Gravitational Convection from a Boundary Source

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Abstract

Elementary analyses of the mean patterns of free convection from a line source and a point source are presented without regard to the specific means by which the gravitational action is produced. The derived functional relationships are then verified and completed through use of velocity and temperature measurements above sources of heat, the generalized form of the results permitting characteristics of the mean flow to be determined over a considerable range of the primary variables. These results should enable meteorologists to evaluate the role of the basic convective process in the more complex movements of the atmosphere.

Introductory Remarks

Atmospheric disturbances are generally so complex in nature that the relative importance of the various factors which they involve can be appraised only by investigating the effect of each factor independently. In conducting an analysis of this nature, one often notes a close similarity between particular aspects of meterological and other flow occurrences, indicating that experience gained in related fields can be adapted to the problem in question. A case in point is the phenomenon of large-scale thermal updrafts in the atmosphere, one aspect of which is the comparatively simple process of gravitational or "free" convection from a boundary source, recently studied experimentally at the Iowa Institute of Hydraulic Research. The writers believe that the results of this investigation will permit meteorologists to evaluate the role of the basic convective mechanism in atmospheric phenomena.

Free convection due to a point source of heat is very simply illustrated by the plume of smoke which rises from a cigarette in otherwise stagnant air. Because of the buoyancy of the heated air in the immediate vicinity of the burning end, a continuous upward current is induced, with a corresponding radial inflow for reasons of continuity. While the initial steadiness of the smoke filament indicates purely laminar motion for a considerable distance above the heat source, the flow thereafter becomes unstable and the filament breaks up into the eddying clouds normally associated with turbulent motion. After the onset of turbulence, the effect of the convection upon the surrounding fluid becomes far more pronounced, the molecular shear and heat transfer of the initial laminar motion becoming dwarfed in scale by the macroscopic mixing action. Whereas the magnitude of the velocity along the vertical axis is then rapidly di-

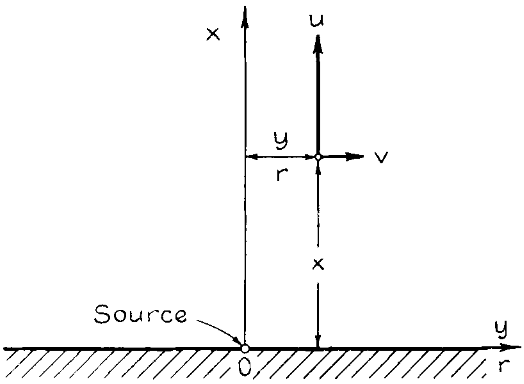

Fig. 1. Definition sketch.

minished, the angle of radial diffusion (and hence the rate of radial entrainment) is accordingly increased.

Although free convection of this nature is usually produced by a source of heat, it is the gravitational rather than the thermal aspects of the flow which are fundamental to the phenomenon. Indeed, essentially the same convection pattern can be produced in a liquid instead of a gas, and without any temperature change whatever. For example, the atmosphere may be replaced by a large body of water, and liquid having a different specific gravity may be introduced at a constant rate at a point on either horizontal boundary — a lighter liquid at the bottom or a heavier one at the top. In each event a similar pattern of convection will result, provided only that the local changes in density are relatively small in magnitude.

In the following pages an approximate analysis is made of the mean pattern of free convection from both a line source and a point source, without regard to the specific means of producing the gravitational action. Experimental evidence is then introduced to verify and complete the derived functional relationships, and generalized diagrams are presented to permit evaluation of the primary flow characteristics over a considerable range of the parameters involved.

Theoretical Analysis of the Mean Flow

If it is assumed that the scale of the initial laminar zone is small compared with that of the subsequent zone of turbulent convection, notation for either two-dimensional flow over a line source or axially symmetric flow over a point source may be reduced to that shown schematically in Fig. 1. Herein the origin O represents the location of either source. The local mean component of velocity in the vertical direction x is denoted by u, and that in the lateral direction y or radial direction r is denoted by v.

In addition to the mean velocity of convection above the source, there will also be a mean local change $\Delta\gamma$ in the weight density of the fluid. This entails, to be sure, a comparable change in the mass density ϱ. However, the following analysis is based upon the assumption that, whereas the unit buoyant force $-\Delta\gamma$ is sufficiently great to produce vertical acceleration, the corresponding variation in the mass density of the fluid undergoing acceleration is sufficiently small, in comparison with the density itself, to be neglected.

