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# Some Aspects of the Flow of Stratified Fluids

## II. Experiments with a Two-Fluid System

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*Abstract*

A description is given of the flow of two superimposed layers of fluid over a barrier. This represents a partial experimental investigation of a problem considered theoretically in Part I. In general three regimes of motion are possible: If the velocities of the fluids are sufficiently small the interface is little disturbed except for a slight depression over the barrier. If the velocities are sufficiently high the interface swells symmetrically over the obstacle. At intermediate speeds a hydraulic jump occurs in the lee of the barrier and the lower layer increases in depth upstream. Two occurrences do not fit into the above description: If the obstacle is small compared to the depth of the lower layer, weak lee waves appear at low speeds, increasing in amplitude as the approach velocity of the fluid is increased. This seems to be the only case in which perturbation theory provides an adequate prediction of the flow. The second anomalous occurrence is the appearance of a "jump down" or hydraulic "drop" in the lee when the speed of the fluid is moderately high, the obstacle large, and the upper fluid relatively thin.

The description of the experiments is supplemented by a theoretical discussion, employing the assumption of a hydrostatic pressure distribution. In general this theory provides a satisfactory explanation of the observed behavior. The paper concludes with a discussion of meteorological implications.

### I. Introduction

Part I of this series of papers (Long, 1953 a) contained a rather general theoretical analysis of the two-dimensional flow of a stratified liquid in a gravity field. Assuming only that the liquid was frictionless and the flow steady, the equations of motion and continuity were integrated once to yield a second-order, partial differential equation. This equation, valid for any arbitrary basic density stratification and velocity distribution, was examined from several viewpoints: In the first place it was assumed that a solution existed for given finite configurations of the boundary. A sufficient condition was then obtained for this solution to be unique. By analogy with hydraulics a regime of flow satisfying the uniqueness criterion was called supercritical.

If only infinitesimal perturbations are considered, uniqueness implies that the flow is in such a state that no free wave can maintain its position with respect to the fixed bottom, but must instead be swept downstream. This means that the wave energy generated by an obstacle in the stream cannot be propagated upstream since the maximum group velocity in such systems is the phase velocity of long waves. In the flow of water in a channel the occurrence of a supercritical regime at some section is a necessary condition for a hydraulic jump.

In the problems of water flow with a free surface, and high speed flow of a compressible fluid, this shock-wave phenomenon is by far the most important characteristic of the motion. In the case of a stratified fluid such as the

[1] This research was supported by the U.S. Weather Bureau.

atmosphere there seems little doubt not only that shocks exist but that they are of very fundamental importance to meteorology. TEPPER (1952), for example, assumes that squall lines are really surges (moving hydraulic jumps) on an inversion surface in the troposphere. He then applies a simplified mathematical theory of such shock waves to actual surface pressure observations and predicts motion and development. Despite some objections (FULKS, 1951), it is likely that Tepper has given a correct physical interpretation of most if not all squall lines.

Further evidence of the existence of hydraulic jumps in the atmosphere is provided by observations of the very violent type of motion that occurs in the lee of the Sierra Nevada Range near Bishop, California (COLSON, 1952). Resemblances between this phenomenon and a crude laboratory model (LONG, 1953 b) are numerous. At Froude numbers roughly equivalent in model and prototype similar violent upward motions occur in both cases in the lee.

The unquestioned importance of stability in the atmosphere has inspired a number of studies of internal gravity waves set up by mountain barriers and other sources of perturbations (LYRA, 1943; QUENEY, 1947; SCORER, 1949; MALKUS and STERN, 1953). In these studies, however, it is first assumed that the amplitudes of the perturbations are infinitely small in order to linearize the differential equations. Streamlines are then drawn in which the amplitudes of the waves are finite. This inconsistent approach raises a number of questions. Such a procedure yields a shock-free flow of a fluid past a barrier when, in fact, it is not known whether such a solution really exists. The experiments described in this paper indicate that hydraulic jumps occur at very moderate Froude numbers from a meteorological viewpoint, provided the obstacle is not too small. The experiments further indicate that the jumps are the only phenomenon of any great importance unless the height of the obstacle is small compared to the depth of the lower fluid. In such cases there are small lee waves which increase in amplitude with the fluid speed. For an obstacle of appreciable size, however, these waves do not simply change in wave length and amplitude but completely disappear at low speeds or are replaced by finite jumps at the higher speeds. It is evident from the experiments that a finite obstacle cannot be considered simply as a "big" infinitesimal barrier.

In Part I mention was made of some preliminary experiments with a three-fluid system in which flow over an obstacle produced a hydraulic jump at each interface. A more basic approach to the experimental problem of investigating stratified flows, however, would be to set up the most simple model possible and attempt to understand it fully before constructing more complex models. Obviously a system of two fluids is simpler than a three-fluid system and the former is the subject of this paper. The latter will be investigated at a later time, along with other models of interest.

Some mention should be made of the reason for the choice of multiple-layer systems instead of those with continuous density gradient. To some extent the choice is one of expediency. Layers of immiscible fluids can be set up with ease and the resulting step-wise density distribution can be maintained indefinitely. After a given run the fluids settle back quickly to their original undisturbed state and the experiments continue without interruption. A fluid with a continuous density gradient may be obtained by a mixture of water and salt, for example, (GÖRTLER, 1943), and some preliminary experiments with such a set-up have already been undertaken. Yet multiple-layer systems have considerable practical interest in meteorology. Rather strong temperature inversions are of frequent occurrence in the atmosphere and, dynamically, they resemble closely interfaces between two homogeneous fluids (adiabatic atmospheric layers[1]).

Moreover, even the two-fluid case probably has considerable application to atmospheric problems. In the preliminary experimental investigation with three fluids it was suggested that the lower two fluids were analogous to the troposphere, and the upper fluid to the stratosphere. When the density difference was the same across each interface, small but significant effects (hydraulic jumps) were noticed at the "tropopause" for atmospheric values of the Froude number and obstacle sizes. The great stability of the stratosphere, how-

[1] The effect of compressibility will be discussed in a forthcoming paper.

ever, would indicate that a closer approximation to the atmosphere would be obtained by increasing the density difference between the middle and upper fluid. When the latter density difference was made four times the difference across the first interface, the "tropopause" was almost undisturbed for reasonable Froude numbers and the fluid system behaved very much like a two-fluid system with an upper rigid or free surface. At the present stage of the research this should be regarded only as a suggestion. It is possible to arrange various layers of immiscible fluids to obtain density distributions more closely resembling those in the atmosphere. Such experiments will help to resolve some of the questions that arise in applications of simple stratified systems to atmospheric problems.

Section 2 of this paper is devoted to a discussion of experiments with a two-fluid system of stratified fluids. A qualitative description is given of the most important phenomena together with a number of photographs. Section 3 contains an introduction to the theory of simple stratified systems. In this section the flow of a single fluid over a barrier is treated exhaustively, since a full understanding of this model leads to a fuller understanding of the more complicated two-fluid problem.

Section 4 presents the theory of the general two-fluid problem of flow over a barrier and a comparison with the experimental observations. The last section discusses the meteorological implications of this work.

## 2. Experimental observations

The apparatus for the experimental work is a channel about 10 ft long, 5 in wide, and 20 in high (fig. 1). The fluids are mixtures of water and salt and of carbon tetrachloride and a commercial cleaning fluid. An obstacle of "easy shape" is drawn by a motor drive at a uniform speed along the bottom.[1] This obstacle extends from one glass wall to the other. The density difference between layers is 0.025 gm $cm^{-3}$ and is not varied. Theoretical considerations indicate that this involves little or no loss

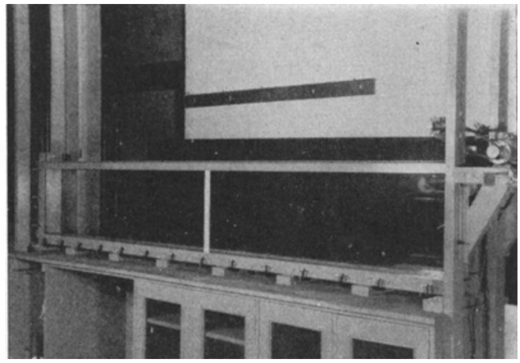

Fig. 1. Experimental channel.

of generality if interest is confined to models and prototypes in which density differences are small.

It follows from dimensional considerations that the phenomena appearing in this experiment depend on a minimum of three non-dimensional numbers

$$F_i = \frac{U}{\sqrt{g\frac{\Delta\varrho}{\varrho}H}}, \quad R_0 = \frac{h_0}{H}, \quad \beta = \frac{b}{H},$$

where $F_i$ is the internal Froude number, $h_0$ is the initial height of the lower layer upstream, $H$ is the total depth of the two fluids, and $b$ is the maximum height of the obstacle. In the following description we will identify each experiment with the values of these three numbers.

