Energy Transformations and Meridional Circulations associated with simple Baroclinic Waves in a two-level, Quasi-geostrophic Model[1]

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Abstract

The changes in geostrophic kinetic energy predicted by the "2 - ½ dimensional" quasi-geostrophic vorticity equations without friction are shown to be compatible with the mechanical energy equation. The second-order effects on a zonal current due to the presence of very simple unstable baroclinic waves are then analysed, using a two-level model of finite lateral width without friction or heat sources. In addition to the poleward transport of sensible heat and the creation of kinetic energy by these waves, it is shown that they are accompanied in this model by a weak meridional circulation. This circulation consists of an indirect cell in middle latitudes with direct cells to the north and south. The possible importance of this mechanism in providing appropriately distributed sources and sinks of relative zonal momentum (and therefore in prescribing the distribution with latitude of the surface zonal winds) is demonstrated with the aid of Widger's observations of the horizontal momentum transfer by eddies during January 1946. Finally it is shown that about 95 per cent of the perturbation energy in the unstable waves of this type comes from the "potential energy" of the basic current, the small remainder coming from the kinetic energy of that current.

1. Introduction

It is now quite generally recognized that the large-scale disturbances of extra-tropical latitudes play an important role in the maintenance of the "general circulation" of those latitudes. Their role as agents for the poleward transport of sensible heat was pointed out some time ago by DEFANT (1921), and a few years later, JEFFREYS (1926) suggested that they were also important in the latitudinal transport of angular momentum. These first considerations have been verified in recent years by the actual measurement of these transports using aerological charts and data [WIDGER (1949), WHITE (1951)].

Theoretical investigations of the dynamic and kinematic properties of these turbulence elements have also been carried out in recent years, using the technique of the linearized perturbation theory, where the disturbances are considered to be small perturbations superimposed on a zonal basic current. These theoretical studies may be divided into two principal types. The first, to which we may apply the adjective "baroclinic", has considered the basic current to be a function of height only, latitudinal variations in this basic current being neglected (CHARNEY, 1947). The second type, which we may call "baro-

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tropic", has neglected the vertical increase of the zonal current but has considered instead the effect of latitudinal variations in this quantity (KUO, 1949).

In both analyses, unstable (amplifying) and stable (damped) types of waves are mathematically possible. However, the kinematics of these theoretical motions are such that the disturbances one encounters on actual weather charts seem to be best described by a combination of the *amplifying baroclinic* wave and the *damped barotropic* wave. The pure amplifying baroclinic wave transports energy northward and generates kinetic energy (KUO, 1952). The pure damped barotropic wave does not change the total amount of kinetic energy, but the kinetic energy of the wave is instead organized into the kinetic energy of the basic zonal motion. This last process is accompanied by latitudinal momentum transports of the type described by Jeffreys (see also STARR, 1948).

The combination of the barotropic and baroclinic effects in a perturbation analysis has not yet been accomplished. Furthermore, the effects of non-adiabatic heat changes and friction have not been included in these theories. The mathematical difficulties involved are not trivial, and it may be that the analysis of the role the large-scale disturbances play may be best accomplished with the aid of high speed computing machines in the following manner. Using an appropriate system of prediction equations for the large-scale motions, including friction (surface skin friction is probably most important) and some simplified representation of the non-adiabatic effects (perhaps as a simple function of latitude and height), the machine could be instructed to forecast the development of the flow pattern with time, beginning, say, with some simple initial situation. One would then examine the forecast after some time to see what types of disturbances appeared and what kind of zonal wind distribution was created.

The present paper is written partly as a prelude to such an experiment (which the writer and his colleague Dr Jule Charney hope to carry out at the Institute for Advanced Study) and also to describe some interesting properties of the simple baroclinic waves with special reference to the importance of these properties for the general circulation in extra-tropical latitudes.

2. The quasi-geostrophic equations and boundary conditions

In investigating the role of baroclinic disturbances in the general circulation, we shall, for simplicity, assume that these motions are adequately described by the geostrophic equations for a "2 1/2 dimensional model" (CHARNEY and PHILLIPS, 1952). With some slight additional simplification, these may be written

$$D_1\,(f+\zeta_1)/Dt - p_0^{-1}\,2f\omega_2 = 0 \qquad (1)$$

$$D_3\,(f+\zeta_3)/Dt + p_0^{-1}\,2f\omega_2 = 0 \qquad (2)$$

$$p_0^{-1}\,2\,\omega_2 - f^{-2}\lambda^2 D_{1,3}\,(\varphi_1-\varphi_3)/Dt = 0 \qquad (3)$$

The subscripts 1, 2, and 3 refer to quantities measured at the 250-, 500-, and 750-mb levels, respectively. The operator D/Dt is defined by

$$D/Dt = \partial/\partial t + \boldsymbol{v}\cdot\nabla$$

where ∇ is the horizontal gradient operator on a constant pressure surface and $\boldsymbol{v}$ is the horizontal (geostrophic) velocity vector. ω_2 is equal to the value of dp/dt at the 500-mb level. φ is the geopotential and λ^2, assumed to be constant, is given by the formula

$$\lambda^2 = \{\Phi_1-\Phi_3\}^{-1} f^2[\Theta_2/(\Theta_1-\Theta_3)]$$

where Θ is potential temperature and $\{\Phi_1-\Phi_3\}$ represents a typical value of the geopotential difference between levels one and three. p_0, which we have taken as 1000 mb, is the (constant) approximate value of the pressure at the ground. We shall describe the motion in a cartesian coordinate system (x to the east, y to the north) so that the relative vorticity ζ is given by $\partial v/\partial x - \partial u/\partial y$. The coriolis parameter, f, is considered to be a constant everywhere except in the term Df/Dt, where we assume that $\partial f/\partial y = \beta =$ constant (ROSSBY, 1939).

We shall represent the extra-tropical atmosphere as being limited "latitudinally" at $y = \pm w$ by fixed vertical walls. In the x-direction we shall assume cyclic continuity, i.e. the motion at $x = L$ is identical with that at $x = 0$.

The solution of (1), (2), and (3) requires boundary conditions at $y = \pm w$ in addition to the cyclic conditions at $x = 0$ and L. The kinematic boundary condition at the walls requires that $v = 0$ at $y = \pm w$. Geostrophically this implies

$$\partial\varphi/\partial x = 0 \quad \text{at} \quad y = \pm w \tag{4}$$

Thus, at any one value of t, φ at the walls must not vary with x. This is not sufficient, however, to determine a unique solution to (1), (2), and (3) since this constant value of φ may vary with time. Another condition may be obtained by integrating the first equation of motion in the form

$$u_t + uu_x + vu_y - fv + \varphi_x = 0$$

with respect to x, with $v = 0$. (The subscripts here indicate differentiation.) We find that all terms disappear except u_t and we get, geostrophically,

$$\int_0^L (\partial^2\varphi/\partial y\, \partial t)\, dx = 0 \qquad y = \pm w \tag{5}$$

This, together with (4), is enough to determine uniquely the solution of (1), (2), and (3).[1]

3. The mechanical energy equation for geostrophic motions

The simplified system of quasi-geostrophic equations (1), (2), and (3) allows accelerations of the motion to occur in spite of the fact that according to the equations of motion, geostrophic balance corresponds to zero acceleration. This by now familiar paradox is of course explainable in that the geostrophic approximation is not introduced directly into the equations of motion but is introduced instead into the vorticity equation after the horizontal divergence (which cannot be measured geostrophically) has been eliminated (CHARNEY, 1948).