The foregoing assumption as to constancy of the density makes it possible to proceed in close accordance with the elementary analysis of flow in boundary layers, jets, and wakes. Three additional approximations are thus involved: the pressure intensity is assumed to be hydrostatically distributed throughout the field of motion; transverse forces are ignored in comparison with those in the vertical direction; and turbulent mixing in the vertical direction is ignored in comparison with that in the horizontal. The fundamental equations of motion thus reduce to the equation of continuity and simplified equations of vertical acceleration and diffusion. The development of the relationships describing convection above a line source will be described in some detail, but for purposes of brevity only the results will be indicated for the case of axial symmetry.

Under the foregoing assumptions, the equation of vertical acceleration in two-dimensional flow reduces to

$$\varrho u \frac{\partial u}{\partial x} + \varrho v \frac{\partial u}{\partial y} = -\Delta\gamma + \frac{\partial \tau}{\partial y} \qquad (1)$$

in which the intensity of vertical shear τ is proportional to the mean product of the turbulent velocity components u' and v':

$$\tau = -\varrho \overline{u' v'}$$

The equation of continuity is simply

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

Finally, the equation of diffusion of the quantity $\Delta\gamma$ has the form

$$u\frac{\partial(\Delta\gamma)}{\partial x} + v\frac{\partial(\Delta\gamma)}{\partial y} = -\frac{\partial w}{\partial y} \qquad (3)$$

in which the rate of lateral diffusion w due to turbulence is proportional to the mean product of the eddy scale l and the lateral velocity of fluctuation:

$$w = -\overline{v'l}\,\frac{\partial(\Delta\gamma)}{\partial y}$$

Since $u = 0$ at $y = \infty$, $v = 0$ at $y = 0$, and $\tau = 0$ at $y = 0$ and $y = \infty$, integration of Eq. (1), through use of Eq. (2), over a horizontal plane from the axis to infinity results in the following form of the momentum relationship:

$$\frac{d}{dx}\int_0^\infty \varrho u^2\,dy = -\int_0^\infty \Delta\gamma\,dy \qquad (4)$$

This states that the vertical gradient of the momentum flux is equal to the buoyancy of a horizontal stratum of unit thickness. As $\Delta\gamma=0$ at $y = \infty$ and $w = 0$ at $y = 0$ and at $y = \infty$, a similar integration of Eq. (3) leads to the following expression for the constancy of the incremental weight flux past all successive planes:

$$\frac{d}{dx}\int_0^\infty u\,\Delta\gamma\,dy = 0 \qquad (5)$$

Finally, after multiplication of each term of Eq. (1) by u, integration of the resulting expression by parts, with the help of Eq. (1), yields the following form of the energy relationship:

$$\frac{d}{dx}\int_0^\infty \frac{\varrho u^3}{2}\,dy = -\int_0^\infty u\,\Delta\gamma\,dy - \int_0^\infty \tau\,\frac{\partial u}{\partial y}\,dy \qquad (6)$$

Because the quantity v^3 is negligible in comparison with u^3, this relationship states that the vertical gradient of the flux of kinetic energy is equal to the rate at which work is done by the buoyant force less the rate at which work is done by the turbulent shear. The last term at the same time represents the rate at which energy is lost to the mean motion through the generation of turbulence.

Although no indication is given as to the form of the functional relationships for u, $\Delta\gamma$, and τ which will satisfy these three necessary conditions, the customary hypothesis of dynamic similarity of the mean (and turbulent) motion at all elevations permits each of the relationships to be considered of the same form for every value of x. It is thus finally assumed that

$$\frac{u}{u_{max}} = f(\eta) \qquad \frac{\Delta\gamma}{\Delta\gamma_{max}} = g(\eta) \qquad \frac{\tau}{\varrho u_{max}^2/2} = h(\eta)$$

Herein $\eta = y/\sigma$, the quantity σ representing some linear characteristic of the velocity profile $u = u(y)$, such as the value of y at which the ratio u/u_{max} has some arbitrary magnitude.

