On the Relation between Vorticity, Deformation and Divergence and the Configuration of the Pressure Field

By SVERRE PETTERSSEN, University of Chicago[1]

(Manuscript received April 27, 1953)

Abstract

Theorems for the vorticity, deformation and divergence are developed for quasi-horizontal frictionless motion. The divergence theorem is used to investigate the validity of the geostrophic approximation, and it is found that the balance between the Laplacian of the pressure field and the vorticity is appreciably influenced by the deformation, while the divergence is relatively unimportant. The vorticity theorem and the deformation theorem are combined to obtain a prediction quantity which is related to the absolute vorticity and permits treatment of the fields which are more general than the geostrophic field. Some charts are produced to show the magnitudes of the various quantities in a mature storm.

1. Introduction

The prognostic equations used in numerical predictions are generally based upon the assumption that the vorticity of the actual wind can be replaced by that of the geostrophic wind. Thus, from the geostrophic wind equation one finds that the relative vorticity C_g is expressed by the formula

$$C_g = \frac{1}{f}\nabla^2 Z \qquad (1)$$

where f is the Coriolis parameter, and $\nabla^2 Z$ is the two-dimensional Laplacian of the geopotential of an isobaric surface. The same formula would apply to an isentropic surface if Z is interpreted as the Montgomery potential.

Since no restrictions are imposed upon Z, it is evident from the derivation of Eq. (1) that the instantaneous wind is everywhere assumed to be well adjusted to the gradient of the contour field though no accelerational mechanism is provided to allow the wind to remain adjusted while the air moves through a variable contour field. It is, therefore, not obvious how the geostrophic approximation can lead to a useful prediction equation, except when the contour field is trivially simple.

On the other hand, experiments with numerical predictions have definitely established that a considerable portion of the changes in the contour field can be predicted by the use of the geostrophic approximation. The source of this success is by no means evident, but it may be due to one or more of the following conditions: (a) the procedure may contain errors which are largely compensating; (b) critical values of the neglected terms may

[1] The research reported in this paper has been sponsored by the Geophysics Research Directorate of the U. S. Air Force Cambridge Research Center under Contract No. AF. 19 (604)—309.

be rare or confined to limited regions, with the result that the over-all correlation between computed and observed changes is relatively high; or (c) the equations of motion may contain some hitherto unexplored mechanism that tends to suppress the influences of the neglected terms.

The purpose of this paper is to explore these relationships.

2. Deformation, vorticity and divergence

With customary symbols the equations of horizontal frictionless motion may be written:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial Z}{\partial x} + fv \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial Z}{\partial y} - fu \end{aligned} \tag{2}$$

It can be shown that these equations hold for motion with a vertical component, provided: (a) that the motion is adiabatic; (b) that $\partial/\partial t$ is interpreted as the local variation at a point fixed in x and y while it moves vertically with an isentropic surface; (c) that $\partial/\partial x$ and $\partial/\partial y$ are the variations along an isentropic surface per unit distance along the horizontal coordinate axes; and (d) that Z is the Montgomery potential.

For the purpose of this discussion it suffices to consider horizontal motion, although the results will apply to adiabatic motion in general.

Now, the vorticity equation represents a theorem concerning the difference between two of the quotients in Eqs. (2). Obviously, similar theorems can be derived for the other combinations of the pertinent quotients.

It is convenient to introduce the quantities

$$\begin{aligned} A &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \qquad & B &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ C &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad & D &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \end{aligned} \tag{3}$$

Here, C is the relative vorticity, D the divergence, while $(A^2 + B^2)^{1/2}$ is the total deformation, which can be shown to be invariant in respect of choice of coordinate axes.

Putting $Q = C + f =$ absolute vorticity, we obtain from Eqs. (2) and (3)

$$\dot{A} + A\,D = -\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + fB + v\frac{\partial f}{\partial x} + u\frac{\partial f}{\partial y} \tag{4}$$

$$\dot{B} + B\,D = -2\frac{\partial^2 Z}{\partial x \partial y} - fA - u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} \tag{5}$$

$$\dot{Q} + QD = 0 \tag{6}$$

$$\dot{D} + \frac{1}{2}DD = -\nabla^2 Z + fC + \frac{1}{2}(C^2 - A^2 - B^2) + v\frac{\partial f}{\partial x} - u\frac{\partial f}{\partial y} \tag{7}$$

Eq. (6) is the well-known vorticity theorem which states that in adiabatic frictionless motion there are no sources or sinks of absolute vorticity in an isentropic surface.