Since any change in the kinetic energy of horizontal motion must be accompanied by horizontal accelerations, a legitimate question may arise as to the extent to which the motions (and accelerations) predicted by the quasi-geostrophic equations satisfy the energy equation. We shall now demonstrate that (1), (2), and (3) do satisfy the mechanical energy equation in a certain sense.

We first derive a form of the mechanical energy equation suitable to our model, following to some extent the technique introduced by STARR (1951 b). Multiplication of the horizontal equations of motion in the x, y, p, t-coordinate system by u and v first yields

$$q_t + \boldsymbol{v} \cdot \nabla q + \omega q_p + \boldsymbol{v} \cdot \nabla\varphi = 0$$

where q is the horizontal kinetic energy per unit mass, $1/2\,(u^2 + v^2)$. It would clearly be meaningless to introduce the geostrophic approximation here, since then the last term would vanish. We therefore postpone this step and utilize the continuity equation in this coordinate system (ELIASSEN, 1949)

$$\nabla \cdot \boldsymbol{v} + \partial\omega/\partial p = 0 \tag{6}$$

to derive another form of the equation:

$$q_t + \nabla \cdot (q + \varphi)\boldsymbol{v} + (\omega q)_p + \varphi\omega_p = 0 \tag{7}$$

(7) may now be specialized to our model by evaluating it at levels one and three and introducing the approximations (CHARNEY and PHILLIPS, 1952)

$$\omega_{p1} = -\omega_{p3} = p_0^{-1}\, 2\,\omega_2, \text{ and}$$
$$(\omega q)_{p1} = -(\omega q)_{p3} \tag{8}$$

Adding the two equations which result, and integrating over the region $-w \leqslant y \leqslant w$, $0 \leqslant x \leqslant L$, we get finally

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\int\!\!\int (q_1 + q_3)\, dxdy + \\ &+ \frac{2}{p_0}\int\!\!\int (\varphi_1 - \varphi_3)\omega_2 dxdy = 0 \end{aligned}\right\} \tag{9}$$

This is the form of the mechanical energy equation which is applicable to our 2 1/2-dimensional model. It states that the total kinetic energy $q_1 + q_3$ may change as a result of a net correlation between the thickness $(\varphi_1 - \varphi_3)$ and ω_2.

The geostrophic assumption *has not been* used in deriving (9). We now propose to demonstrate that (9) is satisfied if both ω_2 and the kinetic energy q are determined and

[1] Except for a constant of integration determined by the average value of $\partial(\varphi_1 + \varphi_3)/\partial t$ over the entire region. The value of this integration constant plays absolutely no part in the conclusions we shall come to in the remainder of this paper; for convenience, therefore, it has been set equal to zero.

measured *geostrophically* by (1), (2), (3) and the boundary conditions (4) and (5).

(1) and (2) may be written

$$f^{-1}\nabla^2\varphi_{1t}+\boldsymbol{v}_1\cdot\nabla(f+\zeta_1)-p_0^{-1}\,2f\omega_2=0$$

$$f^{-1}\nabla^2\varphi_{3t}+\boldsymbol{v}_3\cdot\nabla(f+\zeta_3)+p_0^{-1}\,2f\omega_2=0$$

where we have introduced the geostrophic approximation for $\partial\zeta/\partial t$. Multiplying these two equations respectively by $-\varphi_1$, and $-\varphi_3$, using the identity $\nabla\cdot\varphi\nabla\varphi_t=\varphi\nabla^2\varphi_t+\nabla\varphi\cdot\cdot\nabla\varphi_t$, and then adding, we obtain

$$\begin{aligned}&f^{-1}(\nabla\varphi_1\cdot\nabla\varphi_{1t}+\nabla\varphi_3\cdot\nabla\varphi_{3t})-\\&-f^{-1}\nabla\cdot(\varphi_1\nabla\varphi_{1t}+\varphi_3\nabla\varphi_{3t})-\\&-[\varphi_1\boldsymbol{v}_1\cdot\nabla(f+\zeta_1)+\varphi_3\boldsymbol{v}_3\cdot\nabla(f+\zeta_3)]+\\&+p_0^{-1}\,2f(\varphi_1-\varphi_3)\,\omega_2=0\end{aligned}$$

$\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ in this equation are of course geostrophic. It is then simple to show that when this equation is integrated over the region $-w\leqslant y\leqslant w$, $0\leqslant x\leqslant L$, the boundary conditions (4) and (5) make the contribution from the second and third terms vanish (remember that f as a coefficient is constant). We are then left with

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\int\!\!\int\frac{1}{2f}[(\nabla\varphi_1)^2+(\nabla\varphi_3)^2]\,dxdy+\\&+\frac{2f}{p_0}\int\!\!\int(\varphi_1-\varphi_3)\,\omega_2 dxdy=0\end{aligned}\right\}\quad(10)$$

which, after division by f, is the geostrophic equivalent of (9).

The appearance of only the *geostrophic* kinetic energy in (10), and not the total horizontal kinetic energy is not too surprising. It is, in fact, quite analogous to the case of simple hydrostatic motion, where the energy equation does not include the kinetic energy of the vertical motion. We may remark finally that since (10) was derived without the use of (3), our demonstration of the compatibility of the mechanical energy equation (9) and the quasi-geostrophic equations does not involve any assumption of adiabatic motion.

4. The perturbation equations

The theoretical treatment to date of the non-stationary disturbances of extra-tropical latitudes has been carried out under the assumption that the motion is both adiabatic and frictionless. The linearized perturbation theories then demonstrate the possibility of unstable waves which grow exponentially with time. These unstable waves, especially those which are most unstable (EADY, 1949), are presumably to be identified with the disturbances we actually see on upper air charts. Good evidence for this is to be found in the fact that the pressure, temperature, and vertical motion fields in many actual developing disturbances are quite similar to those in the theoretical perturbations (KUO, 1952).

In the remainder of this paper we will study the kinematic features of the unstable baroclinic waves (although in a very simplified form), and use these results to compute the effects of these motions on the basic zonal current. This will be done by using the perturbation solutions to evaluate the non-linear "stress" terms which appear in the zonally averaged equations for the rate of change of the mean current. We will then attempt to show that the effects induced in this manner are such as to counteract the neglected effects of (surface) friction and non-adiabatic processes on the mean motion. The reasoning therefore embodies the tacit assumption that the overall kinematics of the *disturbances* are not influenced significantly by the neglected physical processes and the mathematical simplifications introduced by the perturbation theory.

The most detailed analysis to date of the kinematics of the unstable baroclinic wave has been made by KUO (1952), who has shown that the waves transfer sensible heat northward, create kinetic energy, and may be important in the vertical transport of zonal momentum. However, Kuo's analysis has the limitation that his calculations are based on the assumption that the perturbations are completely independent of the y-coordinate. Since the latitudinal extent of the actual disturbances is of the same order of magnitude as the x-wavelength of the most unstable wave in this theory, it seems reasonable that the additional kinematic effects introduced by lateral limitations on the motion may not be insignificant.

We are led thus to the study of the properties of small perturbations superimposed on a zonal flow limited at $y=\pm w$ by fixed walls.

For simplicity we will again use the "2 - 1/2 dimensional" equations (1), (2), and (3) as a crude approximation to the more realistic equations employed by CHARNEY (1947) and KUO (1952).