Inasmuch as the unknown functions f, g, and h are independent of elevation, any power or combination thereof must also be constant with x. It is hence convenient to formulate a series of pertinent integrals for substitution into the three primary relationships which the functions must satisfy:

$$I_1 = \int_0^\infty f\,d\eta \qquad I_2 = \int_0^\infty f^2\,d\eta \qquad I_3 = \int_0^\infty f^3\,d\eta$$

$$I_4 = \int_0^\infty g\,d\eta \qquad I_5 = \int_0^\infty fg\,d\eta \qquad I_6 = \int_0^\infty h\frac{df}{d\eta}\,d\eta$$

Introduction of these terms into Eqs. (4), (5), and (6) permits the latter to be rewritten as

$$\frac{d}{dx}\left(I_2\,\varrho\,u_{max}^2\,\sigma\right) = -I_4\,\Delta\gamma_{max}\,\sigma$$

$$I_5\,u_{max}\,\Delta\gamma_{max}\,\sigma = C_x$$

$$\frac{d}{dx}\left(I_3\,\frac{\varrho\,u_{max}^3}{2}\,\sigma\right) = -C_x - I_6\,\frac{\varrho\,u_{max}^3}{2}$$

Simultaneous solution of these relationships will yield the following significant results:

$$\sigma = \frac{I_4 I_6 x}{2 I_2 I_5 - I_3 I_4} \sim x$$

$$u_{max} = \left(\frac{C_x}{\varrho} \frac{I_3 I_4 - 2 I_2 I_5}{I_2 I_5 I_6}\right)^{1/3} \sim x^0$$

$$\Delta\gamma_{max} = -\left(C_x \varrho^{1/2} \frac{I_3 I_4 - 2 I_2 I_5}{I_2 I_5 I_6}\right)^{2/3} \frac{I_2}{I_4 x} \sim x^{-1}$$

Evidently, no matter what forms the distribution curves may take, the convection zone will expand linearly with elevation; the maximum velocity (and hence the velocity along any line of constant y/x) will be independent of elevation; and the maximum incremental weight density (and hence that along any line of constant y/x) will vary inversely with elevation.

With this information it is possible to reach corresponding conclusions as to the variation of the volume flux Q, the momentum flux M, the kinetic-energy flux E, the unit buoyant force F, and the flux W of the incremental weight above a source of length L. Thus, per unit length of source,

$$\frac{Q}{L} = 2\int_0^\infty u\, dy \sim x \qquad \frac{M}{L} = 2\int_0^\infty \varrho u^2\, dy \sim x$$

$$\frac{E}{L} = 2\int_0^\infty \frac{\varrho u^3}{2}\, dy \sim x \qquad \frac{F}{L} = -2\int_0^\infty \Delta\gamma\, dy \sim x^0$$

$$\frac{W}{L} = 2\int_0^\infty u\,\Delta\gamma\, dy = 2 C_x$$

Herefrom it is seen that the flux of volume, the flux of momentum, and the flux of kinetic energy must increase continuously with elevation, for the force producing the convective motion is the same at all successive levels. As the flux of the incremental weight is also the same at every level, the quantity W must indicate the output of the source.

The corresponding analysis of the axially symmetric pattern of convection above a point source proceeds in a closely comparable manner. The resulting integral equations of momentum, diffusion, and energy have the forms

$$\frac{d}{dx}\int_0^\infty \varrho u^2 r\, dr = -\int_0^\infty \Delta\gamma\, r\, dr \qquad (7)$$

$$\frac{d}{dx}\int_0^\infty u\,\Delta\gamma\, r\, dr = 0 \qquad (8)$$

$$\frac{d}{dx}\int_0^\infty \frac{\varrho u^3}{2} r\, dr = -\int_0^\infty u\,\Delta\gamma\, r\, dr - \int_0^\infty \tau \frac{\partial u}{\partial r} r\, dr \qquad (9)$$

Through the assumption of dynamic similarity of the mean flow at all elevations it is found that

$$\sigma \sim x \qquad u_{max} \sim x^{1/3} \qquad \Delta\gamma_{max} \sim x^{-5/3}$$

The following expressions are then obtained:

$$Q = 2\pi \int_0^\infty u\, r\, dr \sim x^{5/3} \qquad M = 2\pi \int_0^\infty \varrho u^2 r\, dr \sim x^{4/3}$$

$$E = 2\pi \int_0^\infty \frac{\varrho u^3}{2} r\, dr \sim x \qquad F = -2\pi \int_0^\infty \Delta\gamma\, r\, dr \sim x^{2/3}$$

$$W = 2\pi \int_0^\infty u\,\Delta\gamma\, r\, dr = 2\pi C_x'$$

Owing to the differences between these expressions and their counterparts for the two-dimensional case, basically different relationships are indicated for the flux of the several characteristic quantities past successive planes above a point source. Although the convection pattern again expands linearly with height above the source, the maximum velocity (and hence that for any constant value of r/x) now decreases continuously, and the incremental weight density decreases even more rapidly. On the other hand, the buoyant force on a stratum of unit thickness becomes steadily greater with increasing elevation, and Q, M, and E all increase at different rates. Once again, however, the flux of the incremental weight is the same at all elevations, so that W

must be equivalent to the output of the source.

