Some Remarks on the Angular Momentum Balance in the Atmosphere

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1. By substitution of the equation of continuity in the equation of the zonal motion, we obtain (VAN MIEGHEM 1950)

$$\frac{\partial}{\partial t}(\varrho\mu) + \frac{1}{r^2 \cos\varphi}\frac{\partial}{\partial x^i}[r^2\cos\varphi(\mu\varrho v^i + p\varepsilon_1{}^i - T_1{}^i)] = 0, \quad (i = 1, 2, 3) \tag{1}$$

where ($x^1 \equiv \lambda$, $x^2 \equiv \varphi$, $x^3 \equiv r$) are the spherical coordinates, $v^i \equiv \frac{dx^i}{dt}$ the contravariant components of the air velocity $\mathbf{v}$, ($u \equiv r\cos\varphi\frac{d\lambda}{dt}$, $v \equiv r\frac{d\varphi}{dt}$, $w \equiv \frac{dr}{dt}$ the zonal, meridional and vertical components of $\mathbf{v}$), $\mu \equiv r\cos\varphi(r\cos\varphi\cdot\omega + u)$ the absolute angular momentum, ω the earth's angular speed, ϱ the specific mass, p the pressure, $T_1{}^i$ the contravariant stresses along the latitude circles, due to internal friction, $\varepsilon_1{}^i \equiv 1$ when $i \equiv 1$ and $\varepsilon_1{}^i \equiv 0$ when $i \neq 1$.

Equation (1) shows that there is no production nor destruction of absolute angular momentum in the atmosphere (VAN MIEGHEM, 1950), but only redistribution of momentum as a consequence of inflow and outflow of west-momentum in the surface easterlies and in the surface westerlies respectively (skin-friction and mountain-effect at the earth's surface).

2. Averaging equation (1) along the latitude circles, one obtains

$$\frac{\partial}{\partial t}\overline{(\varrho\mu)} + \frac{1}{r^2\cos\varphi}\frac{\partial}{\partial x^\alpha}[r^2\cos\varphi(\widehat{\mu}\,\overline{\varrho v^\alpha} + r\cos\varphi\,\overline{\varrho u'' v''^\alpha} - \overline{T_1{}^\alpha})] = 0, \quad (\alpha = 2, 3) \tag{2}$$

where $\overline{X}$ designates the zonal mean value of X and $\widehat{X}$ the corresponding weighted mean value defined by $\overline{\varrho}\widehat{X} = \overline{\varrho X}$. Let X' and X'' be the fluctuations of X with respect to $\overline{X}$ and $\widehat{X}$. It is obvious that $\overline{X'} \equiv 0$, $\overline{\varrho X''} \equiv 0$; hence

$$X'' = \overline{X''} + X', \quad -\overline{\varrho}\,\overline{X''} = \overline{\varrho' X'} = \overline{\varrho' X''} = \overline{\varrho X'} = \overline{\varrho' X}, \tag{3}$$

and

$$\overline{\varrho XY} \equiv \overline{\varrho}\widehat{X}\widehat{Y} + \overline{\varrho X'' Y''}. \tag{4}$$

For the zonally averaged instantaneous absolute angular momentum $\widehat{\mu}$, three modes of redistribution are possible:

a) redistribution of momentum due to mass transport $\overline{\varrho v^\alpha}$ in meridional planes;
b) an eddy flux ($\overline{\varrho u'' v''}$, $\overline{\varrho u'' w''}$);
c) a non convective flux due to the mean zonal components of the viscosity and small-scale eddy stresses. This flux may be disregarded outside the boundary layer.

3. Replacing in (3) X by v, one finds

$$-\overline{\varrho}\,\overline{v''} = \overline{\varrho' v''} = \overline{\varrho' v'} = \overline{\varrho' v} = \overline{\varrho v'} \tag{3a}$$

and identical relations for w.

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The vector $\overline{\mathbf{v}''}$ $(\overline{v''}, \overline{w''})$ may be considered as defining the organization of the mass exchange in meridional planes as a consequence of the large-scale eddying motion (VAN MIEGHEM 1949). When, along the latitude circles, the fluctuations ϱ' and v''^{α} (or v'^{α}) are always of the same *or* the opposite sign, that is to say when the mass exchange in meridional planes due to the large-scale eddies is well organized, the velocity components $\overline{v''^{\alpha}}$ reach a high absolute value.

