

If now $(\Delta t)_0$ and Δt denote the maximum grid distances in time which can be used for a numerical description of $H(\mathbf{r}_0, t)$ and $I(\mathbf{r}, t)$, respectively, we could take $(\Delta t)_0 \gg \Delta t$ if (100) were true. This would so far be an advantage in the numerical work of integration. However, to get a prognosis based on (98) we must also know where the individual particles move in the course of time, a problem which does not arise in (99). If it should now happen that in order to find the displacements $\mathbf{p}(\mathbf{r}_0, t)$, iterative operations of some nature were required with a maximum iteration period T

which could not be taken larger than Δt, then there would certainly be no advantage in using a Lagrangian method. If, however, on the contrary $T > \Delta t$, then under the assumption (100), and provided the operations involved in finding $\mathbf{p}(\mathbf{r}_0, t)$ are not too complicated, there would be an advantage in using (98), because the number of iterations required for a forecast over a certain period would be less than the one required if we used (99).

In the *barotropic* case where $H = \text{const} = 0$, there is no upper limit to $(\Delta t)_0$. Further, it has been shown in this article that it is possible to find in a relatively simple manner the displacements $\mathbf{p}(\mathbf{r}_0, t)$ using an iteration period $T \simeq 24$—28 hours. Since now Δt cannot be taken larger than between 1—2 hours because of the rapidity with which $-\mathbf{v} \cdot \nabla_h \nabla_h^2 \psi$ changes in the atmosphere, there would in fact be an advantage in using a Lagrangian numerical method in the barotropic case.

In the *baroclinic* case $I(\mathbf{r}, t) = -\mathbf{v} \cdot \nabla_h \nabla_h^2 \psi + H(\mathbf{r}, t)$. Since there is no general tendency in the atmosphere for a compensation between these two terms, and further because as a matter of observation

$$\left|\frac{\partial H(\mathbf{r}_0, t)}{\partial t}\right| \ll \left|\frac{\partial H(\mathbf{r}, t)}{\partial t}\right|,$$

the assumption (100) would certainly also be fulfilled in the baroclinic case. Recalling that

$$\frac{D}{dt} \nabla_h^2 \psi = \frac{D^N}{dt} \nabla_h^2 \psi$$

and letting $\mathbf{r}_0^N$ denote Lagrangian particle variables for the smoothed motion, (98) can also be written

$$\frac{\partial}{\partial t} \nabla_h^2 \psi(\mathbf{r}_0^N, t) = H(\mathbf{r}_0^N, t) \qquad (101)$$

It is an observational fact that generally

$$\left|\frac{\partial H(\mathbf{r}_0^N, t)}{\partial t}\right| < \left|\frac{\partial H(\mathbf{r}_0, t)}{\partial t}\right|.$$

It will therefore also for this reason be an advantage to connect the integration problem with (101). Suppose now that T' represents the maximum grid distance in time which can be used for a numerical description of $H(\mathbf{r}_0^N, t)$. This means that for particles moving in the smoothed motion we may consider H an approximate constant for $t \leqq T'$

$$\frac{D^N H}{dt} = 0; \; t \leq T'. \qquad (102)$$

In order to utilize (101) we must now be able to find the trajectories in the smoothed flow, and next H as a function of time. Since H is determined when we know ψ_T, and ψ_T is found in accordance with (91) when we know the displacements together with the initial conditions, we are left therefore essentially with a displacement problem.

A relatively simple displacement rule can be found by considering the system (36), (96), and (102), dividing the latter equations by $-a_N$:

$$\hat{\psi} = \hat{\psi}^{(N)} - \frac{\nabla_h^2 \psi}{a_N}$$

$$\frac{D}{dt}\left(-\frac{\nabla_h^2 \psi}{a_N}\right) = -\frac{H}{a_N} \qquad (103)$$

$$\frac{D^N H}{dt} = 0. \; t \leqq T'.$$

On the basis of this system the following rule can now be proved which is an extension to the baroclinic case of the rule (38):

To find the displacements up to a time $t' \leq T'$, we may at first displace in the velocity field with the streamfunction

$$-\left(\frac{\nabla_h^2 \psi}{a_N}\right)_{t=0} - \left(\frac{H_{t=0}}{a_N}\right) t,$$

and then add from the resulting positions the displacements in the smoothed flow. (104)

It is seen that the only difference from the barotropic displacement rule (38) is that the first displacements must be carried out in a field which has got the additional nonstationary component $-(H_{t=0}/a_N)t$. However, because this field maintains its form and also possesses a simple dependency on time, the displacement will not be difficult to carry out. In most cases it will be sufficient to displace in the stationary field

$$-\left(\frac{\nabla_h^2\psi}{a_N}\right)_{t=0}-\left(\frac{H_{t=0}}{a_N}\right)\frac{t'}{2}$$

because in most cases $(H_{t=0}/a_N)t'$ will be relatively weak in comparison to $-\nabla_h^2\psi_{t=0}/a_N$.