Experimental Procedure and Results

In any experimental program to determine explicit relationships for either two-dimensional or axially symmetric convection from a boundary source, there are evidently four variables which can be controlled independently — two coordinate positions, the mass density of the fluid, and the strength of the source. These four quantities govern the magnitudes of such dependent variables as velocity, weight density, shear, turbulence, and their various combinations. While all of these factors would be of interest in a complete investigation of the phenomenon, attention is focused in the present study upon the major characteristics of the mean motion: the velocity component u and the change $\Delta\gamma$ in weight density. The resulting functional relationships between the dependent and independent variables for the two types of source can be indicated schematically as follows:

$$\begin{matrix} u \\ \Delta\gamma \end{matrix} = f_{1-4}\left(x, \begin{matrix} y \\ r \end{matrix}, \varrho, \begin{matrix} W/L \\ W \end{matrix}\right)$$

Inasmuch as five variables involving three fundamental dimensions appear in each of these four functions, it is to be expected that the dimensionless form of each will contain only two terms. Thus, for two-dimensional conditions, the relationships may be expressed as

$$\frac{u}{(W/L\varrho)^{1/3}} = \varphi_1\left(\frac{y}{x}\right)$$

$$\frac{\Delta\gamma}{(\varrho W^2/L^2x^3)^{1/3}} = \varphi_2\left(\frac{y}{x}\right)$$

For conditions of axial symmetry, the counterparts of these expressions are

$$\frac{u}{(W/\varrho x)^{1/3}} = \varphi_3\left(\frac{r}{x}\right)$$

$$\frac{\Delta\gamma}{(\varrho W^2/x^5)^{1/3}} = \varphi_4\left(\frac{r}{x}\right)$$

In each instance the grouping of terms is seen to be in accordance with the general relationships obtained analytically.

Experiments with heated air to determine the forms of these functions were conducted in closed rooms, particular care being required in the study of axially symmetric conditions to eliminate all sources of disturbance. In this instance the room was approximately circular in plan and had a diameter of 25 feet and an overall height of 11 feet. The heat source was placed at the midpoint of a centrally located platform 8 feet in diameter and a few inches in height. For the two-dimensional study the flow was confined between two parallel walls 4 feet high, 8 feet long, and 4 feet apart, and the source was placed across the midsection of a low platform extending the length of the walls. The heat sources in both cases consisted of recessed gas burners yielding low, blue flames approaching as nearly as practicable the desired point and line concentrations.

Measurements of the temperature distribution were made by means of a copper-constantan thermocouple, with its cold junction placed well away from the heat source, in combination with a potentiometer reading to 0.002 millivolt. Velocity indications were obtained with a specially constructed vane anemometer $1^1/_4$ inch in diameter, the jewel bearings of which permitted the indication of velocities as low as 0.2 foot per second. These instruments were mounted alternately on remotely controlled traversing mechanisms.

The rates of heat output from both the line and the point source were evaluated from the measured distributions of velocity and temperature according to the following thermal counterparts of the equations for W/L and W:

$$\frac{H}{L} = 2c_p\varrho\int_0^\infty u\,\Delta T\,dy \tag{10}$$

$$H = 2\pi c_p\varrho\int_0^\infty u\,\Delta T\,r\,dr \tag{11}$$

for the closely approximate condition that $\Delta\gamma/\gamma = -\Delta T/T$, the corresponding value of W in either case was computed from the resulting conversion equation,

$$W = -\frac{gH}{c_pT} \tag{12}$$

Supplementary tests performed during the axially symmetric study were directed toward