(a). $R_0 < 1/2$. For any $R_0$ in this range and for moderate values of $\beta$, a so-called absolutely subcritical state of flow exists provided the Froude number is small enough. In this state the fluids move over the barrier with little or no turbulence. The interface is level except for a slight symmetrical dip over the barrier. Fig. 2 is a typical photograph of this regime of motion. Small values of $F_i$ and $\beta$, and large values of $R_0$ are favorable for its existence. During the steady motion of an obstacle in the absolutely subcritical state, the upstream and downstream levels do not change sensibly and no deformation of the *free* surface can be seen. The only cases for which the above description of the absolutely subcritical flow does not hold are those in which $\beta$ is small compared to $R_0$. In such cases the absolutely subcritical regime,

[1] In the early experiments an obstacle of semi-circular cross-section was used. The boundary-layer separation effects were very marked, however, and the resulting large energy losses to turbulence were undesirable.

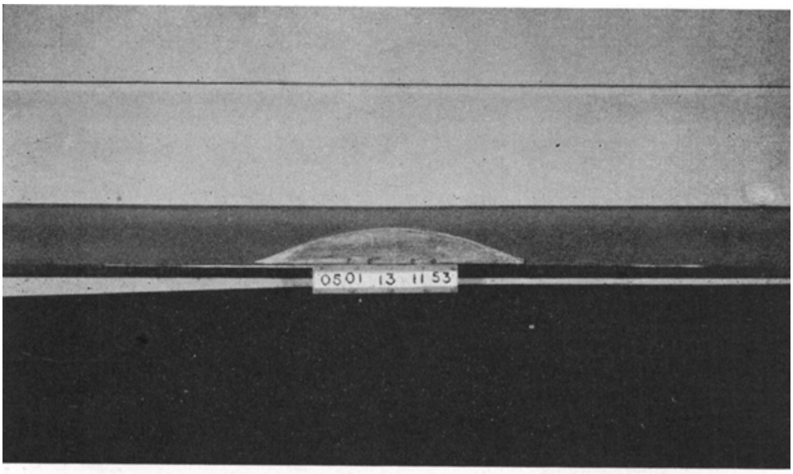


Fig. 2. Absolutely subcritical flow of a two-fluid system. $F_i = .048$, $R_0 = .33$, $\beta = .205$. In all experimental photographs of this paper the flow is from right to left.

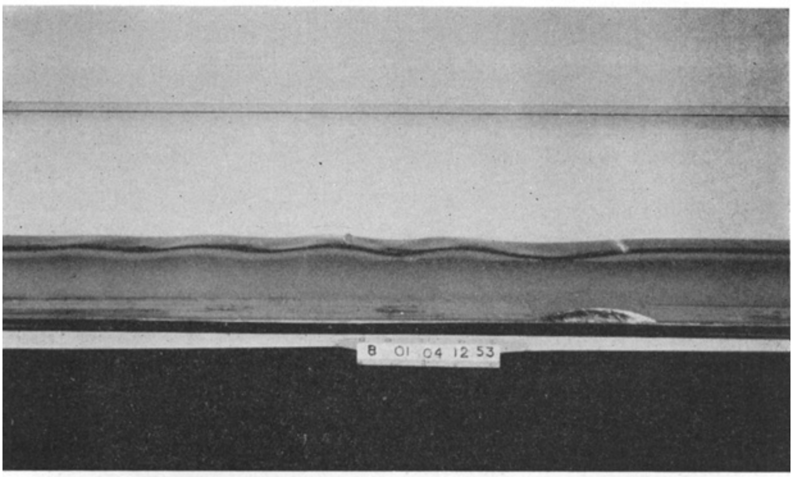


Fig. 3. Wave motion in the lee of a small obstacle. $F_i = .220$, $R_0 = .33$, $\beta = .067$.

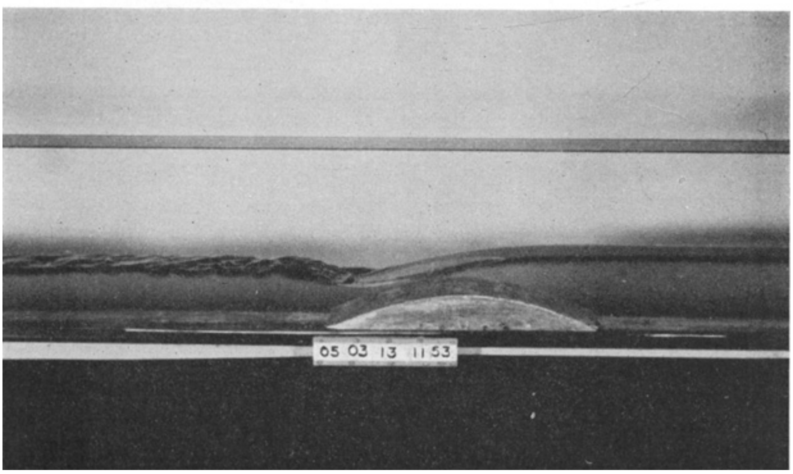


Fig. 4. Moderate hydraulic jump. $F_i = .155$, $R_0 = .33$, $\beta = .205$.

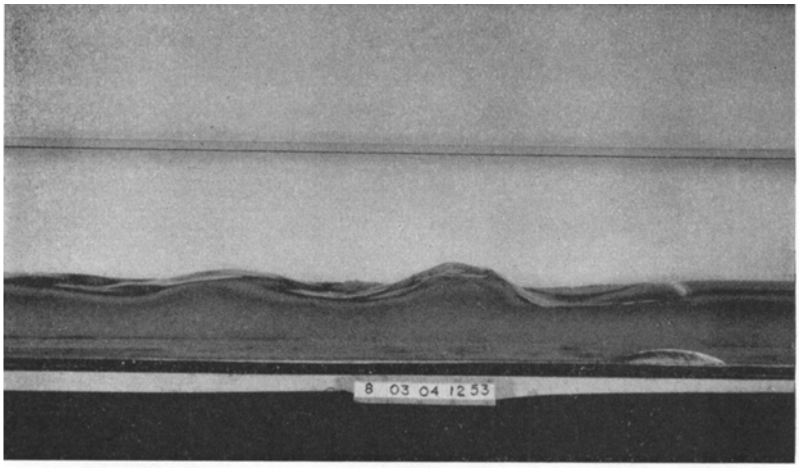


Fig. 5. Undular jump phenomenon. $F_i = .284$, $R_0 = .33$, $\beta = .067$. Except for the higher Froude number the conditions are the same as in fig. 3.

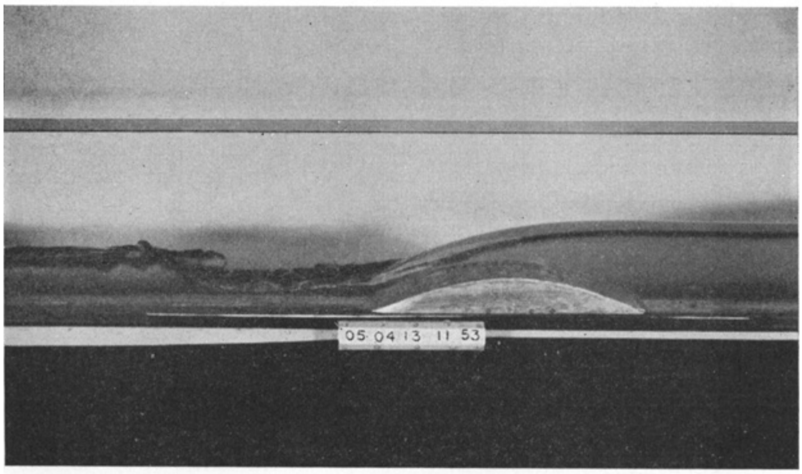


Fig. 6. Strong hydraulic jump. $F_i = .272$, $R_0 = .33$, $\beta = .205$.

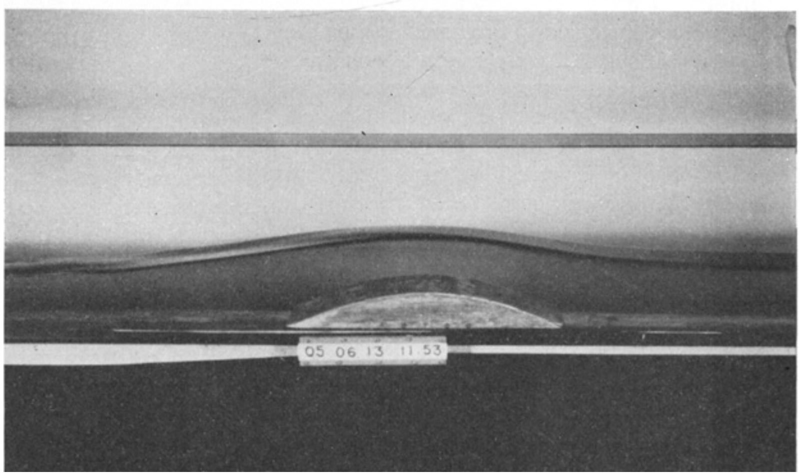


Fig. 7. Absolutely supercritical flow. $F_i = .583$, $R_0 = .33$, $\beta = .205$.

as indicated theoretically in section 4, may involve wave motions in the lee as in fig. 3. The Froude number, .220, is absolutely subcritical for the values $R_0 = .33$, $\beta = .067$.