Again we should remember that the results depend upon the choice of a_N. For if a_N is very large the smoothed flow $\mathbf{v}^{(N)}$ wont be very much different from the actual flow $\mathbf{v}$ in the range of wavelengths containing most of the energy. Then $\mathbf{v}^{(N)}$ would be practically as much time-variable as $\mathbf{v}$, and we could accordingly not displace in the smoothed flow $\mathbf{v}_{t=0}^{(N)}$ much longer than we could have done in the actual flow $\mathbf{v}_{t=0}$. By the proper chcise of a_N, however, the time scale of $\mathbf{v}^{(N)}$ becomes such that we can put as an approximation

$$\mathbf{v}^{(N)}=\mathbf{v}_{t=0}^{(N)};\ t\leq 24\,\text{hours.},\qquad (39)$$

like we did in the barotropic case.

To make a prognosis for the period T' in accordance with what has been outlined above we proceed now in theory as follows.

Using the displacement rule (104) we displace the "thickness"-lines $(\psi_T)_{t=0}=\text{const}$ to find $\psi_T(\mathbf{r}, T')$. This gives us also $H(\mathbf{r}_0^N, T')$. Thereafter, we find

$$\nabla_h^2\hat{\psi}(\mathbf{r}_0^N, T')=\nabla_h^2\hat{\psi}(\mathbf{r}_0^N, 0)+\left[H(\mathbf{r}_0^N, 0)+\right.$$
$$\left.+H(\mathbf{r}_0^N, T')\right]\frac{T'}{2}$$

Knowing the displacements in the smoothed flow we then also know the distribution $\nabla_h^2\hat{\psi}(\mathbf{r}, T')$, say $=F(\mathbf{r}, T')$. By a solution of

$$\nabla_h^2\hat{\psi}(\mathbf{r}, T')=F(\mathbf{r}, T')$$

we find $\hat{\psi}(\mathbf{r}, T')$. The flow in any other level will be determined from

$$\psi=\hat{\psi}+\frac{z}{h}\psi_T.$$

11. A simple non-advective model

If we also include the convection term in the thermodynamic equation it becomes

$$\frac{\partial}{\partial t}\frac{\partial\psi}{\partial z}=-\hat{\mathbf{v}}\cdot\nabla_h\frac{\partial\psi}{\partial z}-\frac{g}{2\Omega_z}\frac{\partial\ln\vartheta}{\partial z}w\quad(105)$$

An equation for vertical velocity is now obtained from an elimination between (80) and (105), giving

$$2\Omega_z\frac{\partial^2 w}{\partial z^2}+\frac{g}{2\Omega_z}\frac{\partial\ln\vartheta}{\partial z}\nabla_h^2 w=F.\quad(106)$$

We shall now make use of the assumption (83). The deviations from a linear distribution with height for ψ which now necessarily must develop because of the inclusion of the convective term in (105) will be supposed to be small enough to justify the use we shall make of (83) below. In this connection we should mention that in a so-called two-parametric model where the solution for ψ is obtained by linear interpolation between the values of ψ in two distinct levels, (83) is by assumption automatically satisfied.

Using (83), the vertically averaged vorticity equation (96) will be unchanged. However, the deviations from a linear distribution with height of velocity makes it natural to define ψ_T as

$$\psi_T=\frac{1}{2}(\psi_{z=h}-\psi_{z=-h})$$

The equation for ψ_T is obtained by integrating(105) from $z=-h$ to $z=h$ and divide by 2. Using (83) we obtain then:

$$\frac{D\psi_T}{dt}=-\frac{g}{2\Omega_z}\frac{\partial\ln\vartheta}{\partial z}\int_{-h}^{h}\frac{1}{2}w\,dz.\quad(107)$$

Smoothing eq. (105) on both sides we obtain in accordance with (48) and (83):

$$\frac{D}{dt}\left(\frac{\partial\psi}{\partial z}\right)^{(N)}=\left(\hat{\mathbf{v}}-\hat{\mathbf{v}}^{(N)}\right)\cdot\nabla_h\frac{\partial\psi}{\partial z}+$$
$$\frac{2}{a_N}\Lambda\left(\hat{\psi},\frac{\partial\psi}{\partial z}\right)-\frac{g}{2\Omega_z}\frac{\partial\ln\vartheta}{\partial z}w^{(N)}.\quad(108)$$

Substituting in (106) $\nabla_h^2 w=a_N(w^{(N)}-w)$ and the expression (87) for F this equation can also be written:

$$2\Omega_z\frac{\partial^2 w}{\partial z^2}-\frac{g a_N}{2\Omega_z}\frac{\partial\ln\vartheta}{\partial z}w=a_N\left[2\left(\hat{\mathbf{v}}-\hat{\mathbf{v}}^{(N)}\right)\cdot\right.$$
$$\left.\cdot\nabla_h\frac{\partial\psi}{\partial z}+\frac{2}{a_N}\Lambda\left(\hat{\psi},\frac{\partial\psi}{\partial z}\right)\right]-\frac{g a_N}{2\Omega_z}\frac{\partial\ln\vartheta}{\partial z}w^{(N)}+$$
$$+a_N z\left(\frac{\partial\mathbf{v}}{\partial z}-\frac{\partial\mathbf{v}^{(N)}}{\partial z}\right)\cdot\nabla_h\frac{\partial\psi}{\partial z}.\quad(109)$$