The basic current will be defined by the process of averaging with respect to x; such averaged quantities will be denoted by a bar, $^-$, or by a capital letter. Deviations from this zonal average will be indicated by a prime superscript, $'$. Using this convention, equations (1), (2), and (3) may now be combined into the following two equations:

$$\left.\begin{aligned}&\left[f\frac{\partial}{\partial t}-(\Phi_{1\,y}+\varphi'_{1\,y})\frac{\partial}{\partial x}+\varphi'_{1\,x}\frac{\partial}{\partial y}\right]\\&[f\beta y+\Phi_{1\,yy}+(\varphi'_{1\,xx}+\varphi'_{1\,yy})-\lambda^2(\Phi_1-\\&\qquad-\Phi_3+\varphi'_1-\varphi'_3)]=0\\&\left[f\frac{\partial}{\partial t}-(\Phi_{3\,y}+\varphi'_{3\,y})\frac{\partial}{\partial x}+\varphi'_{3\,x}\frac{\partial}{\partial y}\right]\\&[f\beta y+\Phi_{3\,yy}+(\varphi'_{1\,xx}+\varphi'_{3\,yy})+\lambda^2(\Phi_1-\Phi_3+\\&\qquad+\varphi'_1-\varphi'_3)]=0\end{aligned}\right\}\quad(11)$$

where f, β, and λ^2 are constants. The equations for the rate of change of the mean motion are obtained by averaging these with respect to x:

$$\left.\begin{aligned}&\Phi_{1\,yyt}-\lambda^2(\Phi_{1\,t}-\Phi_{3\,t})=\\&=\overline{f^{-1}\left[\varphi'_{1\,y}\frac{\partial}{\partial x}-\varphi'_{1\,x}\frac{\partial}{\partial y}\right]}\\&\overline{[(\varphi'_{1\,xx}+\varphi'_{1\,yy})-\lambda^2(\varphi'_1-\varphi'_3)]}\cdot\\&\Phi_{3\,yyt}+\lambda^2(\Phi_{1\,t}-\Phi_{3\,t})=\\&=\overline{f^{-1}\left[\varphi'_{3\,y}\frac{\partial}{\partial x}-\varphi'_{3\,x}\frac{\partial}{\partial y}\right]}\cdot\\&\overline{[(\varphi'_{3\,xx}+\varphi'_{3\,yy})+\lambda^2(\varphi'_1-\varphi'_3)]}\end{aligned}\right\}\quad(12)$$

Thus the time rate of change of the basic current, as described by $\Phi_{1\,t}$ and $\Phi_{3\,t}$, is influenced by "stress" terms which arise from non-linear interactions of the deviations from zonal flow. In the perturbation theory the deviations φ'_1 and φ'_3 are considered to be small, of the first order. (12), in combination with (5) and (4), then states that $\Phi_{1\,t}$ and $\Phi_{3\,t}$ are at most of the second order.

We may now separate out the first order terms in (11), making the additional assumption that Φ_1 and Φ_3 are linear functions of y, i.e. we assume that the basic current velocities $U_1=-f^{-1}\Phi_{1\,y}$ and $U_3=-f^{-1}\Phi_{3\,y}$ are initially independent of y. (Note, however, that we do not place this restriction on the second order quantities $U_{1\,t}$ and $U_{3\,t}$).

$$\left.\begin{aligned}&\left[\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}\right]\\&[\varphi'_{1\,xx}+\varphi'_{1\,yy}-\lambda^2(\varphi'_1-\varphi'_3)]+\\&\quad+[\beta+\lambda^2(U_1-U_3)]\,\varphi'_{1\,x}=0\\&\left[\frac{\partial}{\partial t}+U_3\frac{\partial}{\partial x}\right]\\&[\varphi'_{3\,xx}+\varphi'_{3\,yy}+\lambda^2(\varphi'_1-\varphi'_3)]+\\&\quad+[\beta-\lambda^2(U_1-U_3)]\,\varphi'_{3\,x}=0\end{aligned}\right\}\quad(13)$$

5. Kinematics of the unstable perturbations

For convenience we define

$$\left.\begin{aligned}V&\equiv 1/2\,(U_1-U_3),\\B&\equiv\lambda^{-2}\beta,\\\mu&\equiv\pi/2\,w,\\k&\equiv 2\,\pi/L,\ \text{and}\\\alpha&\equiv\lambda^{-2}(k^2+\mu^2).\end{aligned}\right\}\quad(14)$$

(13) will then be satisfied by wave solutions of the form $\varphi'\sim\cos\mu y\exp ik(x-ct)$. [The $\cos\mu y$ term is present to satisfy the boundary condition (4).] The phase speed c must satisfy the equation

$$\left.\begin{aligned}&c={}^1/_2\,(U_1+U_3)+[\alpha\,(2+\alpha)]^{-1}\\&[-(1+\alpha)\,B\pm\sqrt{B^2-\alpha^2(4-\alpha^2)\,V^2}]\end{aligned}\right\}\quad(15)$$

We see that for a given value of α there are two possible values of c, and that when $\alpha<2$, it is possible that c may be complex. The exact condition for complex c is

$$V^2\alpha^2(4-\alpha^2)>B^2\qquad(16)$$

A complex value of c means that the perturbation either amplifies or damps exponentially with time. We are interested in the amplifying type, which we are taking as representative of the disturbances in the real extra-tropical atmosphere. These are obtained by taking the plus sign in (15) when (16) is satisfied. For this case it is easily shown that the pertur-

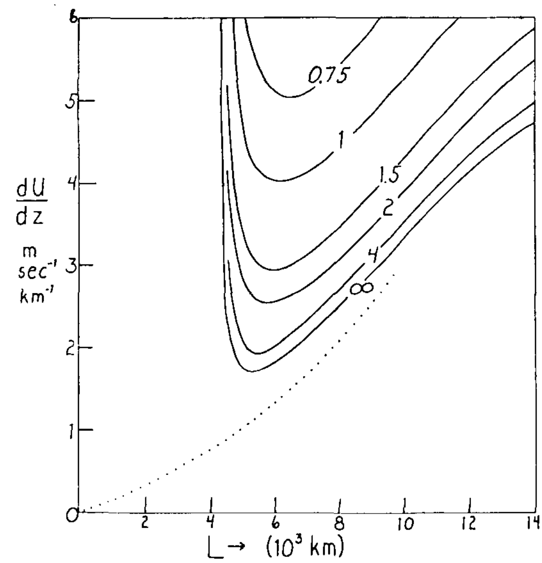

Fig. 1. The time in days required for an unstable wave in the two-level model to double its amplitude, given as a function of the vertical wind shear in the basic current and the wavelength. The dotted line represents the curve $\nu_i = 0$ for the perturbation analyses of Charney and Kuo.

bations φ_1' and φ_3' are given by the following formulae:

$$\left.\begin{aligned}\varphi_3' &= D e^{\nu_i t} \cos \mu y \cos k\,(x - c_r t) \\ \varphi_1' &= \sigma D e^{\nu_i t} \cos \mu y \cos\left[k\,(x - c_r t) + \psi\right]\end{aligned}\right\} \quad (17)$$

The constant D, which determines the absolute amplitude of the disturbance, is completely arbitrary because of the linearity of (13). However, the relative amplitude σ is determined, as is the phase angle ψ:

$$\left.\begin{aligned}\sigma &= \left[(2\alpha V + B)/(2\alpha V - B)\right]^{1/2} \\ \tan\psi &= (2 + \alpha)\,\nu_i/(k\alpha V)\end{aligned}\right\} \quad (18)$$

where $\nu_i = kc_i$ is the "frequency of flight", given by

$$\nu_i = k\left[\alpha^2(4 - \alpha^2)\,V^2 - B^2\right]^{1/2}\left[\alpha(2 + \alpha)\right]^{-1} \quad (19)$$