If the fluid levels and the obstacles are kept constant while the Froude number is increased, the absolutely subcritical regime changes, at a certain Froude number, to one which we may call critical. The interface is no longer symmetric about the crest of the barrier, but continues to lower on the lee. Instead of returning smoothly to the undisturbed downstream level, the interface jumps more or less abruptly to the downstream level as in fig. 4. The interface has a rough appearance but assumes a relatively constant level downstream. With small obstacles the rise to the downstream level takes place over a series of downstream waves, resembling the undular jump in open channel flow. This is illustrated in fig. 5.

If the Froude number is lowered the turbulent jump moves toward the crest, diminishing in intensity. The value of the Froude number when the jump just disappears is taken as the lower limit of the critical regime.

If the Froude number is increased the jump moves downstream and increases in intensity. An example of a stronger jump is given in fig. 6. Unless the jump is very weak, the obstacle acts to block the flow of the lower fluid upstream and the upstream depth increases markedly. This is very evident in fig. 6. The depth ratio was originally .33; at the time of the picture it has been increased to nearly .50 by this blocking action. The increase is accomplished by a gravity wave of elevation which moves further and further ahead of the barrier. If the channel were much longer than our present one this wave would eventually be able to move so far ahead of the obstacle that we could neglect it and use the height of the lower layer just upstream to compute a new value of $R_0$. Experimental comparison with the theory of section 4 indicates that the channel is too short for this, and a true unsteady state exists whenever a strong jump occurs.

If the Froude number is sufficiently high initially, a jump, forming in the lee of the obstacle, is left behind as the barrier moves on. In this state the blocking action does not appear; the fluid level upstream remains at its initial height. The interface swells up over the barrier and is symmetric about the crest. This is called absolutely supercritical and is shown in fig. 7.

Depending on the experimental conditions the absolutely supercritical regime occurs for Froude numbers of the order of .50 and greater. At such speeds the free surface is sensibly undisturbed. If $F_i$ is now raised to 2 or 3, the free surface becomes disturbed and a lee jump may even occur on it (fig. 8). This regime is considered to be outside the area of interest of this paper.

When the initial height of the lower fluid is less than that of the obstacle, two types of flow are possible: If $F_i$ is very high the lower fluid moves symmetrically over the barrier in a swell, or for the larger obstacles, the interface rises over the barrier remaining high on the lee for some distance and then appears to *drop down* violently to a lower downstream level (fig. 9). This particular phenomenon is discussed further below. If the Froude number is smaller, however, the high barrier blocks the lower fluid completely and the lower fluid builds up eventually until it spills over the obstacle, forming a lee jump. This is brought out in figs 10 and 11. In fig. 10 the blocking action has increased the upstream depth considerably. Fig. 11, taken a short time later, shows the spilling over the barrier and the lee jump.

(b). $R_0 > 1/2$. When the lower fluid is deeper than the upper, the absolutely subcritical regime still exists for sufficiently small Froude numbers, and a very small obstacle will again produce lee waves in this regime. If the Froude number is increased, a small jump appears. As in the case of $R_0 < 1/2$, the jump increases in size and moves downstream with an increase in $F_i$. When the obstacle is large or moderately large and the Froude number is high, instead of an ordinary lee jump, or a symmetrical, absolutely supercritical type of motion, the interface rises on the upwind side, reaches a maximum at the crest and then appears to "drop" suddenly on the lee. An example of the hydraulic drop is shown in fig. 12. An increase in $F_i$ merely causes the hydraulic drop to move downstream. In some cases, however, a drop is followed by an orthodox jump in the interface level, but this does not always happen. Although a hydraulic drop is not possible in a one-fluid system it is

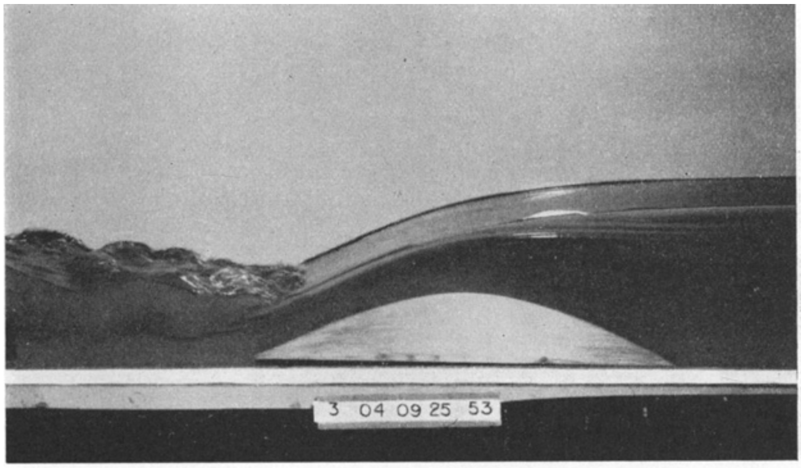


Fig. 8. Jump on free surface. $F_i = 2.20$, $R_0 = .75$, $\beta = .60$.

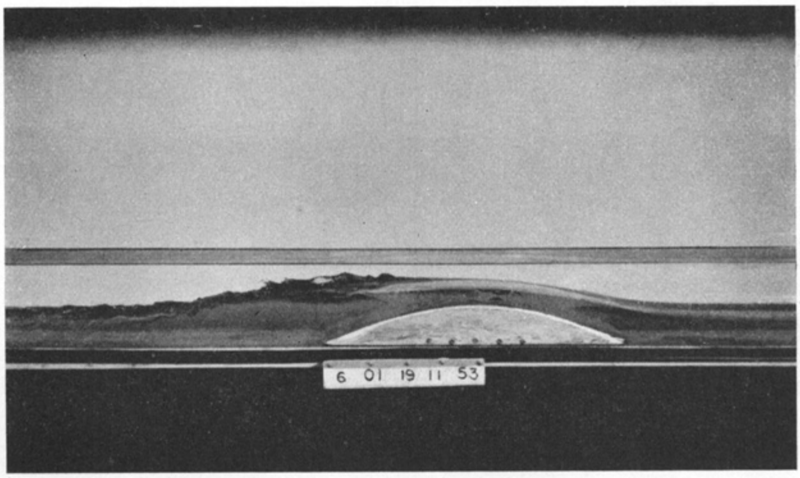


Fig. 9. Hydraulic drop. The obstacle is higher than the upstream fluid depth. The interface remains high on the lee dropping with accompanying turbulence to the downstream level. $F_i = .577$, $R_0 = .33$, $\beta = .50$.

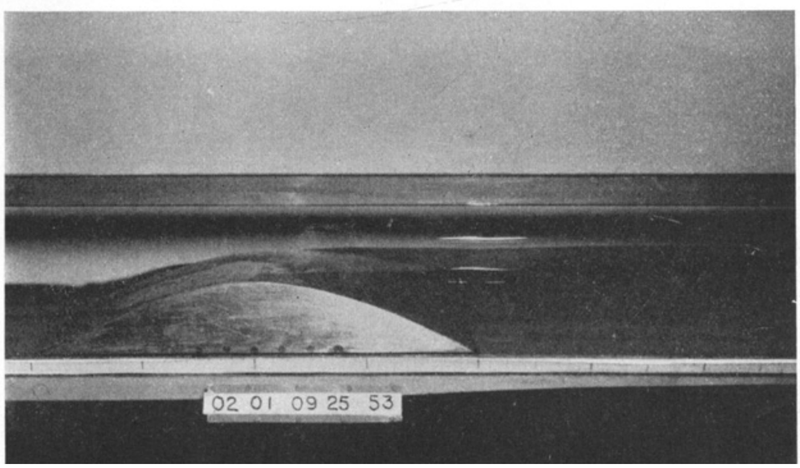


Fig. 10. Complete blocking of lower fluid by a barrier higher than the interface.

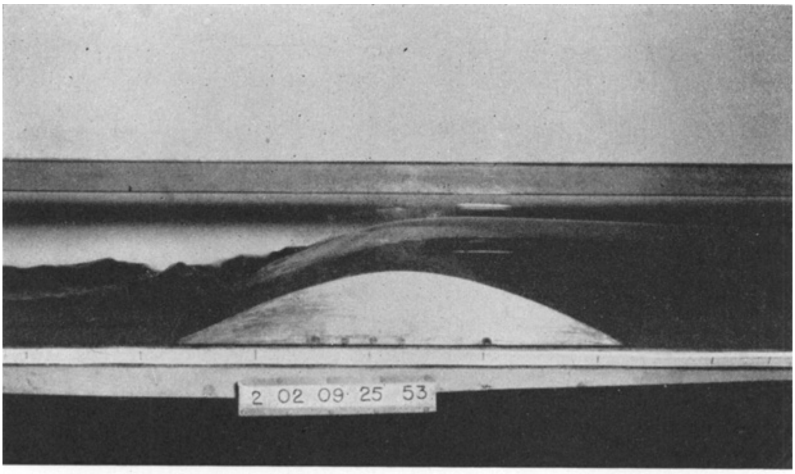


Fig. 11. Final spilling of the fluid over the barrier of fig. 10 and formation of a lee jump.