For instance, in most cases, warm air ($\varrho' < 0$) ascends ($w > 0$) and comes from the south ($v > 0$), while cold air ($\varrho' > 0$) subsides ($w < 0$) and comes from the north ($v < 0$). Hence, along the latitude circles, $\overline{w''} > 0$ and, below 250 mb, $\overline{v''} > 0$. However, along the latitude circles above the 250 mb-level, $\overline{v''} < 0$. From $\overline{\varrho X} = \bar{\varrho}\,\overline{X} + \overline{\varrho' X'}$ and (3), it follows that:

$$\overline{\varrho v} = \bar{\varrho}(\bar{v} - \overline{v''}) \quad \text{and}$$

$$\overline{\varrho w} = \bar{\varrho}(\overline{w} - \overline{w''}) \quad \text{with} \quad \overline{w''} > 0. \qquad (5)$$

The instantaneous mean meridional circulation is generally defined by $(\bar{v}, \overline{w})$ although it would have been better to define this mean circulation by $(\hat{v}, \hat{w})$. In order to avoid confusion, we shall use the definition $(\bar{v}, \overline{w})$ introduced previously.

Now, along a meridian, for continuity reasons, we must have

$$\int_t^{t+\tau} dt \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \overline{\varrho w}\, d\varphi \equiv 0,$$

if the time interval τ is sufficiently large. Whence, by virtue of (5),

$$\int_t^{t+\tau} dt \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \bar{\varrho}\,\overline{w}\, d\varphi > 0 \quad \text{and} \quad \int_t^{t+\tau} \bar{\varrho}\,\overline{w}\, dt \neq 0.$$

This inequality demonstrates the existence of a mean meridional circulation. Moreover, the horizontal gradient of $\bar{\varrho}$ being small, the lifting in equatorial regions ($\overline{w} > 0$) must overcompensate the sinking in higher latitudes ($\overline{w} < 0$).

For the same reasons, along a vertical,

$$\int_t^{t+\tau} dt \int_0^{\infty} \overline{\varrho v}\, dz \equiv 0.$$

On account of the exponential decrease of ϱ with altitude z, the sum of the positive $\varrho v'$, above about 10 km, presumably does not compensate the sum of the negative $\varrho v'$, below 10 km, so that

$$\int_t^{t+\tau} dt \int_0^{\infty} \bar{\varrho}\,\bar{v}\, dz > 0 \quad \text{and} \quad \int_t^{t+\tau} \bar{\varrho}\,\bar{v}\, dt \neq 0.$$

Hence, the upper layer with $\bar{v} > 0$ must be much deeper than the lowest layer with $\bar{v} < 0$, as a consequence of the rapid decrease of $\bar{\varrho}$ with height.

It should be emphazised that this existence proof of a mean meridional circulation defined by the time mean value of $\bar{\varrho}\,\bar{v}$ and $\bar{\varrho}\,\overline{w}$ does not give any information about the order of magnitude of the instantaneous mean meridional velocities $(\bar{v}, \overline{w})$, except that they are very small.

From (5) it follows that, as a result of the large scale eddy exchange of mass in meridional planes: a) the vertical branch of the instantaneous mean meridional circulation $(\bar{v}, \overline{w})$ is weakened in equatorial latitudes ($\overline{w} > 0$, $\overline{w''} > 0$) and strengthened in higher latitudes ($\overline{w} < 0$, $\overline{w''} > 0$); b) the horizontal branch is strengthened in the low troposphere ($\bar{v} < 0$, $\overline{v''} > 0$) and in the stratosphere ($\bar{v} > 0$, $\overline{v''} < 0$), but weakened in the middle and high troposphere ($\bar{v} > 0$, $\overline{v''} > 0$).