Eliminating $w^{(N)}$ between (109) and (108), we obtain

$$2\Omega_z\frac{\partial^2 w}{\partial z^2}-\frac{g a_N}{2\Omega_z}\frac{\partial \ln\vartheta}{\partial z}w=a_N\left(\hat{\mathbf{v}}-\hat{\mathbf{v}}^{(N)}\right)\cdot$$

$$\cdot\nabla_h\frac{\partial\psi}{\partial z}+a_N\frac{D}{dt}\left(\frac{\partial\psi}{\partial z}\right)^{(N)}+a_N z\left(\frac{\partial\mathbf{v}}{\partial z}-\frac{\partial\mathbf{v}^{(N)}}{\partial z}\right)\cdot$$

$$\cdot\nabla_h\frac{\partial\psi}{\partial z}. \qquad (110)$$

This equation is now solved under the earlier mentioned assumption that $\frac{\partial}{\partial t}\frac{\partial\psi^{(N)}}{\partial z}$ and hence $\frac{D}{dt}\frac{\partial\psi^{(N)}}{\partial z}$ may be considered as approximately independent of height.

If next the solution is introduced into (107), the unsymmetrical part of the solution falls out under the integration and we are left with

$$\frac{D\psi_T}{dt}=k_N\left(\hat{\mathbf{v}}-\hat{\mathbf{v}}^{(N)}\right)\cdot\nabla_h\psi_T+k_N\frac{D\psi_T^{(N)}}{dt} \qquad (111)$$

where

$$k_N=\left[1-\frac{Tg\lambda}{\lambda}\right];\ \lambda^2=\frac{h^2 a_N g}{4\Omega_z^2}\frac{\partial\ln\vartheta}{\partial z}. \qquad (112)$$

Assuming an average constant value for $\frac{h^2}{4\Omega_z^2}\frac{\partial\ln\vartheta}{\partial z}$, k_N becomes for the proper choice of N equal to one half:

$$k_N=\frac{1}{2}\ \text{for the proper choice of } N.$$

Under these assumptions it is easy to show that eq. (111) can be written

$$\frac{D^N}{dt}\left(\psi_T-\frac{1}{2}\psi_T^{(N)}\right)=-\frac{1}{2}\left(\hat{\mathbf{v}}-\hat{\mathbf{v}}^{(N)}\right)\cdot$$

$$\cdot\nabla_h\left(\psi_T-\psi_T^{(N)}\right). \qquad (113)$$

Here the source term on the r.h.s. is according to its nature under most circumstances negligible in its effect on the prognosis of the thickness lines and may be neglected with good approximation:

$$\frac{D^N}{dt}\left(\psi_T-\frac{1}{2}\psi_T^{(N)}\right)=0. \qquad (114)$$

Accordingly the quantity $\psi_T-\frac{1}{2}\psi_T^{(N)}$ may be considered conserved in the smoothed motion $\psi^{(N)}$ when N is given the proper value. Experiments have shown that eq. (114) has proved useful in overcoming in a simple way the main errors in the thickness forecasts caused by the advective assumption.

12. The trajectory problem for levels other than the vertically mean level

This problem can be treated with success by means of the theorem (35). We shall illustrate this for the advective model. The streamfunction in an arbitrary level is $\psi=\hat{\psi}+(z/h)\psi_T$. We have therefore the system

$$\psi=\hat{\psi}+(z/h)\psi_T$$

$$\frac{D}{dt}\left(\frac{z}{h}\psi_T\right)=0;\ \frac{D\psi}{dt}\equiv\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla_h. \qquad (115)$$

According to the theorem (35) we may therefore find the displacements in an arbitrary level by at first displacing in the stationary field $(z/h)(\psi_T)_{t=0}$, and then add the displacements in the motion in the mean level which can be found by means of any of the earlier rules.

LIST OF REFERENCES

CHARNEY, J. G., 1948: On the scale of atmospheric motions. Geofysiske Publikasjoner. Vol. XVII No. 3.

ELIASSEN, A., 1948: The quasi-static equations of motion with pressure as independent variable. Geofysiske Publikasjoner. Vol. XVII. No. 4.

FJØRTOFT, R., 1951: Stability properties of large-scale atmospheric disturbances. Compendium of Meteorology.

FJØRTOFT, R., 1952: On a numerical method of integrating the barotropic vorticity equation. *Tellus*, **4**, 3.

NYBERG, A., 1949: An aerological study of large-scale atmospheric disturbances. *Tellus*, **1**, 1.

RICHARDSON, L. F., 1922: The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam. *Phil. Trans. roy. Soc.* London, A, **210**, 307—357.