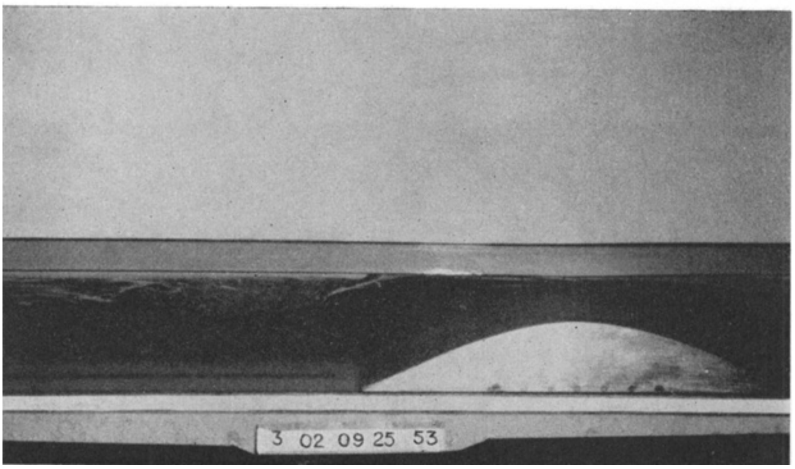


Fig. 12. Hydraulic drop. $F_i = .542$, $R_0 = .75$, $\beta = .60$.

a definite theoretical possibility in multiple fluids. In a two-fluid system it is favored by a relatively deep lower fluid (see section 4).

### 3. Hydraulic analogy

The theory of the flow of two fluids over an obstacle is developed in section 4 of this paper. The analogous problem of the flow of water over a barrier is much simpler and it seems advisable to develop first the theory of the latter system. Since this problem has long been of interest to hydraulic engineers, the present section probably does not contain any original material. The procedure is quite different, however, from the standard approach and seems to contain a number of advantages.[1]

Fig. 13 is a drawing of a channel in which water of basic depth, $H$, approaches a barrier of variable height, $b(x)$, over the otherwise horizontal bottom. The water has a vertical depth, $h(x)$, over the topography and the flow with respect to the obstacle is assumed steady. The fluid is considered frictionless and the pressure distribution hydrostatic. In the experimental work of this paper an obstacle is towed through a resting fluid. Relative motion

[1] I am indebted to Professor George S. Benton for a number of invaluable suggestions regarding the material of this section.

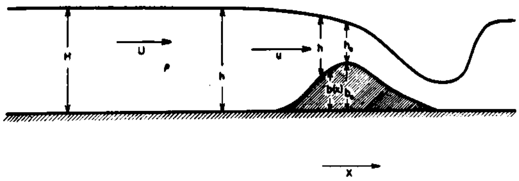

Fig. 13. Theoretical model of single-fluid flow over a barrier.

of the fluid and the boundaries occurs as the liquid passes over the barrier. It is not likely that the resulting energy changes will be important, since turbulence is largely avoided by using a smooth obstacle of "easy shape". The hydrostatic assumption is certainly satisfied upstream where the vertical motion is zero. In the vicinity of the barrier the paths of the fluid particles are necessarily curved and there will be vertical centripetal accelerations.

Some mention may be made of the effect of dissipative forces and centrifugal forces. In the absence of a jump there will still be some loss of energy, as the fluid passes over the barrier, by friction with the moving obstacle. This is accompanied in open channel flow by a loss of total head (total depth of the fluid). Consequently, in a steady state the water level on the downstream side must be somewhat lower than that on the upstream side. If the channel is infinite in length, however, sufficiently far upstream and downstream the levels must be equal if the obstacle, starting from rest, has moved a finite length of time. The conclusion is inevitable that a steady state can never be achieved because of the continuing adjustment of upstream and downstream levels. While this effect will be very small in laminar motion, a jump, occurring upstream or in the lee of the obstacle, causes considerable turbulence and an enormous increase in energy loss. The relative increase of upstream head is accomplished by a blocking action of the barrier. A long wave of elevation moves upstream, raising the approach level sufficiently to permit the jump to remain in a steady state. It follows that for large jumps, since the approach Froude number is large, the elevation wave cannot move very far ahead of the barrier in the limited length of the experimental channel. A quasi-steady state cannot be assumed and the theoretical deductions in this paper are not applicable. The blocking effect is certainly worthy of separate study, however, since it will be operative in the flow of air over mountain barriers.

The neglect of the curvilinear nature of the flow by assuming a hydrostatic pressure distribution is not completely justified unless the characteristic radius of curvature of the boundary form is considerably greater than the depth of the fluid. This is not fulfilled in the experimental work and some error results. The centrifugal forces act upward in the flow over a barrier so that the effect is to reduce gravity apparently. This should tend to reduce the critical Froude number at which lee jumps begin to occur in the lee. The experiments do not indicate an appreciable error in this regard provided a real shock wave occurs for a given height of the obstacle and fluid depth. As mentioned in section 4 a shock wave predicted by hydrostatic theory does not materialize in the experiments if the barrier is too small. Instead lee waves or so-called undular jumps (without accompanying turbulence) are found. Since a truly hydrostatic pressure distribution permits only infinitely long waves, hydrostatic theory obviously breaks down in such cases. On the whole, however, if this limitation is kept in mind, the hydrostatic assumption is much more useful and valid than small perturbation theory.

In the model of fig. 13 the Bernoulli equation for a steady state is

$$\frac{p}{\varrho g} + \frac{q^2}{2g} + z = \text{constant}, \qquad (1)$$

where $z$ is the height of the point in question above the horizontal bottom, and $q$ is the speed at that point. Using the hydrostatic assumption to compute the pressure, and neglecting the vertical velocity[1], (1) becomes

$$\frac{u^2}{2g} + h + b(x) = \text{constant}. \qquad (2)$$

In hydraulics the sum of the first two terms of this expression is called the specific energy, $e$, of the fluid,

$$e = \frac{u^2}{2g} + h. \qquad (3)$$

[1] The ratio $v^2/u^2 = o(h^2/\lambda^2)$, where $\lambda$ is the distance between two consecutive points at which the wave profile meets the undisturbed level. The hydrostatic assumption therefore implies that the kinetic energy due to vertical motion is negligible. (See LAMB 1932, p. 258).

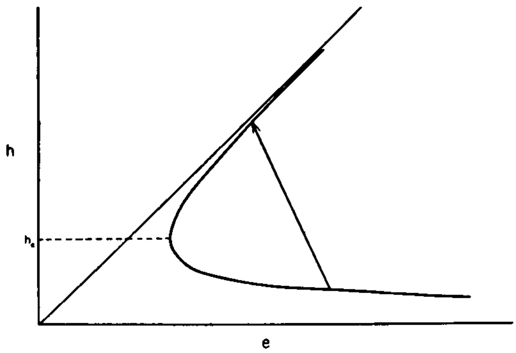


Fig. 14. Specific energy diagram.

Since the equation of continuity requires a constant discharge, Q, at each section,

$$Q = uh = \text{constant}. \tag{4}$$

Substituting for $u$ in (3) we find that

$$e = \frac{Q^2}{2gh^2} + h = e(h). \tag{5}$$

Thus, for a given discharge, $e$ is a function of the depth of the fluid, $h$. A plot of this relationship is given in fig. 14. For a given Q, any fluid depth is possible but at a certain depth, $h_c$, the specific energy is a minimum. The minimum occurs when $u^2 = gh$. Depths less than $h_c$ are called supercritical and the flow is then in such a state that a hydraulic jump is possible at some downstream section. The jump is characterized by a sudden increase in the depth of flow downstream, accompanied by turbulence. The flow downstream of the jump must have a depth greater than the critical depth, $h_c$, and so lies in the upper branch of the specific energy curve. Some of the energy is lost through turbulence as illustrated qualitatively by the arrow in fig. 14. It should be noted that in the absence of frictional or turbulent losses the specific energy changes only by the virtue of changes in the height of the bottom of the channel. As the fluid moves over an obstacle, for example, it loses specific energy on the upwind side and gains it on the downwind side. Quantitatively, from (2) and (5),

$$\frac{de}{dx} = \frac{de}{dh}\frac{dh}{dx} = \left(1 - \frac{Q^2}{gh^3}\right)\frac{dh}{dx} = -\frac{db}{dx}, \tag{6}$$

or

$$\left(1 - \frac{u^2}{gh}\right)\frac{dh}{dx} = -\frac{db}{dx}. \tag{7}$$

This equation shows that $u^2$ can equal $gh$ (i. e. the flow can become critical) only where $db/dx$ vanishes. Since the specific energy changes only during passage over the barrier, if the flow becomes critical anywhere, it must do so at the crest where $b = b_c$. Moreover we may show that a critical condition at the crest necessarily implies a change of regime from the upstream to the downstream side of the barrier. Thus, differentiating (7) with respect to $x$,

$$\frac{3u^2}{gh^2}\left(\frac{dh}{dx}\right)^2 + \left(1 - \frac{u^2}{gh}\right)\frac{d^2h}{dx^2} = -\frac{d^2b}{dx^2}, \tag{8}$$

at the top of the barrier. Since it is critical there the second term on the left vanishes. However, the right-hand side is not zero so that $dh/dx$ does not vanish at the crest. Hence, if it is subcritical upstream and critical at the crest, it will be supercritical just on the lee. If it is supercritical upstream, and critical at the crest it must be subcritical on the lee side.