Thus in the amplifying wave ($\nu_i > 0$) the φ_1' wave is larger in amplitude than the wave in φ_3' and lags behind it (for $\nu > 0$). The real phase speed is given by c_r:

$$c_r = 1/2\,(U_1 + U_3) - \left[\alpha\,(2 + \alpha)\right]^{-1}(1 + \alpha)\,B$$

To obtain an idea of the magnitude of this instability, the time $\tau = \nu_i^{-1} \ln 2$ required for a given perturbation to double in size is shown in fig. 1 as a function of the wind shear and the wave length L. Numerical values of the other parameters used are as follows:

$$\left.\begin{aligned}\{\Phi_1 - \Phi_3)\} &= 9.81 \times 8.16 \times 10^3\ \mathrm{m^2\,sec^{-2}} \\ \Theta_2/(\Theta_1 - \Theta_3) &= 316/36 \sim 8.8 \\ \mu &= 3/R_e \\ dU/dz &= 2\,gV\,\{\Phi_1 - \Phi_3\}^{-1}\end{aligned}\right\} \quad (20)$$

R_e is the radius of the earth, so the distance $2\,w$ corresponds to sixty degrees of latitude. The constant values of f and β were evaluated at 45° latitude.

For reference it may be of interest to note that the mean value of dU/dz on the winter meridional cross sections published by HESS (1948) and by PALMÉN and NEWTON (1948) is about 2.3 m sec⁻¹ km⁻¹, the maximum value of dU/dz on Hess's section being about 4.3 m sec⁻¹ km⁻¹ at 35° N. This corresponds to a time of between one to three days for a doubling of the size of the disturbance.

The wave length of maximum instability seems to be about 6,000 km. This corresponds to a wave number of about five at 45 degrees latitude.

Although the stability criterion (16) has been derived on the assumption that V is independent of latitude, it is interesting to examine the summer and winter cross sections of HESS (1948) to see at what latitudes (16) is satisfied. The most favorable conditions are obtained by choosing L so that $\alpha^2 = 2$. Replacing α^2 by 2 in (16) we find that the criterion for instability is satisfied when

$$\frac{\cos\varphi}{\sin^2\varphi} < \frac{4\,\Omega\,R_e \Theta_2}{\{\Phi_1 - \Phi_3\}\,(\Theta_1 - \Theta_3)}\,|V| \sim 0.1\,|U_1 - U_3|$$

where $U_1 - U_3$ is in m sec⁻¹ and φ is latitude. Ω is the angular velocity of the earth and R_e the radius of the earth. This relationship is shown in fig. 2, and may perhaps be taken as a partial explanation of the seasonal changes in cyclone activity, both in intensity and latitude.

A more exact solution of the quasi-geostrophic perturbation equations has been made by CHARNEY (1947) and KUO (1952). Their results differ from those above primarily in that sufficiently short waves may always be unstable. In fig. 1 this would mean that the region of instability would be delineated by the dotted line. However, Kuo has shown that the values of ν_i in the added region of instability are very small. Since in this discussion we shall be most interested in the waves with large ν_i, it seems reasonable to accept the results we have derived here as a reasonably good first approximation to the kinematics of the baroclinic waves.

6. Second order changes in the basic current

Equation (12) and the boundary condition (5) tell how the basic current will change with time. The perturbation analysis has given us expressions for the perturbations φ_1' and φ_3' which are correct to the first order. We are therefore in a position to evaluate the right side of (12) correct to the second order, and thereby obtain Φ_{1t} and Φ_{3t} correct to the second order.

In using (17) to evaluate the right side of (12), we find that the terms involving $\varphi_{xx}' + \varphi_{yy}'$ drop out, i.e. the perturbation advection of perturbation vorticity is zero. This is a consequence of our assumption that U_1 and U_3 are (initially) independent of y. The other terms, representing the advection of perturbation thickness by the perturbation velocities, do not drop out. We find then that (12) may be written

$$\left.\begin{aligned} \Phi_{1\,\xi\xi t} - \gamma\,(\Phi_{1t} - \Phi_{3t}) &= -1/2\,A \sin 2\,\xi \\ \Phi_{3\,\xi\xi t} + \gamma\,(\Phi_{1t} - \Phi_{3t}) &= 1/2\,A \sin 2\,\xi \end{aligned}\right\} \quad (21)$$

where

$$\left.\begin{aligned} A &= [\lambda^2 D^2 (\exp 2\,\nu_i t)\,\alpha\,(2+\alpha)\nu_i] \\ &\quad [f\mu\,(2\,\alpha V - B)]^{-1} \\ \gamma &= \mu^{-2}\lambda^2 \end{aligned}\right\} \quad (22)$$

and we have introduced the new dimensionless variable

$$\xi \equiv \mu y = (\pi/2)\,(y/w)$$

The solution of (21) satisfying the lateral boundary condition (5) at $\xi = \pm\pi/2$ is

$$\left.\begin{aligned} \Phi_{1t} = -\Phi_{3t} &= \frac{A}{2\,(2+\gamma)} \\ &\left[\frac{\sinh\sqrt{2\,\gamma}\,\xi}{\sqrt{2\,\gamma}\cosh\sqrt{2\,\gamma}\,\pi/2} + \frac{1}{2}\sin 2\,\xi\right] \end{aligned}\right\} \quad (23)$$

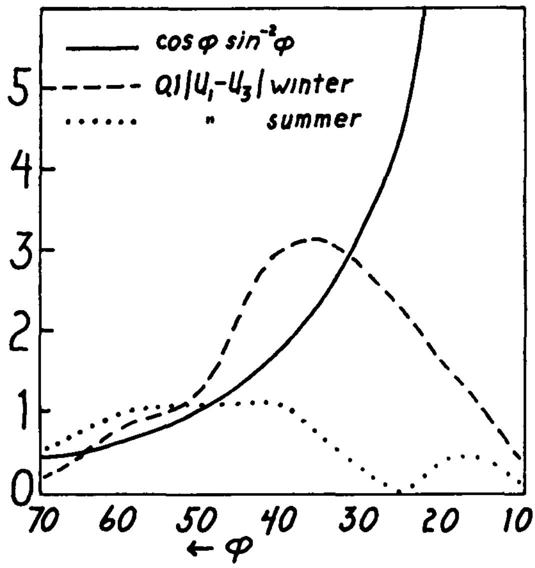

Fig. 2. The satisfaction of the stability criterion for baroclinic waves in the two-level model as a function of latitude and season. The curves of 0.1 $(U_1 - U_3)$ (in m sec^{-1}) are taken from the cross sections of Hess.

Fig. 3 contains a picture of these solutions. In the northern half of the region $\Phi_{1t} - \Phi_{3t}$ is positive, while in the southern half it is negative. This corresponds to an average warming up of the northern half and a cooling of the southern half of the region. Therefore, as has been shown previously by Kuo (1952), one effect of the unstable waves is to counteract the latitudinal variation in heating and cooling due to non-adiabatic causes.