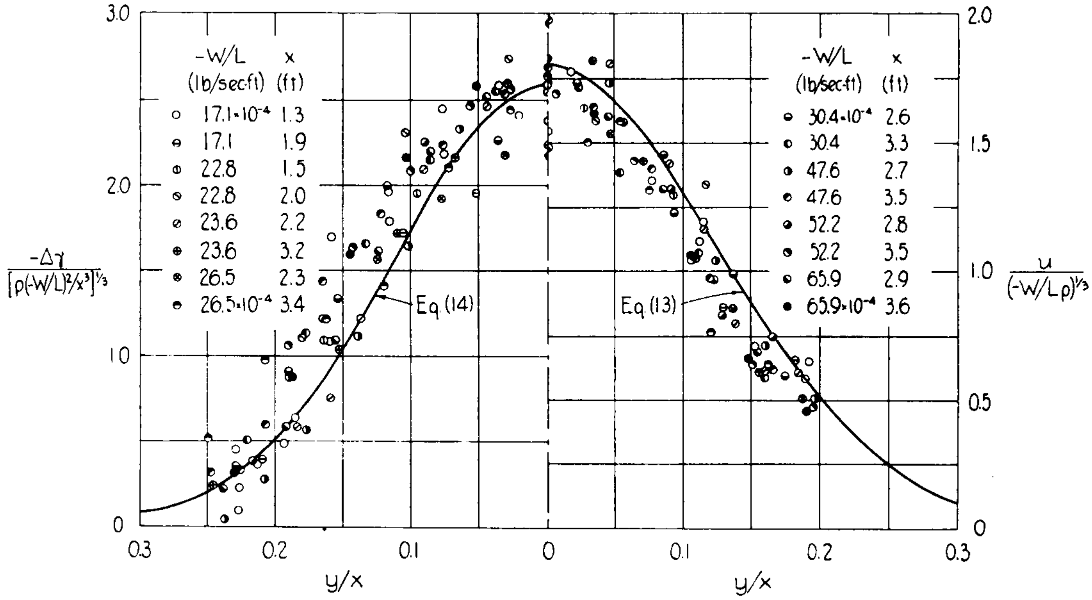

Fig. 2. Distribution functions for two-dimensional convection.

determination of the level at which, under ordinary circumstances, the initial laminar flow could be expected to become unstable. Because of the low rates of heat input required to yield readily measurable conditions of transition, it was necessary to use as the limiting heat source a burning cigarette (from a package carefully calibrated for rate of heat release), the smoke serving as a convenient tracing agent. The critical Reynolds number determined in this manner was of the form

$$\mathbf{R}_c = \frac{x_c}{\sqrt[3]{\rho\nu^3/(-W)}} \approx 10^5$$

in which ν is the kinematic viscosity. As can be determined herefrom, the critical height x_c was small in every case for which detailed measurements were made.

All velocity and temperature traverses were carried entirely across the convection zone, the plotted curves being essentially symmetrical about the maximum but often indicating a slight deviation of the flow from the vertical. In analyzing the results, the reference axis was adjusted to agree with the axis of symmetry. Upon reduction of data to the pertinent dimensionless forms, composite plots were made of velocity and weight distributions for both the two-dimensional and the axially symmetric cases, to verify the assumed similarity of flow at different levels and with different strengths of source and to determine the explicit distribution functions. For ease in comparison as well as economy of space, these plots are shown in half section: in Fig. 2 for two dimensions and in Fig. 3 for axial symmetry.

At once apparent is the appreciable scatter of points which seems to typify experimental studies of this nature — in part because of the difficulty of precise measurement and in part because of the sensitivity of such flow to slight disturbances. The scatter is not appreciably systematic, however, nor is it difficult to construct mean curves to indicate the distribution functions. The curves shown are normal-probability functions which best fit the data and at the same time satisfy Eqs. (4) and (5), and Eqs. (7) and (8), respectively. At least as a first approximation, there seems little doubt that the plotted curves describe the basic free-convection phenomenon. To be remarked, however, is the fact that use of the Gaussian function — not to mention the simplified analysis itself — loses significance as y/x or r/x becomes large.