With these considerations in mind we may discuss the various possibilities listed in Table 1, not all of which are possible steady states. They may occur instantaneously, however, if a given obstacle is accelerated from zero very rapidly to a given speed. In (a) the obstacle speed is very slow and the flow is everywhere subcritical. This is an obvious steady state. If

**Table 1. Regimes of motion in flow of single fluid over an obstacle**

| *Approach* | *Upstream Jump* | *Crest* | *Downstream Jump* | *Downstream* |
|---|---|---|---|---|
| (a) Subcritical | — | Subcritical | — | Subcritical |
| (b) Subcritical | — | Critical | yes | Subcritical |
| (c) Subcritical | — | Supercritical | yes | Subcritical |
| (d) Supercritical | yes | Subcritical | — | Supercritical |
| (e) Supercritical | yes | Critical | — | Supercritical |
| (f) Supercritical | — | Supercritical | — | Supercritical |

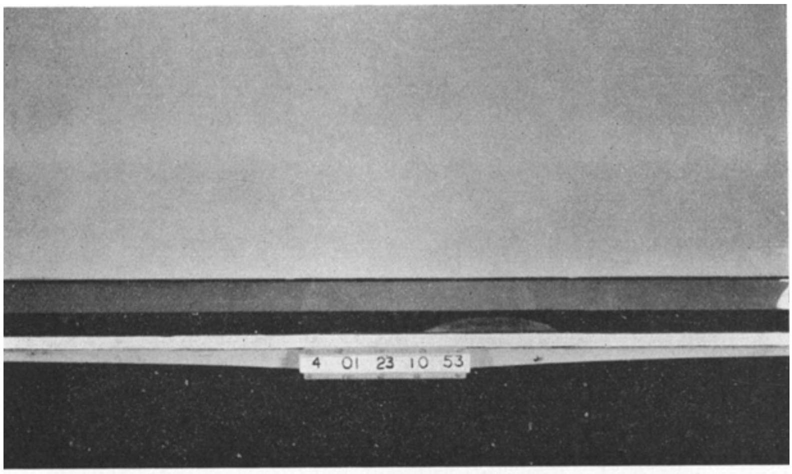


Fig. 15. Subcritical flow of single fluid over a barrier. $F = .044$, $\beta = .773$.

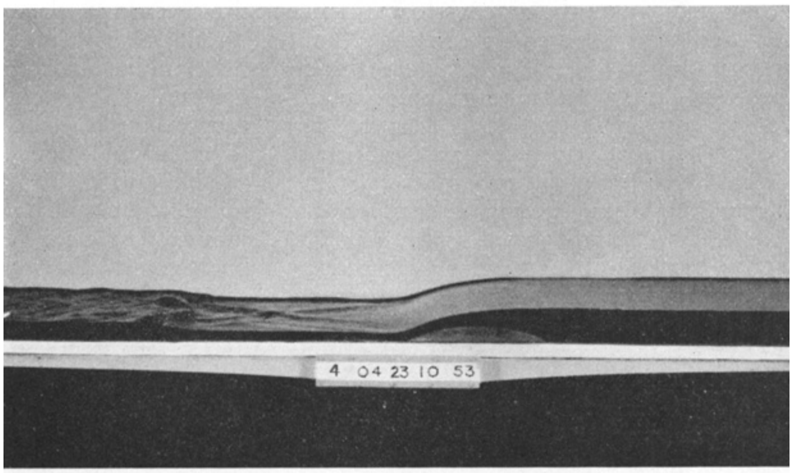


Fig. 16. Lee jump in single-fluid system. $F = .394$, $\beta = .773$.

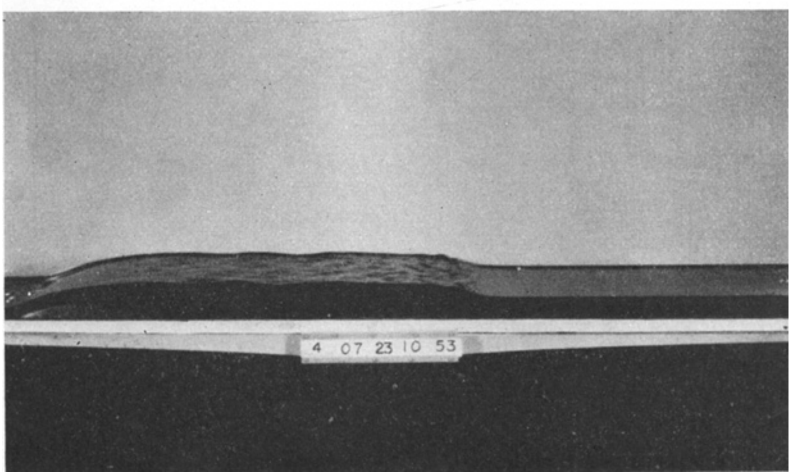


Fig. 17. Upstream jump in single fluid system. $F = .937$, $\beta = .773$. Experimentally it was not possible to move the obstacle fast enough to obtain a Froude number greater than 1.0. Consequently, this figure does not represent a condition that quite falls in the region (d) of fig. 18. Obviously since $F$ is slightly less than 1.0 the upstream jump could not maintain its position with respect to the obstacle because the fluid would the be jumping from subcritical to subcritical. Thus the jump must be moving down the channel faster than the obstacle following it.

the speed is increased further (b), the flow will become critical at the crest of the barrier and, necessarily, supercritical somewhere on the lee. It is apparent from the specific energy diagram that the fluid cannot return to the subcritical state required downstream except by a jump on the lee. If the obstacle speed is increased further (c) the flow will become supercritical instantaneously at the crest, although remaining subcritical upstream and downstream. This is not a steady state, of course, and a wave of elevation[1] will progress upstream raising the upstream level sufficiently to permit a critical condition at the crest. A steady state will ultimately be established with a lee jump.

If the speed of the barrier is so high that the flow is moderately supercritical upstream (d), the loss of specific energy in flowing over the barrier may reduce the system to subcritical at the crest. This is also unsteady and a jump upstream must form. We may see this by considering the rate of propagation of disturbance energy originating at the barrier. Since the flow in the vicinity of the crest is subcritical, the group velocity of the longer waves, tending to move upstream, exceeds the fluid velocity and energy is propagated upstream. This energy cannot move indefinitely upstream, however, because of the postulated supercritical flow far upstream. The resulting convergence of energy is then available to support a jump. The fluid will then jump to subcritical somewhere upstream followed by critical flow at the crest, passing smoothly to the supercritical condition downstream.

The obstacle speed may now be increased sufficiently to permit a critical condition at the crest (e), and therefore, a subcritical condition for some distance to the lee. Since a smooth transition to the downstream supercritical condition is not possible, a discontinuity of flow must occur somewhere. We may have a jump upstream but it does not seem possible to exclude a hydraulic drop in the lee on the basis of the tools developed so far for analyzing the model. If the principle of conservation of momentum is employed, the drop is seen to require a negative dissipation of energy and is thus impossible. This resort is not available for multiple-fluid systems and in the subsequent sections we must employ experimental evidence to indicate which of these possibilities will be realized.

The last possibility (f) for the one-fluid system will be obtained if the initial speed of the obstacle is faster than that required for (e). The flow will then be supercritical everywhere.

It is of interest to consider the form of the free surface in the various cases of Table 1. In case (a) there will be a slight draw-down over the obstacle returning to the original level downstream (fig. 15). In (b) and (c) the draw-down over the obstacle is followed by a lee jump and an abrupt rise to the undisturbed downstream level (fig. 16). In cases (d) and (e) the upstream jump is followed by a gradual lowering of the surface toward the crest (fig. 17). In (f) the surface will rise smoothly over the obstacle. Such a phenomenon is called a "swell" in hydraulics.