Eqs. (4) and (5) may be combined to give a deformation theorem, and Eq. (7) may be referred to as the divergence theorem. It will be seen that the fields of deformation and divergence normally contain sources or sinks.

Eq. (7) shows that the relation between the relative vorticity and the configuration of the contour field is rather complex, and there can be no question of computing the vorticity from the Laplacian of Z when its magnitude is small. What we are concerned to compute is, however, the absolute vorticity, and the computational procedure will be determined by the accuracy required.

It is doubtful whether the Laplacian of Z can be computed with an error less than, say, 15 per cent, and, for the sake of argument, we shall be satisfied to compute the absolute vorticity with the same accuracy.

Putting

$$v\frac{\partial f}{\partial x} - u\frac{\partial f}{\partial y} = -\beta U$$

where β is the Rossby parameter, and U is the zonal speed of the wind, we obtain from (7)

$$Q = C + f = (2\nabla^2 Z + f^2 + A^2 + B^2 + 2\dot{D} + D^2 + 2\beta U)^{1/2} \tag{8}$$

There is much synoptic evidence in support of the view that in the large-scale currents in middle and high latitudes, $2\dot{D}$ and D^2 are one or two orders of magnitude less than f^2, and may, therefore, be omitted. Furthermore, the last term on the right of Eq. (8) will contribute less than 15 per cent to the absolute vorticity, unless the zonal wind exceeds about 100 m. sec.$^{-1}$. In the following this term will be omitted also, and Eq. (8) reduces to

$$Q = (2\nabla^2 Z + f^2 + A^2 + B^2)^{1/2} \tag{9}$$

In this form the divergence theorem has lost its predictive quality; instead it may be used for computational purposes.

It will be seen that the customary approximation (1) is satisfactory only when

$$C^2 = A^2 + B^2$$

In this case the local wind field can be represented by a uniform translation superimposed upon a straight current with lateral shear, and the vorticity is determined by the Laplacian of Z.

We shall next consider the case when $A^2 + B^2$ is very much smaller than C^2. In this case

$$Q = (2\nabla^2 Z + f^2)^{1/2} \tag{10}$$

and, again, the vorticity is determined by the Laplacian, but the formula differs from approximation (1). From Eqs. (1) and (10) we obtain

$$C_g/C = 1 + C/2f$$

showing that the geostrophic approximation overestimates the vorticity in cyclonic motion while the reverse is true in anticyclonic fields.

In the general case, the quantity $C^2 - A^2 - B^2$ may be positive or negative, and in hyperbolic contour patterns the Laplacian of Z may be balanced mainly by the deformation.

From the foregoing discussion it follows that if the problem is to compute the absolute vorticity, the geostrophic approximation may be unsatisfactory. Furthermore, it is of interest to note that the quantity $C^2 - A^2 - B^2$ derives from the convective parts of the accelerational terms in Eqs. (2), and represents, therefore, an accelerational mechanism that will enable the wind to remain adjusted while the air moves through a variable pressure field.

3. Prediction equations without divergence

The form that the prediction equations assume depends to some extent upon the wind approximation used. As a first orientation we shall consider the well-known non-divergent model. The prediction equations pertaining to this model are obtained from Eq. (6) by putting $D = 0$, and otherwise manipulating in the customary manner. Depending upon which of the foregoing approximations are used, the following equations are obtained:

$$\frac{d}{dt}\left(\frac{\nabla^2 Z}{f} + f\right) = 0 \quad \text{from (1)} \tag{11}$$

$$\frac{d}{dt}\left(2\nabla^2 Z + f^2 + A^2 + B^2\right) = 0 \quad \text{from (9)} \tag{12}$$

Combining now Eqs. (4) and (5), we obtain, since the motion is non-divergent:

$$\frac{1}{2}\frac{d}{dt}\left(A^2 + B^2\right) = -\left(\frac{\partial^2 Z}{\partial x^2} - \frac{\partial^2 Z}{\partial y^2}\right) A - 2\frac{\partial^2 Z}{\partial x \partial y} B \tag{13}$$

plus some small terms which depend upon the meridional variation of the Coriolis parameter. Since it is not our intention to discuss the influences of a variable Coriolis parameter on the deformational field, these terms will be omitted.