If the probability curves plotted in Fig. 2 are assumed to represent the true distributions of u and $\Delta\gamma$, the corresponding expressions

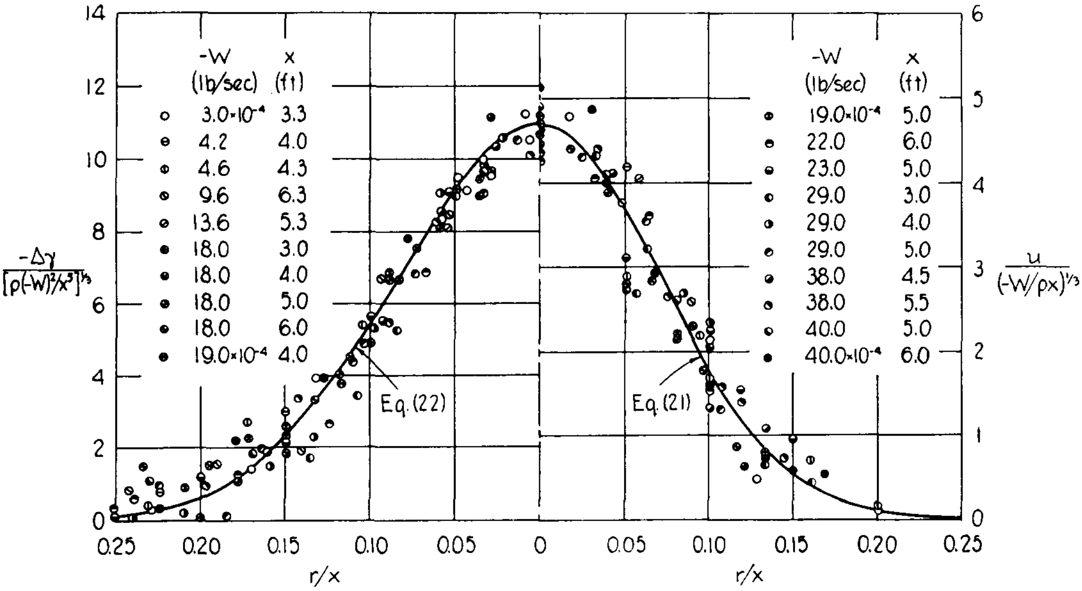

Fig. 3. Distribution functions for axially symmetric convection.

$$u = 1.80\left(\frac{-W/L}{\varrho}\right)^{1/3} exp\left(-32\frac{y^2}{x^2}\right) \quad (13)$$

$$\Delta\gamma = -2.6\left[\frac{\varrho(-W/L)^2}{x^3}\right]^{1/3} exp\left(-41\frac{y^2}{x^2}\right) \quad (14)$$

can be used conveniently to evaluate other characteristics of the mean pattern of motion. Thus, since $\psi = \int u dy$, the stream function is expressible implicitly as

$$\psi = 1.80\left(\frac{-W/L}{\varrho}\right)^{1/3}\int_0^y exp\left(-32\frac{y^2}{x^2}\right) dy \quad (15)$$

in which values of the integral are obtainable from probability tables. The lateral velocity of entrainment just beyond the diffusion zone, otherwise available from the condition that $v = -\partial\psi/\partial x$, is instead computed from the expression $2v_e = -d(Q/L)/dx$ as

$$v_e = \mp 0.28\left(\frac{-W/L}{\varrho}\right)^{1/3} \quad (16)$$

Specific relationships for Q, M, E, and F per unit length of source are obtained by integration as follows:

$$\frac{Q}{L} = 0.57\left(\frac{-W/L}{\varrho}\right)^{1/3} x \quad (17)$$

$$\frac{M}{L} = 0.72\left(\frac{-W}{L}\right)^{2/3}\varrho^{1/3} x \quad (18)$$

$$\frac{E}{L} = 0.53\,\frac{-W}{L}\,x \quad (19)$$

$$\frac{F}{L} = 0.72\left(\frac{-W}{L}\right)^{2/3}\varrho^{1/3} \quad (20)$$

For conditions of axial symmetry these expressions take the following forms:

$$u = 4.7\left(\frac{-W}{\varrho x}\right)^{1/3} exp\left(-96\frac{r^2}{x^2}\right) \quad (21)$$

$$\Delta\gamma = -11.0\left[\frac{\varrho(-W)^2}{x^5}\right]^{1/3} exp\left(-71\frac{r^2}{x^2}\right) \quad (22)$$

$$\psi = 0.024\left(\frac{-Wx^5}{\varrho}\right)^{1/3}\left[1 - exp\left(-96\frac{r^2}{x^2}\right)\right] \quad (23)$$

$$v_e = -0.041\left(\frac{-W}{\varrho}\right)^{1/3}\frac{x^{2/3}}{r} \quad (24)$$