Given the undisturbed fluid level, $H$, the height of the obstacle, $b_c$, and the speed of the approaching current, $U$, it should be possible to predict which of the possibilities of Table 1 will be realized. Obviously, the deciding factor is the condition that the flow be critical at the summit. Using this, plus the energy and continuity equations we have, at the crest,

$$\frac{U^2}{2g} + H = \frac{u_c^2}{2g} + h_c + b_c, \qquad (9)$$

$$\frac{u_c^2}{gh_c} = 1, \qquad (10)$$

$$u_c h_c = UH. \qquad (11)$$

Eliminating $u_c$ and $h_c$,

$$F^2 - 3F^{2/3} = 2\left(\frac{b_c}{H} - 1\right), \quad F^2 = U^2/gH. \qquad (12)$$

A graph of this equation is shown in fig. 18. The regimes are labeled in accordance with the system employed in Table 1.

## 4. Theory of a two-fluid system

In the case of a two-fluid system we assume that the density difference between the two layers, $\Delta\varrho/\varrho$, is very small. The approach velocities are assumed to be the same in both fluids. It is not difficult to see intuitively that velocities accompanying marked effects at the interface will be so small as to produce little

[1] As in fig. 17 this wave of elevation may break forming an upstream bore.

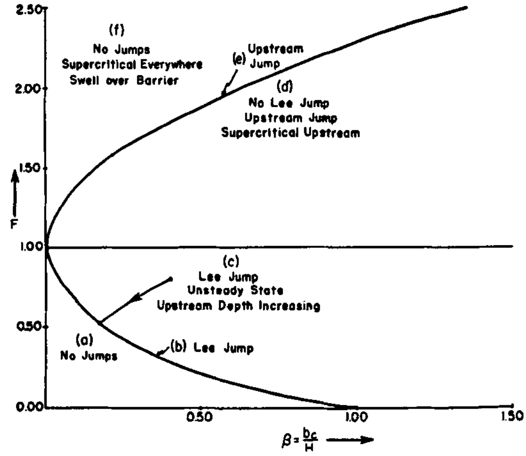


Fig. 18. Critical curve of a one-fluid system. Compare Table 1.

disturbance at the free surface. Kinematically, therefore, it should be possible to replace the free surface with a rigid surface. If this is done it is necessary to consider pressure variations ($\pi$) on this upper rigid surface, just as the dynamic effects of the very slight changes in height of the free surface would also be important. In the following development, therefore, the model of fig. 19 is adopted in which the two-fluid system is capped by a rigid surface. A parallel development with a free surface yields identical results provided conditions are such that the flow is well within the absolutely subcritical range with respect to deformations of the free surface. This puts generously high upper limits on the size of the obstacle and the internal Froude number. These limits may be computed from fig. 18.

With the hydrostatic assumption the energy equations in this model are

$$\pi + \varrho' g H + \varrho' \frac{v^2}{2} = \text{constant}, \tag{13}$$

$$\pi - \varrho' g(h + b - H) + \varrho g(h + b) + \frac{\varrho u^2}{2} = \text{constant}. \tag{14}$$

The total energy transport through a given section is

$$E_t = (Q + Q')\,\pi + Q'\varrho' g H + Q'\varrho' \frac{v^2}{2} + Q\varrho \frac{u^2}{2} + \\ + Q\varrho g(h+b) \tag{15}$$

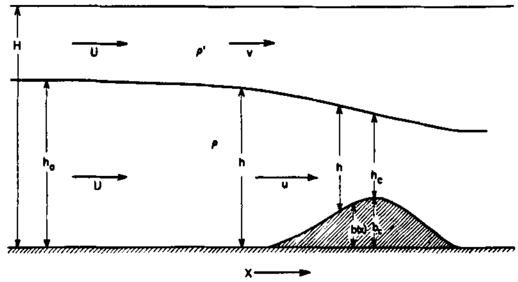


Fig. 19. Theoretical model of two-fluid flow over a barrier.

$$- Q\varrho' g(h + b - H) = E_{to} = \text{constant}.$$

Evaluating $E_{to}$ upstream and substituting for $\pi$ from equation (13),

$$\frac{Q'^2 \varrho'}{2} [(H - h_0)^{-2} - (H - h - b)^{-2}] + \\ \frac{Q^2 \varrho}{2} [h^{-2} - h_0^{-2}] + (\varrho - \varrho')\, g\, (h - h_0 + b) = 0. \tag{16}$$

If we define the quantities

$$F_i^2 = U^2/g \frac{\Delta \varrho}{\varrho} H, \quad R_0 = \frac{h_0}{H}, \quad \alpha = \frac{h}{H}, \quad \beta = \frac{b}{H} \tag{17}$$

and assume the same approach velocity, $U$, in both layers, equation (16) becomes

$$\frac{F_i^2}{2} \left[ \frac{R_0^2}{\alpha^2} - \frac{(1 - R_0)^2}{(1 - \alpha - \beta)^2} \right] + \alpha + \beta - R_0 = 0. \tag{18}$$

This relationship gives $\alpha$ as a function of $\beta$, i. e. the profile of the interface during passage over the obstacle. Curves of this equation for $R_0$ of 0.30 and 0.70 are shown in figs 20 and 21.

(a) $R_0 < 1/2$. Fig. 20 is typical of flows with a lower layer less than one-half the total depth. If $F_i$ exceeds 0.50, the interface will swell over a barrier returning to its original depth on the lee. This theoretical behavior is confirmed by experiment, as for example in fig. 7. If $F_i < .50$, the interface will tend to lower slightly as the fluid ascends the barrier. For example, if $F_i = .10$ and the obstacle has a maximum height less than .155, the interface lowers to a minimum at the summit, returning smoothly to $R_0$ on the lee. This is the situation in fig. 2.

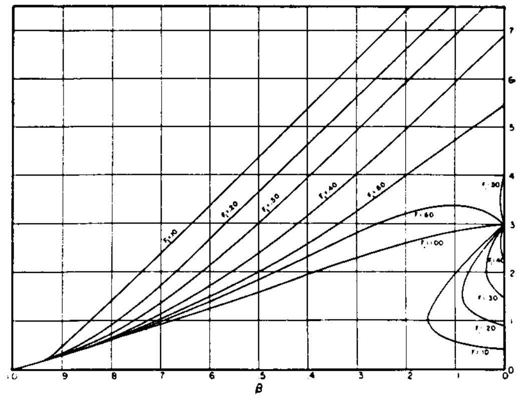

Fig. 20. Thickness of lower layer, $\alpha(x)$, during passage of two-fluid system over a barrier of height, $\beta(x)$. The upstream depth of the lower fluid is $R_0 = .30$.

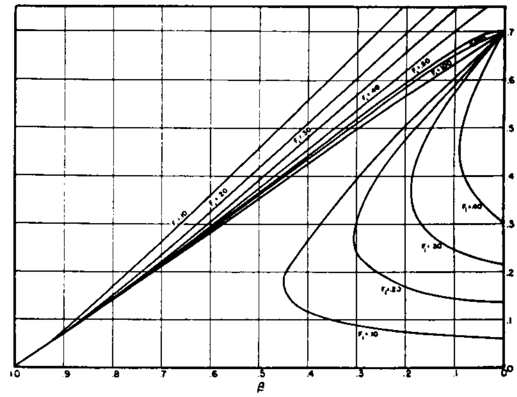

Fig. 21. Thickness of lower layer, $\alpha(x)$, during passage of two-fluid system over a barrier of height, $\beta(x)$. The upstream depth of the lower fluid is $R_0 = .70$.

If the barrier is precisely .155, the interface will continue to lower on the lee. From the one-fluid analogy we would expect a very small (theoretically infinitesimal) hydraulic jump just on the lee, the interface returning to a downstream level slightly less than $R_0$ because of the energy loss. If the obstacle is appreciably greater than .155, a steady state is not possible. This causes a change in $R_0$, i. e. a blocking effect takes place that increases $R_0$ to a level that permits a steady state between some upstream section and the crest of the obstacle. Thus, if the obstacle has a maximum height .30 and is held at a speed corresponding to $F_i = .10$, additional curves of equation (18) show that the upstream depth of the lower fluid will tend to increase from .30 to .50. The system will then have attained a quasi-steady state upstream, the interface will lower as the fluid ascends the barrier and will continue to lower on the lee. Since the blocking effect has created a differential in the upstream and downstream levels, the interface can take a finite jump on the lee to the lower energy level downstream, as in figs 4 and 6. We will see later how we can predict the new value of $R_0$, knowing the approach Froude number and the height of the obstacle.