It is of interest to note that if the wind is allowed to approach the geostrophic wind, or some value proportional thereto, the right hand side of Eq. (13) converges to zero, and the total deformation tends to be conserved. This is true not because A and B are small but because of the particular combination in which they appear in Eq. (13).

More specifically, let k be a factor of proportionality between the actual wind and the geostrophic wind. Then, regardless of how much the geostrophic wind varies in x and y, the right hand side of Eq. (13) tends to zero provided that k is constant or varies slowly. Now, in the large-scale currents k is not far from unity and its spatial variation is small. For such current systems Eq. (13) reduces to

$$\frac{d}{d}(A^2 + B^2) = 0 \tag{14}$$

and Eq. (12) to

$$\frac{d}{d}(2\nabla^2 Z + f^2) = 0 \tag{15}$$

Eq. (12) states that the absolute vorticity is conserved, while Eq. (15) goes further to state that a certain portion of this vorticity is conserved also. Thus, although the deformation terms in Eq. (9) may affect the value of the vorticity, these terms are of no consequence for the prediction equation, provided that the motion is non-divergent. It will thus be seen that the deformation theorem expresses a real mechanism through which the influence of some of the terms left out of the geostrophic assumption is suppressed in the prediction equations.

It remains now to explain the difference between Eqs. (11) and (15). For sake of brevity we put $\nabla^2 Z = L$. Eqs. (11) and (15) may then be written

$$\dot{L} = -\dot{f}(f - C_g) \qquad (11a)$$

$$\dot{L} = -\dot{f}f \qquad (15a)$$

where, as before, C_g is the geostrophic vorticity.

It will be seen that the difference is immaterial when the geostrophic vorticity is much smaller than f. This is often the case over large areas of the charts, and this may be one of the reasons why the geostrophic approximation has yielded fair predictions. On the other hand, when $C_g > 0$ and comparable with f (which is usually the case in mature storms) the difference between Eqs. (11a) and (15a) becomes important, and when $C_g > f$ it is difficult to see how Eq. (11a) can correspond to any real process. Finally it may be noted that Eq. (11a) tends to exaggerate the changes when $C_g < 0$ while the reverse is true when $C_g > 0$.

In some of the experiments with numerical predictions it has tacitly been assumed that f in the first term within the parentheses of Eq. (11) can be kept constant while the second $\dot{f}$ is allowed to vary. The justification for this has been that it makes very little difference whether or not the first f is differentiated (but see Section 5). However, if the first f is kept constant, Eq. (11) happens to become identical to Eq. (15), with the fortunate result that one of the shortcomings of the geostrophic approximation is compensated for by treating f both as a constant and as a variable.

4. Prediction equations with divergence

If the model is allowed to contain divergence, the absolute vorticity could still be computed by the aid of Eq. (9) provided that $2\dot{D}$ and D^2 are much smaller than f^2.

Instead of Eq. (6) we may write

$$\frac{1}{2}\frac{d}{dt}Q^2 + Q^2 D = 0$$

Substitution from Eq. (9) now gives

$$\frac{d}{dt}(2\nabla^2 Z + f^2 + A^2 + B^2) = -2D(2\nabla^2 Z + f^2 + A^2 + B^2) \qquad (16)$$

Eqs. (4) and (5) may be combined to give

$$\frac{d}{dt}(A^2 + B^2) = -2D(A^2 + B^2) - 2\left[\left(\frac{\partial^2 Z}{\partial x^2} - \frac{\partial^2 Z}{\partial y^2}\right)A + 2\frac{\partial^2 Z}{\partial x \partial y}B\right] \qquad (17)$$

where, as before, the small terms depending upon ∇f have been omitted.

Now, for motion on a sufficiently large scale (see Section 3), the last term on the right of Eq. (17) may be omitted, and Eqs. (16) and (17) combined to give

$$\frac{d}{dt}(2\nabla^2 Z + f^2) = -2D(2\nabla^2 Z + f^2) \qquad (18)$$

Again, one finds a certain portion of the absolute vorticity which lends itself to treatment.

In order to use Eq. (18) in numerical forecasting it is necessary to construct a model, or system of models, in which the distribution of D is included. We shall not discuss such models here. Instead, we shall be concerned to discuss the relative importance of the terms containing D and $\dot{f}$. Putting again $\nabla^2 Z = L$, Eq. (18) may be written

$$\dot{L} + 2LD = -f^2(D + V_N \beta/f) \qquad (19)$$

where V_N is the meridional component of the wind, and β is the Rossby parameter.