$$Q = 0.153\left(\frac{-W}{\varrho}\right)^{1/3} x^{5/3} \quad (25)$$

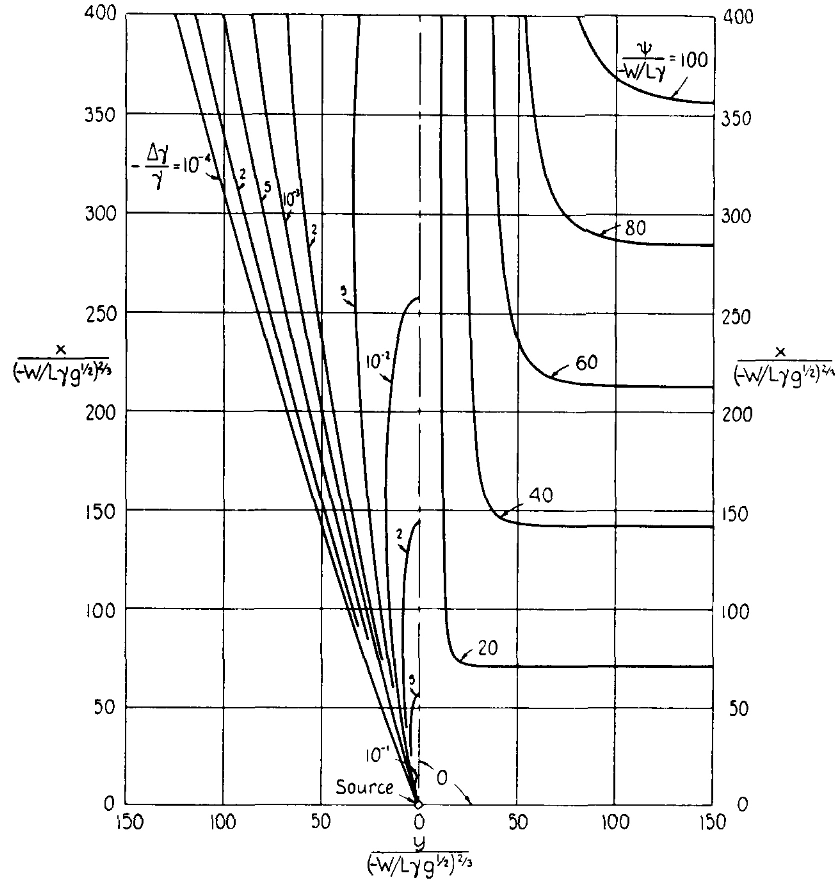

Fig. 4. Convection pattern over a line source.

$$M = 0.36\,(-W)^{2/3}\,\varrho^{1/3}\,x^{4/3} \tag{26}$$

$$E = 0.57\,(-W)\,x \tag{27}$$

$$F = 0.49\,(-W)^{2/3}\,\varrho^{1/3}\,x^{1/3} \tag{28}$$

So far as the mean flow is concerned, it remains only to indicate in generalized coordinates the pattern of stream lines and lines of constant incremental weight for the two cases. This, however, requires three rather than two dimensionless groups of variables in each instance — the additional variable which is needed being a characteristic length to serve as a reference coordinate. Such a length is found in the quantity $(W/L\gamma g^{1/2})^{2/3}$ for two dimensions, and in the corresponding quantity $(W/\gamma g^{1/2})^{2/5}$ for axial symmetry. Upon introduction of these quantities, the dimensionless coordinate relationships for ψ and $\Delta\gamma$ become, for the line source,

$$\frac{\psi}{W/L\gamma} = \varphi_5\left[\frac{x}{(W/L\gamma g^{1/2})^{2/3}},\ \frac{y}{(W/L\gamma g^{1/2})^{2/3}}\right]$$

$$\frac{\Delta\gamma}{\gamma} = \varphi_6\left[\frac{x}{(W/L\gamma g^{1/2})^{2/3}},\ \frac{y}{(W/L\gamma g^{1/2})^{2/3}}\right]$$

and, for the point source,

$$\frac{\psi}{W/\gamma} = \varphi_7\left[\frac{x}{(W/\gamma g^{1/2})^{2/5}},\ \frac{r}{(W/\gamma g^{1/2})^{2/5}}\right]$$

$$\frac{\Delta\gamma}{\gamma} = \varphi_8\left[\frac{x}{(W/\gamma g^{1/2})^{2/5}},\ \frac{r}{(W/\gamma g^{1/2})^{2/5}}\right]$$

The corresponding plots of stream lines and lines of constant incremental weight are shown in half section in Figs. 4 and 5.