If the obstacle is higher than .30, in the case of fig. 20, and $F_i < .50$, the barrier blocks the lower fluid completely until it builds up to obstacle height, spills over and forms a hydraulic jump on the lee as in figs 10 and 11. If $F_i > .50$ it may be expected from the curves that the interface will swell over the barrier, returning smoothly to its equilibrium height. This is confirmed by experiments, however, only if the height of the barrier is not too much greater than $R_0$. In fig. 9, however, the interface does not return smoothly on the lee but appears to drop down with considerable turbulence. A possible explanation is the following: In fig. 9 we observe that the interface traces a curve close to the $F_i = .60$ cu ve in fig. 20 up to the crest, $\beta = .50$. This isr to be expected since there should be very little loss of energy in this region. As the fluid attempts to move down the lee of the obstacle, boundary-layer separation and resulting turbulence will abstract energy. Fig. 9 shows that the interface stays at its maximum height for a considerable distance in the lee which means that the interface describes a line, $\alpha + \beta \simeq$ const $\simeq .70$. In an intuitive sense we may assume that the loss of energy to turbulence, throws the fluid system from an energy level, corresponding to $F_i = .60$ to a lower energy curve close to that of the $F_i = .40$ curve in fig. 20. This, however, would require a deep lower fluid on the lee (.70), whereas the experimental set-up requires the fluid to return to the vicinity of .30. This can be, and apparently is, accomplished by the occurrence of a shock wave, in this case a hydraulic drop, to form a connection with the downstream level. According to this intuitive argument, this phenomenon requires a nearby curve of lower energy level at the point of the diagram corresponding to the crest of the barrier, so that a modest loss of energy to turbulence will permit a return of the interface height along an entirely different path than the approach. This requirement is met for a large barrier in this case. If the barrier has a maximum height of, say, only .15, the $F_i$ lines at the crest are fanned out sufficiently so that there is no nearby curve and the interface returns to close to the equilibrium height downstream along much the same curve as it described on the upwind side. It is evident from fig. 20 that the higher the barrier the more easily is this shift to a different branch of the $F_i$ curves accomplished.

The above argument lacks rigor because the various $F_i$ curves in fig. 20 do not correspond to different energy levels for given approach conditions but rather to different approach conditions and correspondingly different energy levels. It does not seem possible to verify the argument at this time because the present theory is inadequate to permit a calculation of alternate energy curves for the given approach conditions, $R_0$ and $F_i$.

(b) $R_0 > 1/2$. It has been stated that the behavior of a system in which $R_0 > 1/2$, is much the same as for shallower lower fluids except that for large or moderately large obstacles and high Froude numbers the symmetrical regime is missing and a hydraulic drop occurs in the lee. The suggestion above for the occurrence of a drop may also be applied in this case. We see from fig. 21 that the $F_i$ curves are very tightly packed in the upper part of the diagram. Thus, for example, in fig. 12 the curve traced by the interface should follow the $F_i = .542$ curve until the crest. The energy losses to turbulence in the lee will permit the interface to describe a curve, $\alpha + \beta = .90$ approximately. The return to a height somewhere near the equilibrium, $R_0 = .75$, downstream would require such a hydraulic drop as appears in fig. 12.

It would be desirable to have curves analogous to fig. 18 for a two-fluid system. Such curves are obtained by assuming, as in the single-fluid case, that conditions are just critical at the top of the barrier. Thus, if we differentiate (18) with respect to $x$,

$$\left.\begin{aligned}\frac{d\alpha}{dx}\left\{1 - F_i^2\left[\frac{R_0^2}{\alpha^3} + \frac{(1-R_0)^2}{(1-\alpha-\beta)^3}\right]\right\} &= \\ = -\frac{d\beta}{dx}\left\{1 - \frac{F_i^2\,(1-R_0)^2}{(1-\alpha-\beta)^3}\right\}&.\end{aligned}\right\} \quad (19)$$

At the top of the obstacle, therefore, either the slope of the interface vanishes or

$$F_i^2\left[\frac{R_0^2}{\alpha_c^3} + \frac{(1-R_0)^2}{(1-\alpha_c-\beta_c)^3}\right] = 1, \qquad (20)$$

where the subscript $c$ indicates that the section is at the top of the barrier. If there has been no energy loss upstream from the barrier, then at the top we have also from (18)

$$\frac{F_i^2}{2}\left[\frac{R_0^2}{\alpha_c^2} - \frac{(1-R_0)^2}{(1-\alpha_c-\beta_c)^2}\right] + \alpha_c + \beta_c - R_0 = 0. \qquad (21)$$

Between equations (20) and (21) we may eliminate $\alpha_c$ and, for a given $R_0$, obtain a relationship between $F_i$ and $\beta_c$ when the conditions at the crest of the barrier are just critical. It is obvious from the considerations of the previous section that this relationship is of fundamental interest. For a given $R_0$ the resulting curve will be analogous to the curve of fig. 18 for a single fluid. These curves are given in fig. 22 for steps of 1/10 in $R_0$. They resemble in general the curve of fig. 18 except that the upper branches differ. It follows from the preceding discussion that, for a given approach depth, $R_0$, of the lower fluid, points on or below the appropriate $R_0$ curve in fig. 22 represent possible quasi-steady states. If the given obstacle height and Froude number correspond to a point below the curve, the conditions are absolutely subcritical and no shock wave will occur. If the point falls on the curve a lee jump occurs (or lee waves for very small obstacles). If the point is somewhat above the curve the upstream depth should increase to the $R_0$ of the curve passing through this point. If $F_i$ is sufficiently large either the symmetrical, absolutely supercritical regime will occur or a hydraulic drop will be experienced in the lee.

In the case of a single fluid it is well known that a finite amplitude long wave tends to "break" with accompanying turbulence (LAMB, 1932). This follows from the consideration that the speed of propagation of a given height of the wave is proportional to the height. The high portions then eventually catch up with the troughs and the wave configuration becomes unstable. Since this is intimately connected with the hydraulic jump phenomenon, it is of

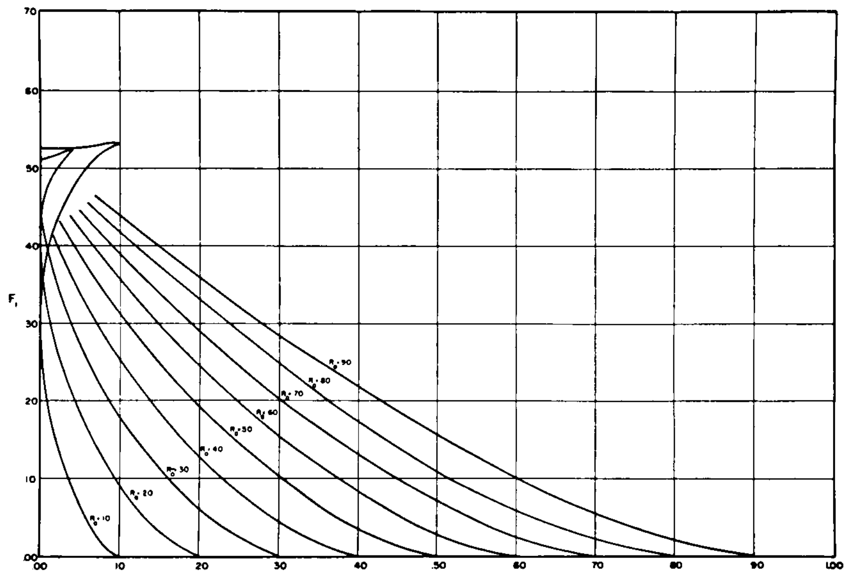

Fig. 22. Critical curves of a two-fluid system

considerable interest to investigate the behavior of finite long waves in a two-layer system. In particular it is of interest to investigate the hydraulic drop observed in the experiments under certain conditions when the upper fluid becomes more shallow than the lower.

It is easily shown that the speed, $c_0$, relative to some coordinate system, of an infinitesimal, long wave at the interface is given by

$$\frac{(u_{10}-c_0)^2}{g'h_{10}}+\frac{(u_{20}-c_0)^2}{g'h_{20}}=1, \qquad (22)$$

where $g' = g\Delta\varrho/\varrho$, and, for convenience, we denote the lower layer by subscript 1 and the upper by subscript 2. Solving for $c_0$,

$$\left.\begin{aligned} c_0 &= \frac{u_{10}h_{20}+u_{20}h_{10}}{H} - \\ &- \left[\frac{g'h_{10}h_{20}}{H}-\frac{h_{10}h_{20}}{H^2}(u_{10}-u_{20})^2\right]^{1/2}, \end{aligned}\right\} \qquad (23)$$

where we confine our attention to the wave tending to move upstream (toward negative $x$). Let us now consider a long wave of finite amplitude in which the height of the interface at some section is say $h_{10}+\eta$. Since the interface has an infinitesimal slope in a long wave, the interface is nearly horizontal in the vicinity of this section and, moreover, nearby portions of the interface are at infinitesimally small distances above or below $h_{10}+\eta$. The speed of propagation of nearby wave heights should therefore be given by equation (23) if we substitute the heights and velocities in the vicinity of this section. Thus

$$\left.\begin{aligned} c &= \frac{u_1(h_{20}-\eta)+u_2(h_{10}+\eta)}{H} - \\ &- \left[\frac{g'(h_{10}+\eta)(h_{20}-\eta)}{H} - \right. \\ &\left. - \frac{(h_{10}+\eta)(h_{20}-\eta)(u_1-u_2)^2}{H^2}\right]^{1/2}. \end{aligned}\right\} \qquad (24)$$