In middle latitudes β/f is about 1.5×10^{-7} m^{-1} and the divergence term on the right is not negligible against the β-term unless D is less than about $2\, V_N\, 10^{-8}$ m^{-1}.

There is much evidence to indicate that in cases of appreciable development D is at least as large as 10^{-5} sec^{-1}, and in such cases the term containing β is very much smaller than D. Furthermore, the second term on the left of Eq. (19) shows that the effect of the divergence increases with the Laplacian of the contour field.

5. Examples

In order to determine whether the various expressions for the vorticity give significantly different results, the absolute geostrophic vorticity ($Q_g = C_g + f$) and the absolute vorticity (Q) as given by Eq. (9), were computed and compared. In computing Q, A and B were replaced by their geostrophic values.

In addition the quantity

$$P = 2\nabla^2 Z + f^2 \qquad (20)$$

was evaluated since it appears to be the appropriate quantity to be used in the prediction equation.

To obtain figures and dimensions comparable with the geostrophic vorticity, the square root of P was computed. It will be seen from Eqs. (9) and (20) that when $\nabla^2 Z$ is large and negative while $A^2 + B^2$ is small, P may be negative and $P^{1/2}$ imaginary. This, however, is no real inconvenience since it is P and not $P^{1/2}$ that enters into the prediction equation.

The Laplacian of Z and similar quantities were evaluated by the aid of the rectangular grid shown in Fig. 1, in which $H = 444{,}000$ m.

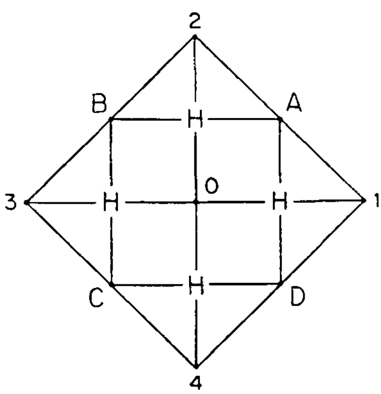

Fig. 1. Showing the grid and the subscripts in the interpolation formulas.

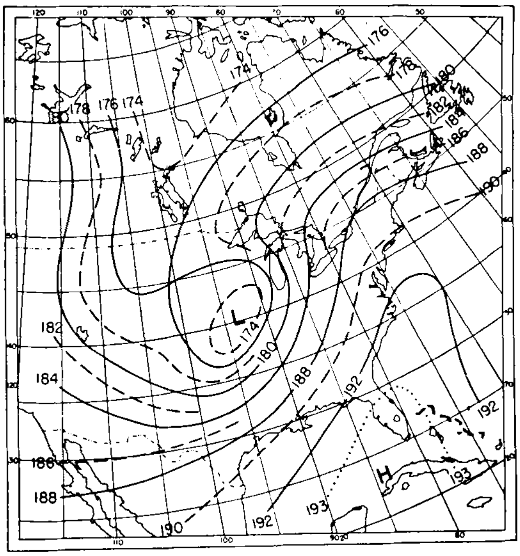

Fig. 2. The contours of the 500 mb. surface, 26 Nov. 1952, 0300 G.M.T. Heights of contour lines in hundreds of geopotential feet.

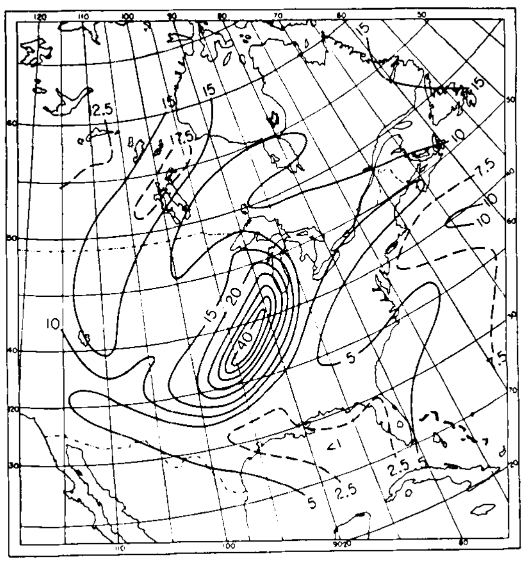

Fig. 3. The absolute geostrophic vorticity corresponding to Fig. 2. Units 10^{-5} sec^{-1}.