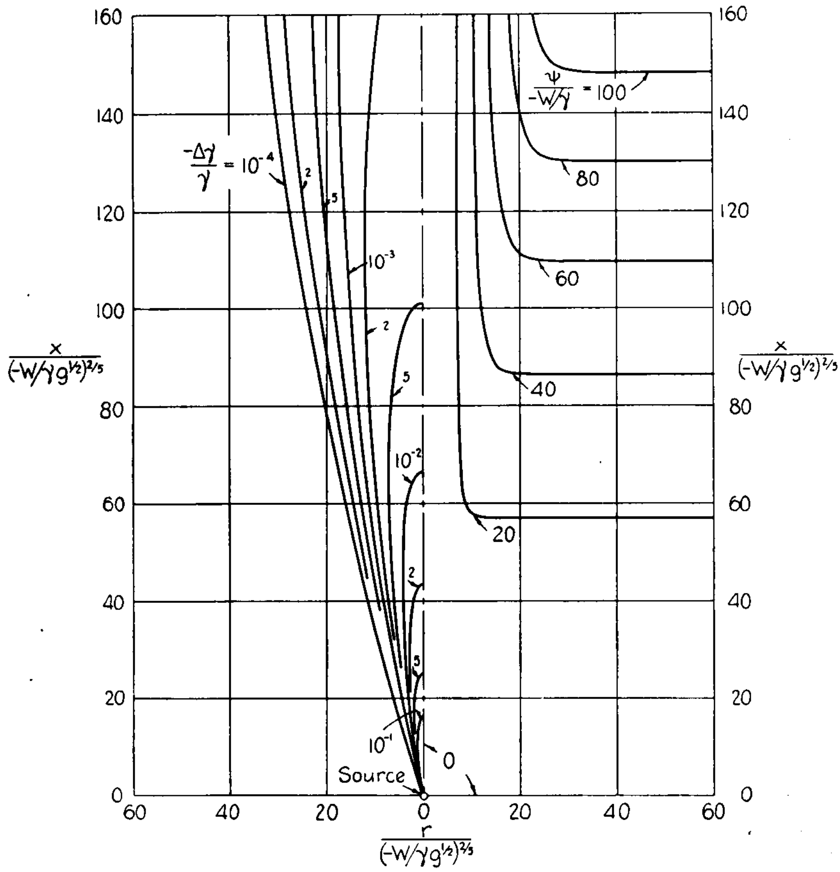

Fig. 5. Convection pattern over a point source.

As can be seen at once from these undistorted diagrams, the zones of pronounced vertical motion and change in weight density are concentrated about the area of symmetry and expand only slowly with elevation. Because of the fact that the measurements were restricted to these zones and that the analysis itself is not exact, the outlying portions of the stream-line patterns are purely schematic. Aside from this qualitative indication of the pattern of inflow, the diagrams are considered to represent with good approximation the basic characteristics of convection at any scale. They may readily be interpolated or extrapolated, as all curves in any diagram are geometrically similar by virtue of the dynamic similarity of the convection mechanism at successive elevations.

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Conclusions

From the foregoing analytical and experimental studies of gravitational convection from both a line source and a point source, the following conclusions may be drawn:

(1) The motivating force in gravitational convection is the positive or negative buoyancy of the fluid, and the primary characteristics of the convection pattern can hence be analyzed without regard to the means whereby the buoyant effect is produced.

(2) Through use of an approximate method of analysis similar to that for boundary layers, wakes, and jets, it is possible to express as proportionalities the mean characteristics of gravitational convection for both

two-dimensional and axially symmetric turbulent motion.[1]
(3) Laboratory measurements of the mean convection patterns above boundary sources of heat have verified the approximate analysis and provided explicit relationships for the primary flow characteristics.
(4) Although both the analysis and the experiments represent a considerable simplification of the actual state of motion, the correlated results should permit evaluation of the role played by the primary convection process in more involved natural phenomena.

[1] Shortly before proof of the foregoing paper was received, there came to the authors' attention a German wartime publication on the same subject, the analysis proceeding from assumptions as to the secondary rather than the primary characteristics of the motion: SCHMIDT, W., 1941, Turbulente Ausbreitung eines Stromes erhitzter Luft, *Z. angew. Math. Mech.* **21**, 265—278, 351—363.

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