The size of the disturbance necessary to have the correct magnitude of this effect is readily estimated. From the hydrostatic equa-

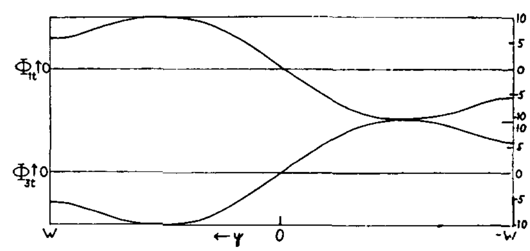

Fig. 3. Graph of the solutions Φ_{1t} and Φ_{3t} of the averaged vorticity equations. The scale on the right side shows the corresponding value of the mean height tendency in m day^{-1} when T_t^* is taken as 0.5 °C day^{-1}.

tion we first obtain the approximate relation $\partial \bar{T}_2/\partial t \sim R^{-1}(\Phi_{1\,t} - \Phi_{3\,t})$ where $\bar{T}_2$ is the 500-mb temperature and R is the gas constant for dry air. Defining $T_t^* \equiv (2/\pi)\int_0^{\pi/2} \bar{T}_{2\,t}\, d\xi$ as the average value of the warming up of the northern half of the region by the disturbances, we find

$$T_t^* = \frac{A\left(1+\gamma - \operatorname{sech}\sqrt{2\gamma}\,\pi/2\right)}{\pi R \gamma\,(2+\gamma)}$$

A reasonable estimate of the order of magnitude T_t^* must have in order to balance the yearly average cooling over the northern half of the northern hemisphere is about 0.5 °C per day (see BJERKNES, 1933). With $\mu = 3/R_e$ and $\lambda^2 = 1.17 \times 10^{-12}$ m^{-2} [see (20) and the definition of λ^2 following (3)], we find that $\gamma \sim 5.3$, and that

$$T_t^*\,(\text{deg day}^{-1}) \sim 15.4\, A\ (\text{m}^2\,\text{sec}^{-3}) \quad (24)$$

Defining v_0 as the maximum value of v at level three (750 mb) we may write

$$v_0 = [f^{-1}\varphi'_{3\,x}]_{\max} = f^{-1} k\, D e^{\nu_i t}$$

Introducing this into (22) we then obtain

$$v_0^2 = A k^2 \mu\,(2\alpha V - B)\,[f\lambda^2\alpha\,(2+\alpha)\,\nu_i]^{-1}$$

Taking A as equal to $(30.8)^{-1}$ m^2 sec^{-3} and the following values in addition to those in (20),

$$k = 2\pi\,(6\times 10^6)^{-1}\ \text{m}^{-1}$$
$$V \sim B \sim 14\ \text{m sec}^{-1}$$
$$\nu_i = (\ln 2/1.245)\ \text{days}^{-1}$$

we find that v_0 is about 10 m sec^{-1}. In other words, a disturbance of a size such that v_0 is 10 m sec^{-1} will cause a second-order effect on the mean flow of the correct order of magnitude to balance the average latitudinal variation in heating and cooling. This is a rather small value of v_0, but we have assumed that the perturbation is of the most efficient type (i.e. that with the maximum value of ν_i). Furthermore, the observed eddy motion in the atmosphere is partly due to features such as longitudinal irregularities in orography which probably do not contribute significantly to the latitudinal transport of energy.

7. The meridional circulation

In this section we shall see that the presence of the baroclinic waves implies the existence of a small meridional circulation. Our assumption of quasi-geostrophic motion does not allow us to use the second equation of motion to discuss the creation of a meridional circulation and we must therefore demonstrate its existence in another manner. Averaging (1), we first obtain

$$\bar{\omega}_2 = (2f^2)^{-1} p_0 \Phi_{1\,yyt} = (2f^2)^{-1} p_0 \mu^2 \Phi_{1\,\xi\xi t}$$

which, when we use (23), yields the following formula for $\bar{\omega}_2$:

$$\left.\begin{array}{c} \bar{\omega}_2 = \dfrac{p_0 \mu^2 A}{4 f^2\,(2+\gamma)} \\ \left[\dfrac{\sqrt{2\gamma}\,\sinh\sqrt{2\gamma}\,\xi}{\cosh\sqrt{2\gamma}\,\pi/2} - 2\sin 2\xi\right] \end{array}\right\} \quad (25)$$

We now integrate the continuity equation (6) with respect to x, define

$$\bar{v}_1 \equiv p_0^{-1}\, 2\int_0^{p_0/2} \bar{v}\,dp$$

$$\bar{v}_3 \equiv p_0^{-1}\, 2\int_{p_0/2}^{p_0} \bar{v}\,dp$$

and, as we have already done in deriving (1) and (2), assume that ω vanishes at $p = 0$ and $p = p_0$. $\bar{v}_1$ and $\bar{v}_3$ are then related to $\bar{\omega}_2$ by the formulae

$$\frac{\partial \bar{v}_1}{\partial \xi} = -\frac{\partial \bar{v}_3}{\partial \xi} = -\frac{2\,\bar{\omega}_2}{\mu p_0}$$

These may now be integrated with respect to ξ from $\xi = -\pi/2$ to $\xi = \xi$, using the boundary condition that $\bar{v}_1$ and $\bar{v}_3$ are both equal to zero at $\xi = -\pi/2$.

$$\left.\begin{array}{c} -\bar{v}_1 = \bar{v}_3 = \dfrac{\mu A}{2 f^2\,(2+\gamma)} \\ \left[\dfrac{\cosh\sqrt{2\gamma}\,\xi}{\cosh\sqrt{2\gamma}\,\pi/2} + \cos 2\xi\right] \end{array}\right\} \quad (26)$$

(Note that the boundary condition that $\bar{v}_1$ and $\bar{v}_3$ also vanish at $\xi = \pi/2$ is satisfied automatically.)

The meridional circulation given by (25) and (26) is shown in fig. 4, with A taken to correspond to $T_t^* = 0.5$ °C day^{-1}. It consists of three "cells", a "direct" cell at the northern and southern ends, and a larger "indirect" cell in the central part of the region. It is very weak and therefore questionable whether it could be detected with the present observational network.

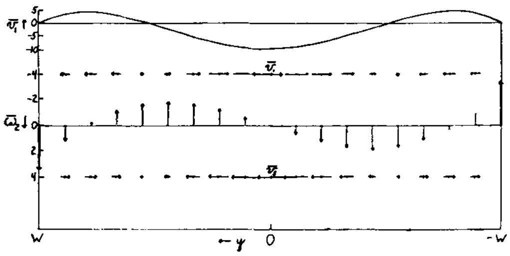

Fig. 4. The meridional circulation due to the baroclinic waves. The arrows indicate the direction and intensity of the flow. Values of $\overline{\omega}_2$ and $\overline{v}_1$ when T_t^* is taken as 0.5 °C day^{-1} are indicated by the scales on the left side; $\overline{\omega}_2$ is in units of mb day^{-1} and $\overline{v}_1$ (top line) is in units of cm sec^{-1}.

The existence of this meridional circulation was not incorporated into our equations (1), (2), and (3). It if had been, the meaned equations (12) would have included a term $fB\overline{v}$. Having obtained an explicit expression for $\overline{v}$, it is now possible to justify its neglect in (12). If we compute the ratio of the mean values over the interval $-\pi/2 \leq \xi \leq \pi/2$ of the square of the neglected term $f\beta\overline{v}$ with the square of the retained term $\Phi_{1\,yyt}$ we obtain a value of about .03. This crude comparison demonstrates that the presence of the mean circulation (25) and (26) is relatively unimportant in the solution of the zonally averaged vorticity equations (12).