4. The flux of ω-momentum assumes the form:

$$\begin{cases} r^2 \cos^2\varphi\,\omega\,\overline{\varrho v} = r^2 \cos^2\varphi\,\omega\,\bar{\varrho}\,(\bar{v} - \overline{v''}), \\ r^2 \cos^2\varphi\,\omega\,\overline{\varrho w} = r^2 \cos^2\varphi\,\omega\,\bar{\varrho}\,(\overline{w} - \overline{w''}). \end{cases} \qquad (6)$$

In virtue of (3), one obtains for the eddy flux of angular momentum

$$\begin{cases} r \cos\varphi\,\overline{\varrho u'' v''} = r\cos\varphi \cdot \\ \qquad \cdot \left[\bar{\varrho}\,(\overline{u' v'} - \overline{u''}\,\overline{v''}) + \overline{\varrho' u' v'}\right], \\ r \cos\varphi\,\overline{\varrho u'' w''} = r\cos\varphi \cdot \\ \qquad \cdot \left[\bar{\varrho}\,(\overline{u' w'} - \overline{u''}\,\overline{w''}) + \overline{\varrho' u' w'}\right]. \end{cases} \qquad (7)$$

Hence, the eddy flux of angular momentum may be decomposed into:

a) a flux $(\bar{\varrho}\,\overline{u' v'}, \bar{\varrho}\,\overline{u' w'})$ due to the correlation between the velocity fluctuations only. It seems now well established that the flux $\bar{\varrho}\,\overline{u' v'}$

represents the bulk of the horizontal poleward transport of momentum, repuired in order to compensate the outflow in the surface westerlies and to maintain the strong upper westerly flow (MINTZ 1951, STARR and WHITE 1951, 1952). The flux $\bar\varrho\,\overline{u'w'}$ is presumably downward (WHITE 1950), so that it counterbalances the upward flux of angular momentum due to the mean meridional circulation (BJERKNES 1951).

b) a flux $(-\bar\varrho\,\overline{u''}\,\overline{v''},\ -\bar\varrho\,\overline{u''}\,\overline{w''})$ due to the organization of the large-scale eddy exchange of mass in meridional planes. In the case of a meridional tilt to the east of throughs and ridges, $-\bar\varrho\,\overline{u''} = \overline{\varrho' u'} = \overline{\varrho' u} < 0$; consequently this flux is directed to the south and to the earth's surface.

c) a flux $(\overline{\varrho' u' v'},\ \overline{\varrho' u' w'})$ due to the correlations between both the velocity and density fluctuations.

The fluxes *b* and *c* are, as a rule, disregarded and therefore the flux *a* is generally called the eddy flux.

On the other hand, the horizontal and vertical components of the convective flux of relative angular momentum may be expressed in the following way;

$$\left\{\begin{aligned} r\cos\varphi\,\bar\varrho\,\hat u\,\hat v &= r\cos\varphi\,\bar\varrho\,\cdot \\ &\cdot\left[\overline{\bar u\,\bar v} + \overline{u''}\,\overline{v''} - \overline{u''}\,\bar v - \bar u\,\overline{v''}\right], \\ r\cos\varphi\,\bar\varrho\,\hat u\,\hat w &= r\cos\varphi\,\bar\varrho\,\cdot \\ &\cdot\left[\overline{\bar u\,\bar w} + \overline{u''}\,\overline{w''} - \overline{u''}\,\bar w - \bar u\,\overline{w''}\right], \end{aligned}\right. \tag{8}$$

account being taken of the general formula $\overline{\varrho X} = \bar\varrho\,\bar X + \overline{\varrho' X'} = \bar\varrho\,\hat X$ and of (3). This convective flux has four constituent fluxes:

a) a flux $(\bar\varrho\,\bar u\,\bar v,\ \bar\varrho\,\bar u\,\bar w)$ due to the instantaneous mean circulation $(\bar v,\ \bar w)$ in the meridional planes. The time mean value of $\overline{\bar u\,\bar v}$ for a large period seems to be very small (STARR and WHITE, 1952);

b) a flux $(\bar\varrho\,\overline{u''}\,\overline{v''},\ \bar\varrho\,\overline{u''}\,\overline{w''})$ due to the organization of the large-scale eddy exchange of mass in meridional planes. It is remarkable that this flux drops out in the sum of the fluxes (7) and (8);

(c) a flux $(-\bar\varrho\,\overline{u''}\,\bar v,\ -\bar\varrho\,\overline{u''}\,\bar w)$ proportional to the mean meridional mass transport $(\bar\varrho\,\bar v,\ \bar\varrho\,\bar w)$ and the correlation between the fluctuations of the density of the zonal wind;

d) a flux $(-\bar\varrho\,\bar u\,\overline{v''},\ -\bar\varrho\,\bar u\,\overline{w''})$ proportional to the mean zonal wind and the "organization" vector $(\overline{v''},\ \overline{w''})$.

This last flux is presumably more important than the former.