In order to obtain the speed of propagation in terms of the approach speed, $U$, of the two fluids we use the continuity equations at the section in question, namely

$$\frac{\partial}{\partial t}(h_{10}+\eta)+\frac{\partial}{\partial x}[u_1(h_{10}+\eta)]=0, \qquad (25)$$

and

$$\frac{\partial}{\partial t}(h_{20}-\eta)+\frac{\partial}{\partial x}[u_2(h_{20}-\eta)]=0. \qquad (26)$$

In view of the quasi-steady state the time derivatives may be expressed as

$$\frac{\partial}{\partial t}=-c\frac{\partial}{\partial x}. \qquad (27)$$

If we confine our attention to values of $\eta/H$, small compared to one, then to the order of $\eta/H$, the $c$ in equation (27) may be replaced by the $c_0$ of equation (23). Using this and integrating equations (25) and (26),

$$u_1(h_{10} + \eta) - c_0\eta = Uh_{10}, \tag{28}$$

and

$$u_2(h_{20} - \eta) + c_0\eta = Uh_{20}. \tag{29}$$

Substituting for $u_1$ and $u_2$ in equation (24) and retaining terms only of the order of $\eta/H$,

$$c = U - \left(\frac{g'h_{10}h_{20}}{H}\right)^{1/2}\left[1 + \frac{3}{2}\frac{(1 - 2R_0)}{R_0(1 - R_0)}\frac{\eta}{H}\right], \tag{30}$$

where $R_0$ is $h_{10}/H$. Therefore, the speed of propagation of the elevation $\eta$, relative to the equilibrium speed $U$ of the fluids, is

$$\left(\frac{g'h_{10}h_{20}}{H}\right)^{1/2}\left[1 + \frac{3}{2}\frac{(1 - 2R_0)}{R_0(1 - R_0)}\frac{\eta}{H}\right], \tag{31}$$

a result analogous to that of Airy for a single fluid (LAMB, 1932). This result shows that a crest moves faster than a trough provided the equilibrium depth of the lower fluid is less than one-half the total depth. On such an interface one would expect the lower fluid to thicken through a hydraulic jump as in a one-fluid system. If the upper fluid is deeper than the lower, however, a trough will move more quickly than a ridge and the resulting shock wave should take the form of a hydraulic drop. This suggests roughly why hydraulic drops occur in the experiments when the upper fluid becomes shallow. The value of the analysis has two severe limitations, however: First, the wave while finite, was considered small; second no account was taken of the shear between the two fluids arising in passage over the barrier. A full discussion of the initial value problem of a finite gravity wave in a two-fluid system will appear in a fortcoming paper. This independent approach yields results in agreement with equation (31) to the order of $\eta/H$.

## 5. Meteorological implications

It was pointed out in Part I that the eventual goal of the investigations of this series of papers was an understanding of small-scale atmospheric flow over mountains and hills. This has been sidetracked somewhat by the complexity of the general problem of statified fluids and it has become necessary, as in the present paper, to explore first extremely simple systems before attempting to simulate the vertical distribution of wind and density in the atmosphere, complex terrain structures, effect of the earth's rotation, etc. The outstanding questions of similarity may be listed as follows:

(a) What is the effect of neglecting the earth's rotation? This question has been discussed briefly by LONG (1953 a, 1953 b). Obviously rotation may be neglected if the horizontal dimensions of the hill or mountain are small enough. If we require that the Rossby number be large compared to one, for example, rotation may be neglected provided the horizontal dimension of a ridge in the direction of the current is 10 km or less. This appears to be too severe a requirement, however, as is apparent from the following considerations: If a two-fluid system flows over a ridge in a rotating system, the rotation will not be important in determining the regimes of motion and critical Froude numbers if the kinetic energy of the velocity component parallel to the range is small compared to the kinetic energy of the basic motion, i. e. if $v^2/u^2 < 1$. Considering the lower fluid only, at the crest $v$ will be of the order of $fA/h_0$, where $f$ is the Coriolis parameter, $A$ is the cross-sectional area of the ridge from base to crest, and $h_0$ is the height of the inversion at the upwind base of the ridge. Since, for the atmosphere, $u^2 = o(10^6 \text{ cm}^2 \text{ sec}^{-2})$ and $A = o\,(bL/4)$, where $b$ is the maximum height of the ridge and $L$ is the total horizontal length, we must have $L^2 < 16 \times 10^6 \times 10^8\ h_0^2/b^2\ \text{cm}^2$. If we take a fairly typical value, $h_0/b = 2$, for example, we have $L < 8 \times 10^7$, so that the ridge may be as wide as 100 km before the effect of rotation is appreciable upwind of the crest. If there is a tendency for supercritical flow on the lee, the lower layer may thin considerably and higher $v$-velocities generated. This will have an influence on the nature of lee waves or lee jumps rather than on the criteria for the various regimes of flow.

(b) What is the effect of the compressibility of air? Since the model employs incompressible liquids, a question arises as to the effect of compressibility. In a forthcoming study of this problem it appears that the atmosphere may be considered incompressible if we substitute potential temperature gradients ($\Delta\Theta/\Theta$) for density gradients and if the ratio $gb/c^2 < 1$

where $b$ is the height of the barrier and $c$ is the speed of sound. This requires $b < 10^6$, which is satisfied by mountains 1 km high or less. Again this is probably too severe a restriction. A computation employing the methods of section 4, but with two compressible fluids, shows very little effect of compressibility for barriers several times larger than this.

(c) To what extent do inversion surfaces in the atmosphere resemble the interfaces and free surface in the models? In most of the past investigations of flow over mountains it has been assumed that the atmosphere has a more or less uniform potential temperature gradient. In such a case no vertical height dimension, short of the total depth of the atmosphere, enters into the analysis. This is at complete variance with the consideration of this paper in which heights of inversions, particularly the tropopause, are used to define Froude numbers whose critical values are decisive for the occurrence of the laminar flows, jumps, and blocking. This important difference cannot be resolved at the present time, but future experiments should be able to provide a rather decisive answer. It is possible, for example, by the use of water and salt mixtures and/or layers of immiscible fluids, to reproduce faithfully typical potential temperature distributions in the atmosphere up to levels (perhaps 80,000 ft) where we may feel confident any effects of topography disappear.

In conclusion, let us assume for the purpose of discussion that the model of this paper is essentially similar to the atmosphere, and that the free surface corresponds to the tropopause and the interface to a tropospheric inversion surface. What suggestions regarding the flow of air over barriers are contained in the results of this paper? In the first place we may say that, unless the tropospheric inversion is quite high and the barrier quite small, the main phenomenon of interest is not lee waves of the classical type but rather shock waves resembling hydraulic jumps, surges or bores. This results from the fact that the pressure distribution in the model is closely hydrostatic except in the vicinity of the jump. The hydrostatic assumption should be at least as good in the atmosphere, so that the shock phenomenon should be typical of air flow over most mountains and hills, provided inversions are present. Lee waves of the classical type should be confined to hills with maximum heights considerably less than the height of the first inversion surface.

Lee jumps should form in the lee if the internal Froude number exceeds a certain value, depending on the height of the inversion and the height of the obstacle. For lower tropospheric inversions the critical value of $F_i$ is of the order of .20. In the atmosphere we usually have a shear of wind with height which is not reproduced in the experiment. If, however, we use the mean velocity of the tropospheric air, for the purposes of the Froude number, and take $(gH \bigtriangleup\Theta/\Theta)^{1/2} \cong 10^4$, jump phenomena require winds of the order of 20 m sec$^{-1}$. On the basis of this computation we would expect hydraulic jump phenomena to be a common feature of flow over the larger surface barriers. Such phenomena as absolutely supercritical flow and hydraulic drops probably do not occur since they require mean winds of the order of 50 m sec$^{-1}$.

It is of some interest to speculate about one possible effect of rotation on the flow of air over a range as large as the Appalachians. If, for example, a quasi-steady jump would form on the eastern side, by virtue of flow over the range with a tropospheric inversion, it is conceivable that the resulting convergence on the lower layer in the jump could lead to sizeable increases of relative vertical vorticity and a possible source of cyclogenesis. Such an effect would require that columns of air in the lower layer would lose to the ground by friction much of the relative negative vorticity generated by the vertical shrinking of the columns in passage over the mountains. It is suggestive that the Cape Hatteras region is a favorite site for cyclogenesis. An exploratory experiment with a barrier moved in a rotating system of two fluids, did not show any such effect. This should not be regarded as conclusive, however, because the frictional action between the barrier and the lower layer of fluid may be entirely different in the experiment and in the atmosphere.

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