The latitudinal transport of energy in our model, as reflected in the distribution of $\Phi_{1\,t} - \Phi_{3\,t}$ [reference should here be made to (31)], can be shown to be brought about by both a horizontal eddy-transport and by the meridional circulation (25), (26). [A discussion of the manner in which this transport may be broken up into contributions of these two types has been given by STARR (1951 b).] The transport of energy by the meridional circulation is always in the same direction as $\overline{v}_1$, and therefore gives a poleward transport in the northern and southern portions of the region and an equatorward transport in the central portion. The eddy transport on the other hand, is always poleward, and in the central part is much larger than the contribution from the meridional circulation.

Some measurements of the actual eddy transport of sensible heat across latitude circles have been made by WHITE (1951). He has also summarized the amounts of energy which must be transported across latitude circles in order to counteract the effects of radiation. His data suggest that the observed eddy flux is slightly too large in middle latitudes and perhaps too small in higher and lower latitudes to account for the required transports. Although the data are perhaps not representative enough to place too much reliance on this last remark, it is interesting to note that if the discrepancies were to be made up by a meridional circulation, the meridional circulation would be similar to that in fig. 4.

Since most of the heat transport in regions removed from the walls is accomplished by the eddies, it is not surprising that fig. 4 is vastly different from that hypothetical meridional circulation which would provide *all* of the required transport of energy. An example of such a circulation has been computed by BJERKNES et al. (1933), and, as one would expect, it is much more intense than that in fig. 4.

8. Changes in zonal momentum

The second-order changes in Φ_1 and Φ_3 given by (23) show that the baroclinic waves are modifying the distribution of mean zonal momentum. Differentiating (23) with respect to y, we see that the changes in the basic current velocities are given by

$$\left.\begin{aligned} U_{3\,t} = -U_{1\,t} &= \frac{A\pi}{2f(2+\gamma)} \\ &\left[\frac{\cosh\sqrt{2\gamma}\,\xi}{\cosh\sqrt{2\gamma}\,\pi/2} + \cos 2\,\xi\right] \end{aligned}\right\} \quad (27)$$

The distribution of $U_{3\,t}$ when A is taken as $(15.4)^{-1}$ m^2 sec^{-3} (corresponding to $T_t^* = 1$ °C day^{-1}) is shown in fig. 5.

Before discussing what implication this might have for the general circulation, we note the following two facts about (27):

1. Comparing (27) with (26), we see that
$$U_{1t} - f\bar{v}_1 = 0, \text{ and } U_{3t} - f\bar{v}_3 = 0 \quad (28)$$
Therefore the velocity changes (27) in the basic current are to be explained by the presence of the "implicit" meridional circulation (25), (26).[1]
2. Since $U_{3t} = -U_{1t}$, there is no change in the average velocity $1/2\,(U_1 + U_3)$ at any value of ξ. Therefore our perturbations do not include any mechanism for a net transport of momentum across latitude circles as studied by KUO (1949).

Attempts to explain the existence of the subtropical easterlies, middle-latitude westerlies, and polar easterlies in the surface zonal wind profile have long fascinated meteorologists. In general, all such attempts have begun with the familiar model of HADLEY (1735) for the trade wind zone and have then attempted to extend this type of reasoning to higher latitudes, bringing in various types of frictional effects as needed in order to "explain" the successive appearance of the westerlies and polar easterlies at the surface.

In 1948 STARR drew attention again to the suggestion of JEFFREYS (1926) that the angular momentum budget of different latitude belts might be greatly influenced by asymmetries in the large-scale horizontal flow patterns. These asymmetries could give non-vanishing values of ϱuv (ϱ = density) when integrated around a latitude circle and from the ground up to the top of the atmosphere, and thereby give a net flux of momentum across that latitude. The several synoptic investigations of this process which have since been made have shown that this transport does exist. Furthermore, the net transport of momentum into the various latitude belts by this process accounts rather well for the loss or gain in momentum which each of the latitude belts experiences through the effects of surface friction and the torque due to the pressure differential across large mountain ranges. [A summary of these results and a general discussion of their interpretation has been made by STARR (1951 a).] However, as soon as one examines the momentum budget of a typical latitude belt in more detail, it becomes apparent that other types of momentum transfer must exist in addition to the horizontal transport by the correlations in u and v. Let us consider for the moment a latitude belt in middle latitudes. This ring is losing angular momentum at the surface of the earth but gains an approximately equal amount through the convergence of the horizontal eddy transport. But the latter effect is small near the surface and reaches its maximum value at the level of the tropopause (STARR, 1951 a), so that we are faced with a surplus of angular momentum in the upper layers of the belt and a deficit in the lower layers. The situation in latitudes with surface easterlies is reversed; there we have an excess of momentum near the surface and a deficit at higher altitudes.

As can easily be seen by writing the angular momentum equation for a ring-shaped region limited by two latitude circles and two values of the height, there are three principal mechanisms available by which the angular momentum may be redistributed intramurally within a complete latitude belt (from the surface to the top of the atmosphere) so that the angular momentum budget for each layer in the latitude belt is also balanced:

1. A vertical transport by small-scale turbulence, for example, that brought about by cumulus activity. This process can certainly lead to a downward transport, which is what is needed in the region of the surface westerlies, but it is difficult to imagine that this process will provide an upward transport in the regions with surface easterlies.
2. A vertical transport by large-scale turbulence due to correlations in the zonal and vertical velocity components when measured on a synoptic scale. Momentum transport by this process need not be in the direction of decreasing momentum, and conceivably could be directed upwards in low and high latitudes and downwards in middle latitudes. Some measurements of this in middle latitudes have been made by WHITE and COOLEY (1952), and their results suggest that some of the downward transport in middle latitudes may be by this process.

[1] The results derived here are much in the spirit of those derived by ELIASSEN (1952); we have merely emphasized the meridional circulation brought about by the eddy heat transport, while Eliassen has analysed that due to axially symmetric non-adiabatic heat and cold sources.

3. A meridional circulation consisting of a net poleward motion in layers where there is a deficit of momentum (e.g. in the lower levels of middle latitudes) and an equatorward motion in the layers where there is an excess of momentum. PALMÉN and ALAKA have used this mechanism with some success in discussing the details of the momentum budget in low latitudes (1952).

The absence of correlations in u and v in our solution (due to the fact that U_1 and U_3 were assumed to be initially independent of latitude) does not allow us to say anything theoretically about the *total* transport of momentum into a given latitude belt. Furthermore, the crudeness of our two-layer model will not allow us to use it to study the vertical redistribution of momentum in great detail. However, it is possible to apply our results to the atmospheric layer below the 500-mb level, i.e. the lower half of the atmosphere. We shall see that the weak implicit meridional circulation (25), (26) (which in our theory is to be considered as a *result* of the presence of the baroclinic waves) provides local sources and sinks of relative momentum of the correct order of magnitude to help balance the local excesses and deficits of momentum at different altitudes which result from the observed horizontal eddy transport and exchange of momentum with the earth.