5. Finally, disregarding viscosity and small-scale turbulence, the instantaneous total flux of absolute angular momentum is defined by the horizontal (positive to the north) and the vertical (positive in zenith direction) components given below:

$$\begin{aligned} \overline{\mu\varrho v} &\equiv \omega r^2\cos^2\varphi\,\bar\varrho\,\bar v + \\ &+ r\cos\varphi\,(\bar\varrho\,\hat u\,\hat v + \overline{\varrho\, u''\, v''}) = \\ &= r\cos\varphi\,\bar\varrho\,(\omega r\cos\varphi\,\bar v + \overline{\bar u\,\bar v} + \overline{u'\,v'}) - \\ &- r\cos\varphi\left\{\bar\varrho\left[\omega r\cos\varphi\,\overline{v''} + \bar u\,\overline{v''} + \right.\right. \\ &\left.\left. + \overline{u''}\,\bar v\right] - \overline{\varrho'\,u'\,v'}\right\} \end{aligned} \tag{9}$$

and

$$\begin{aligned} \overline{\mu\varrho w} &\equiv \omega r^2\cos^2\varphi\,\bar\varrho\,\bar w + \\ &+ r\cos\varphi\,(\bar\varrho\,\hat u\,\hat w + \overline{\varrho\, u''\, w''}) = \\ &= r\cos\varphi\,\bar\varrho\,(\omega r\cos\varphi\,\bar w + \overline{\bar u\,\bar w} + \overline{u'\,w'}) - \\ &- r\cos\varphi\left\{\bar\varrho\left[\omega r\cos\varphi\,\overline{w''} + \bar u\,\overline{w''} + \right.\right. \\ &\left.\left. + \overline{u''}\,\bar w\right] - \overline{\varrho'\,u'\,w'}\right\}, \end{aligned} \tag{10}$$

with the conditions

$$\overline{w''} > 0, \quad \left\{\begin{array}{l} \overline{v''} > 0, \text{ below } \sim 250 \text{ mb}, \\ \overline{v''} < 0, \text{ above } \sim 250 \text{ mb, and} \end{array}\right.$$

$\overline{u''} \lessgtr 0$, depending upon the sign of the meridional tilt of troughs and ridges.

Hitherto, only the fluxes in parenthesis in the last member of (9) and (10) have been considered and the time mean values of $\bar v$, $\overline{\bar u\,\bar v}$ and $\overline{u'\,v'}$ computed (STARR and WHITE, 1951, 1952). We believe that the instantaneous zonal mean value $\overline{v''}$ may not be disregarded *a priori* and therefore should be computed. Moreover, the triple correlation $\overline{\varrho'\,u'\,v'}$ should be investigated numerically in order to establish its order of magnitude.

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The Variation of Gravity within the Earth

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Abstract

The different methods to determine the values of gravity within the Earth are reviewed.

By using the equation

$$g = g_0 \left(\frac{r_0}{r}\right)^2 - \frac{4}{3}\,\pi G \varrho\, \frac{r_0^{\,3} - r^3}{r^2}$$

and substituting the density values tabulated by Ramsay the variation of gravity has been computed (table 1 and fig. 1). The g value of 982 gal at the surface of the Earth is attained without any correction to the density distribution.

RAMSAY (1949) has recently developed a theory in which the Earth's core and mantle are assumed to be chemically identical. The discontinuity of seismic waves at a depth of 2,900 km. is interpreted in the usual way as a jump in density. But the jump in density is attributed to phase transition under pressure and not to the appearance of a new material such as iron. Olivine has a ionic structure in the mantle; in the core it is in a dense metallic state. The density figures given by Ramsay are slightly different from figures given by BULLEN (1940 and 1942) as the density values vary in the mantle from 3.29 to 6.02 and in the core from 8.92 to 18.5. The corresponding figures in Bullen's papers are 3.32 to 5.68 and 9.43 to 17.20. No gravity values have been computed from the density variation given by Ramsay, as far as it is known, and it is therefore the purpose of this paper to compute new gravity values.

The value of gravity at a distance r from the center of the Earth is given by

$$g_r = \frac{4\pi}{r^2}\, G \int_0^r \varrho_r \cdot r^2\, dr \qquad (1)$$

where G is the constant of gravitation and ϱ_r the density at the point considered.

BENFIELD (1937) used this formula regarding the Earth as a non-rotating sphere, an assumption which was justified by the uncertainty