The interpolation formulas used are

$$\nabla^2 Z = K\frac{Z_1 + Z_2 + Z_3 + Z_4 - 4Z_0}{H^2}$$

$$\frac{\partial^2 Z}{\partial x^2} - \frac{\partial^2 Z}{\partial y^2} = K\frac{Z_1 + Z_3 - Z_2 - Z_4}{H^2}$$

$$\frac{\partial^2 Z}{\partial x\,\partial y} = K\frac{Z_A + Z_B - Z_C - Z_D}{H^2}$$

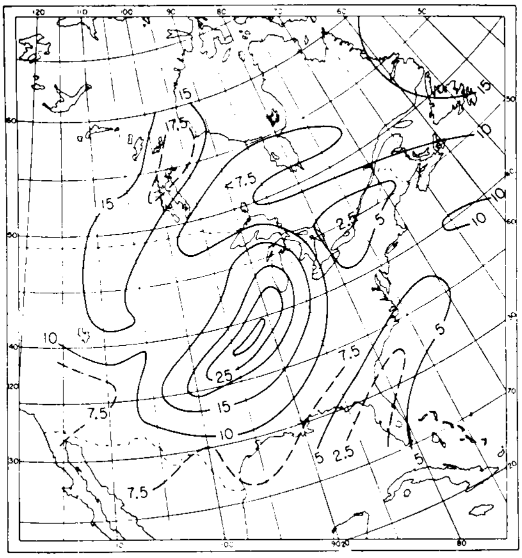

Fig. 4. The absolute vorticity corresponding to Fig. 2, computed from Eq. (9). Units 10^{-5} sec^{-1}.

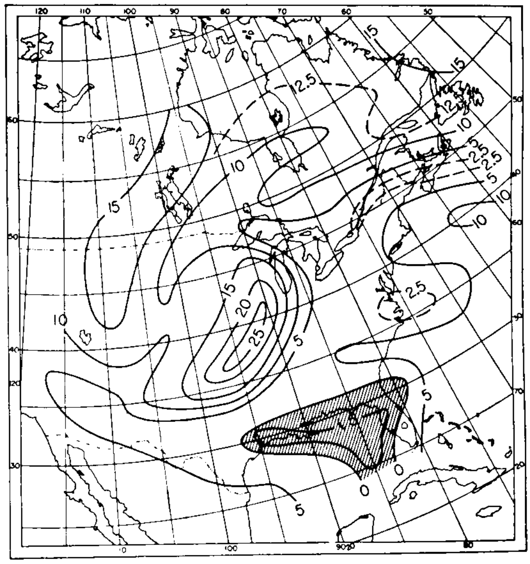

Fig. 6. The distribution of $(2\nabla^2 Z + f^2)^{1/2}$ corresponding to Fig. 2. Units 10^{-5} sec^{-1}.

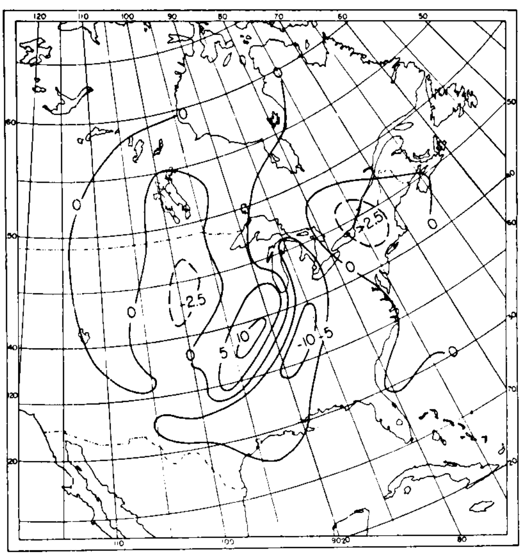

Fig. 5. The difference between Figs. 3 and 4.

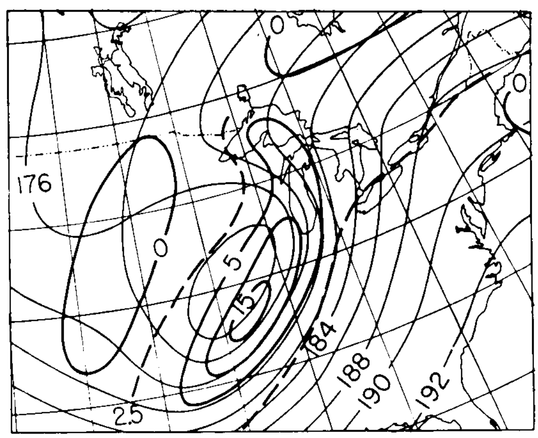

Fig. 7. The difference between Fig. 3 and Fig. 6 superimposed upon Fig. 2.

where K is a factor of conversion to rational units, and the subscripts are those shown in Fig. 1.