WIDGER (1949) has computed the geostrophic eddy flux of angular momentum per unit height during the month of January 1946 at the surface, 700-mb, and 500-mb levels. His observations also include the observed change in momentum during the month together with estimates of the loss (or gain) of momentum due to skin friction and the mountain effect (WHITE, 1949). Let us reckon H, the horizontal momentum transport by the uv correlations during January 1946 from the surface to the 500-mb level, as positive when directed northward. Let $\triangle$ be the change during January 1946 in the angular momentum of the volume limited by two latitude circles, the surface of the earth, and the 500-mb surface. Finally, we shall count E, the total amount of momentum gained from the surface of the earth by a latitude belt during the month, as positive when it represents a gain for the atmosphere. For each latitude belt we may then summarize the angular momentum requirements for the layer surface – 500 mb in the equation

$$\triangle = H_S - H_N + E + Q \qquad (29)$$

where the subscripts S and N refer to values at the southern and northern latitudes of the belt. The quantity Q then represents an additional "source" of momentum due to some or all of the three mechanisms described above.

Values of $H_S - H_N$, $\triangle$, E, and Q [computed from (29)] have been taken from Widger's data and are entered in Table I.[1] The Q values have then been converted into a value of $\partial U/\partial t$ by dividing them by the approximate mass of each ring and its mean distance from the axis of the earth. These values of $\partial U/\partial t$ are entered in the last column of Table I and may be thought of as the net relative acceleration required from the three processes listed above if the momentum budgets for these rings of air are to be satisfied as shown by (29).

Table I. Angular momentum budget for the layer 1,000—509 mb for the month of January 1946. [After WIDGER (1949)].

Lat. Belt	$H_S - H_N$ (in units of 10^{29} g cm² sec⁻¹)	E	$\triangle$	Q	$\partial U/\partial t$ (m sec^{-1} day^{-1})
15—20	— 309	4770	2	— 4459	— 2.2
20—25	— 622	2470	51	— 1797	— 1.0
25—35	— 747	1832	272	— 813	— 0.2
35—45	491	— 4495	100	4104	1.6
45—55	1119	— 3187	— 573	1495	0.8
55—65	200	214	— 131	— 546	— 0.5
65—75	156	280	— 209	— 645	— 1.2

The possible importance of the baroclinic waves and their associated meridional circulations in giving the required sources and sinks of momentum to satisfy (29) can be seen by referring to fig. 5. In addition to the theoretical values of U_{3t} obtained from (27), this figure has indicated on it the values of $\partial U/\partial t$ in Table I. [To make this comparison possible $\xi = 0$ was placed at latitude 45N and $\xi = -\pi/2$ and $\pi/2$ at 15N and 75N, in agreement with the choice of $2w$ as equal to sixty degrees of latitude. Furthermore, A in

[1] The values of H and $\triangle$ have been revised slightly so as to apply to the layer surface – 500 mb rather than for the layer surface – 7.5 km as used by Widger.

(27) was taken as equal to $(15.4)^{-1}$ m^2 sec^{-3} to correspond more closely to winter conditions.] The momentum changes given by (27) are clearly of the correct order of magnitude and even have the approximately correct variation in latitude.[1]

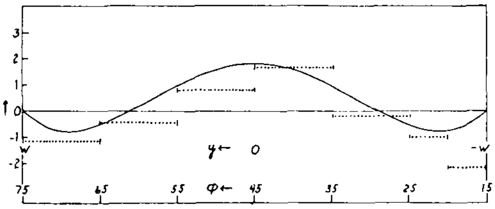

Fig. 5. U_{3t} in m sec^{-1} day^{-1} when T_t^* is taken as 1 °C day^{-1}. The dotted horizontal line segme ts are the values of $\partial U/\partial t$ computed from Widger's data in table I.

The meridional circulation (25), (26), as shown in fig. 4, is of the same general type as that derived from the usual qualitative considerations based on the effect of skin friction on the *observed* surface wind distribution (e.g. the frictional drag on the middle latitude westerlies produces a poleward drift in the lower layers and, from mass continuity reasons, a return equatorward flow at upper levels). However, our argument has not proceeded from an observed distribution of surface wind, but has instead tried to show that *the quasi-geostrophic baroclinic waves are associated with weak meridional circulations which, in combination with the horizontal eddy flux of momentum, prescribes the main features of the observed surface zonal wind distribution.* Thus, for example, if we considered the initial state of our basic current to be such that U at the surface was everywhere zero (this does not affect the second-order effects of the unstable waves), the meridional circulation set up by the waves would create a system of surface winds with easterlies in high and low latitudes and westerlies in between.

[1] The boundary condition (5) places an artificial restraint on the theoretical values of U_t at the northern end of the region. This undesirable feature could be removed at the polar end by using spherical coordinates at the expense of some additional complication in the computations. The proper procedure at the equatorial end is less obvious since the quasi-geostrophic theory cannot be justified so readily for small values of the Coriolis parameter.

The role played by large-scale correlations in ϱuw (or in ωu) in providing a vertical transport of momentum in our model can be computed, but it is negligible in comparison to the effect of the meridional circulation (25), (26). On the other hand, the observations by WHITE and COOLEY (1952) seem to demonstrate that a downward transport of momentum of this type does exist in middle latitudes and that it is of the proper order of magnitude. It may be that the value of this correlation in a perturbation analysis is greatly influenced by the assumption that U_1 and U_3 are independent of latitude. Or it may be, as is perhaps indicated by Kuo's analysis of this effect in the baroclinic waves of infinite lateral extent, that the two-level model is too crude to include this effect explicitly but includes it implicitly in the form of a meridional circulation. In any case, it is clear that the simple two-level model we have used does contain a mechanism for the vertical redistribution of zonal momentum, and may therefore be used with some hope of success in the computational experiment described at the end of section 1.

9. Energy transformations in the unstable waves

In section 3 we have shown that the quasi-geostrophic equations (1), (2) and the boundary conditions (4) and (5) are consistent with the mechanical energy equation. It is perhaps of some interest to apply this fact to the system we have used in studying the unstable baroclinic perturbations.

Introducing the expression (3) for ω_2 into (10), we find that the energy equation may be rewritten

$$\frac{\partial}{\partial t}\int\int\left[q_1+q_3+\frac{\lambda^2}{2f^2}(\varphi_1-\varphi_3)^2\right]dx\,dy=0 \quad (30)$$

where the integration is over $-w \leqslant y \leqslant w$ $0 \leqslant x \leqslant L$ and we have used the fact that the integral of $\boldsymbol{v}\cdot\nabla h$ (h arbitrary) over this region vanishes when $\boldsymbol{v}$ is geostrophic. The term $1/2\, f^{-2}\lambda^2\,(\varphi_1-\varphi_3)^2$ clearly represents some type of potential energy for the system. The kinetic energy q_1+q_3 and the potential energy $1/2 f^{-2}\lambda^2\,(\varphi_1-\varphi_3)^2$ may each be divided up into a part representing the energy of the basic current and a part representing that of the perturbation. Thus (30) becomes

$$\left.\begin{aligned}&\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{-\pi/2}^{\pi/2}\Big\{\Big[U_1^2+U_3^2+f^{-2}\lambda^2(\Phi_1-\Phi_3)^2\Big]+\\&+\Big[\overline{u_1'^2+v_1'^2+u_3'^2+v_3'^2+f^{-2}\lambda^2(\varphi_1'-\varphi_3')^2}\Big]\Big\}d\xi=0\end{aligned}\right\}\quad(31)$$

The rate of change of the basic current energies may be computed from (27) and (23). We find

$$\left.\begin{aligned}&\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{-\pi/2}^{\pi/2}(U_1^2+U_3^2)\,d\xi=\\&=-X\frac{2\,(2+\alpha)\tanh\sqrt{2\gamma}\,\pi/2}{\pi\sqrt{2\gamma}\,(2+\gamma)}\end{aligned}\right\}\quad(32)$$

$$\left.\begin{aligned}&\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{-\pi/2}^{\pi/2}\frac{\lambda^2}{f^2}(\Phi_1-\Phi_3)^2\,d\xi=\\&=-X(2+\alpha)\left[\frac{1}{2}-\frac{2\tanh\sqrt{2\gamma}\,\pi/2}{\pi\sqrt{2\gamma}\,(2+\gamma)}\right]\end{aligned}\right\}\quad(33)$$

where

$$X=[AV\mu]/[f(2+\alpha)]$$

The time rate of change of the perturbation kinetic and potential energies are readily computed from (17):

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{-\pi/2}^{\pi/2}(\overline{u_1'^2}+\overline{v_1'^2}+\overline{u_3'^2}+\overline{v_3'^2})\,d\xi=X\alpha\quad(34)$$

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{-\pi/2}^{\pi/2}\frac{\lambda^2}{f^2}\overline{(\varphi_1'-\varphi_3')^2}\,d\xi=X\left(\frac{2-\alpha}{2}\right)\quad(35)$$

It is easily seen, by adding (32) – (35), that the energy equation is indeed satisfied.

In (32), the factor multiplying $X(2+\alpha)$ has the value .026. Since this term, representing a loss of kinetic energy by the basic current, is exactly balanced by the last term in (33), we may say that of the total increase in perturbation energy 95 per cent comes from the potential energy of the basic current, the other 5 per cent coming from the kinetic energy of the zonal motion. These figures are obtained when $\gamma=\lambda^2\mu^{-2}$ has the value 5.3. For smaller γ, e.g. smaller values of w or smaller values of f, the fraction of perturbation energy which comes from the kinetic energy of the basic current increases.

The rate of production of kinetic energy per unit area, given approximately by one half the value of (34), has a value of about 0.375×10^{-3} m^2 sec^{-3}, when A is taken equal to $(30.8)^{-1}$ m^2 sec^{-3}. If we multiply this by 10 tons (the approximate mass of the atmosphere per square meter) we get a value of about 3.75 joules sec^{-1} m^{-2} as the rate at which the disturbances are creating kinetic energy. This can be compared with BRUNT's estimate (1939) of about 5 joules sec^{-1} m^{-2} as the average rate of dissipation of kinetic energy in the atmosphere.

10. The vertical transport of entropy

Computations of the gain and loss of heat by radiation show that the upper half of the troposphere loses heat while the lower half gains heat by this process if we consider conditions averaged with respect to latitude. EADY (1949) has pointed out that the positive correlation between the vertical velocity and potential temperature in the unstable baroclinic waves must give a net upward transport of entropy which will act to balance this effect of radiation.

Our equations prescribe Φ_t at only two levels, and therefore define a temperature at only one level. Thus they cannot be used for a direct computation of the time rate of change of the mean entropy in the upper and lower halves of the atmosphere. However, if Θ_1^* and Θ_3^* are defined as the mean values of the potential temperature in the upper and lower halves of the atmosphere, the following formula holds for adiabatic motion:

$$\frac{\partial\Theta_1^*}{\partial t}=-\frac{2}{\pi\,p_0}\int_{-\pi/2}^{\pi/2}\overline{\omega_2\Theta_2}\,d\xi$$

Θ_2 may then be expressed in terms of $(\varphi_1-\varphi_3)$ with the aid of the equation of state and the hydrostatic equation. For a disturbance of the type we have considered, with T_t^* taken as 0.5°C day^{-1}, we find that $\Theta\partial^*/\partial t$ is about

$0.25°C\ day^{-1}$. Therefore any complete theory of the existing distribution of temperature with height in the atmosphere must include this effect of the large-scale extratropical disturbances.

Since this effect of the disturbances is proportional to the square of their amplitude, this upward transport of entropy should be greater in the winter season than in the summer season. This should cause the lapse rate to be more stable in winter than in summer. Evidence that this is so can be readily seen in the meridional cross sections of Hess for the winter and summer seasons (HESS, 1948).

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REFERENCES

BJERKNES, V., and collaborators, 1933: *Physikalische Hydrodynamik*. Berlin, J. Springer.

BRUNT, D., 1939: *Physical and dynamical meteorology*, 2nd ed. Cambridge, University Press.

CHARNEY, J. G., 1947: The dynamics of long waves in a baroclinic westerly current. *J. Meteor.*, **4**, 135–162.

— 1948: On the scale of atmospheric motions. *Geofys. Publ.*, **17**, no. 2.

— and N. A. PHILLIPS, 1952: Numerical integration of the quasi-geostrophic equations for barotropic and simple baroclinic flows. *J. Meteor.*, **10**, 71–99.

DEFANT, A., 1921: Die Bestimmung der Turbulenzgrössen der atmosphärischen Zirkulation aussertropischer Breiten. *S. B. Akad. Wiss. Wien*, Abt. II a, 130, 383–403.

EADY, E. T., 1949: Long waves and cyclone waves. *Tellus*, **1**, no. 3, 33–52.

ELIASSEN, A., 1949: The quasi-static equations of motion with pressure as independent variable. *Geofys. Publ.*, **17**, no. 3.

— 1952: Slow thermally or frictionally controlled meridional circulations in a circular vortex. *Astrophys. Norv.*, **V**, no. 2, 19—60.

HADLEY, G., 1735: Concerning the cause of the general trade winds. *Phil. Trans. Roy. Soc.* London, **39**.

HESS, S. L., 1948: Some new meridional cross sections through the atmosphere. *J. Meteor.*, **5**, 293–300.

JEFFREYS, H., 1926: On the dynamics of geostrophic winds. *Quart. J. roy. meteor. Soc.*, **52**, 85–104.

KUO, H. L., 1949: Dynamic instability of two-dimensional non-divergent flow in a barotropic atmosphere. *J. Meteor.*, **6**, 105–122.

KUO, H. L., 1952: Three-dimensional disturbances in a baroclinic zonal current. *J. Meteor.*, **9**, 260–278.

PALMÉN, E., and M. A. ALAKA, 1952: On the budget of angular momentum in the zone between equator and 30° N. *Tellus*, **4**, 324–331.

PALMÉN, E., and C. W. NEWTON, 1948: A study of the mean wind and temperature distribution in the vicinity of the polar front in winter. *J. Meteor.*, **5**, 220–226.

ROSSBY, C.-G., and collaborators, 1939: Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semipermanent centers of action. *J. mar. Res.*, **2**, 38–55.

STARR, V. P., 1948: An essay on the general circulation of the earth's atmosphere. *J. Meteor.*, **5**, 39–43.

— 1951 a: The physical basis for the general circulation. *Compendium of meteorology*. Boston, Amer. meteor. Soc., 541–550.

— 1951b: Application of energy principles to the general circulation. *Compendium of meteorology*. Boston, Amer. meteor. Soc., 568–574.

WHITE, R. M., 1949: The role of mountains in the angular momentum balance of the atmosphere. *J. Meteor.*, **6**, 353–355.

— 1951: The meridional flux of sensible heat over the northern hemisphere. *Tellus*, **3**, 82–88.

— and D. S. COOLEY, 1952: The large scale vertical eddy-stress in the free atmosphere. *Trans. Amer. geophys. Union*, **33**, 502–506.

WIDGER, W. K., 1949: A study of the flow of angular momentum in the atmosphere. *J. Meteor*, **6**, 291–299.