The synoptic situation chosen for the test was the one that occurred over the United States on 26 November 1952. Fig. 2 shows the contours of the 500 mb. surface at 0300 C.M.T. The storm associated with the closed contours in the central part of the chart is one of appreciable intensity and it is evident that both the vorticity and the deformation are large.

In Fig. 3 is shown the distribution of the absolute geostrophic vorticity Q_g computed from Fig. 2. It will be seen that the maximum absolute geostrophic vorticity near the center of the storm is about four times the Coriolis parameter, indicating a relative geostrophic vorticity of about $3f$.

The absolute vorticity Q, computed from Eq. (9), is shown in Fig. 4, and the difference $Q_g - Q$ is shown in Fig. 5. It will be seen that the difference is appreciable except in areas where both Q_g and Q are close to the Coriolis parameter.

Fig. 6, shows the distribution of the square root of P, computed from Eq. (20). Negative values of P were found along the coast of the Gulf of Mexico (within the area bounded by the heavy curve) and here the complex values of $P^{1/2}$ are indicated.

Fig. 7 shows the difference $Q_g - P^{1/2}$ superimposed upon Fig. 2. This chart shows the difference in the geostrophic advection of the two quantities. It will be seen that the difference is appreciable. Finally, Fig. 8 shows the distribution of the deformation $(A^2 + B^2)^{1/2}$. The difference between Q and $P^{1/2}$ is entirely due to this quantity.

The foregoing charts will suffice to show that the geostrophic vorticity may differ appreciably from other quantities which could be used in the prediction equations.

6. Conclusions

The foregoing discussion appears to justify the following conclusions.

a. Although the geostrophic wind may be a fair approximation to the true wind, the geostrophic vorticity may differ appreciably from the actual vorticity in storms of appreciable intensity, and so may the advection of these vorticities.

b. On account of the deformation theorem, the effect of the deformational field on the vorticity changes may be eliminated if the motion is on a sufficiently large scale.

c. The quantity $P = 2\nabla^2 Z + f^2$ appears to be the most satisfactory prediction quantity to be used in non-divergent as well as divergent motions.

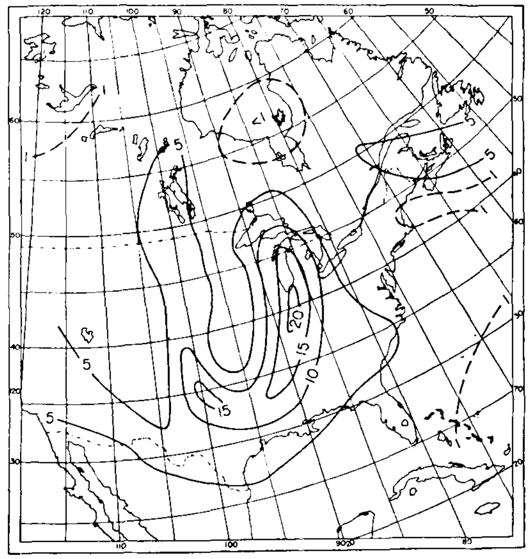

Fig. 8. The total geostrophic deformation $(A^2+B^2)^{1/2}$ corresponding to Fig. 2. Units 10^{-5} sec^{-1}.

d. The effect of the divergence is proportional to P and cannot be neglected unless the divergence is less than about 10^{-6} sec^{-1}.

Acknowledgements

In a recent paper, Dr Leon Sherman (1952) has kindly acknowledged a discussion between him and the writer concerning the divergence theorem, and the writer wishes to record hiis acknowledgement.

The writer wishes to express his indebtedness to Messrs P. M. Breistein and H. D. Parry and Miss D. L. Bradbury for extensive assistance in computing a large number of synoptic charts from which the case reproduced here has been selected.

REFERENCE

Sherman, L., 1952: On the scalar-vorticity and horizontal divergence equations. *Journ. of Meteorology*. **9**, No. 5, pp. 359—366.