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An alternative view on the role of the β-effect in the Rossby wave propagation mechanism

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ABSTRACT

The role of the β-effect in the Rossby wave propagation mechanism is examined in the linearised shallow water equations directly in momentum–height variables, without recourse to potential vorticity (PV). Rigorous asymptotic expansion of the equations, with respect to the small non-dimensionalised β parameter, reveals in detail how the Coriolis force acting on the small ageostrophic terms translates the geostrophic leading-order solution to propagate westward in concert. This information cannot be obtained directly from the conventional PV perspective on the propagation mechanism. Furthermore, a comparison between the β-effect in planetary Rossby waves and the sloping-bottom effect in promoting topographic Rossby waves shows that the ageostrophic terms play different roles in the two cases. This is despite the fact that from the PV viewpoint whether the advection of mean PV gradient is set up by changes in planetary vorticity or by mean depth is inconsequential.

Keywords: Rossby wave propagation, momentum-pressure perspective, role of ageostrophic flow, beta plane, sloping bottom

1. Introduction

The Rossby wave is a cornerstone concept in geophysical fluid dynamics, and a comprehensive understanding of the physical mechanisms that control Rossby wave propagation is therefore of central importance to the field. Since its introduction (Rossby et al., 1939), the Rossby wave has been generally understood from the vorticity perspective: the circulation associated with a potential vorticity (PV) anomaly advects a background PV gradient – such as the planetary vorticity gradient β – and generates fresh anomalies in quadrature with the original anomaly, yielding propagation to the left of the background vorticity gradient. This is the conventional view of Rossby wave motion, presented in all textbooks. While this perspective is mathematically simple and physically intuitive, it yields little insight into the fundamental forces at play or into the role played by divergence, which requires an examination of Rossby wave dynamics directly in momentum–pressure space. This problem has received much less attention, notable exceptions being (Gill 1982, Section 12) and Cai and Huang (2013). Our aim here is to provide a simple yet rigorous derivation of Rossby waves directly from the momentum and continuity equations, which is not offered in Gill (1982), together with a simpler physical picture than that presented in Cai and Huang (2013).

The relevant small parameter intrinsic to the β-plane approximation is $\tilde{\beta} \equiv \beta L/f_0 = \Delta f/f_0$, where L is a meridional length scale over which the Coriolis parameter variation Δf is small compared to its central value f_0. $\tilde{\beta}$ can be regarded as the Rossby number measuring the ratio between the Rossby wave and Coriolis frequencies, $O(\tilde{\beta}) = O(\omega_{Ro}/f_0)$.

In Section 2, it is shown that in the context of a shallow-water β plane model, an asymptotic expansion in powers of $\tilde{\beta}$ readily leads to the conventional Rossby wave dispersion relation. More importantly, each of the terms in the expansion has a simple physical interpretation, giving insight into the propagation mechanism and the role of the divergence field. In the zeroth order the solution is in stationary geostrophic balance, and the ageostrophic terms responsible for propagation only come in at first order. The solution is thus automatically quasi-geostrophic, and the propagation is slow (order $\tilde{\beta}$) because the mechanisms responsible for propagation are entirely contained in the higher-order corrections and are the residual result of opposing effects. We also show (Section 3) that a very

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similar approach can be taken for topographic Rossby waves, that is, in the case of a f plane shallow-water model with a sloping bottom, in which the relevant small parameter is the non-dimensional measure of the bottom slope. In Section 4, we compare the conventional PV perspective with the present alternative perspective in more detail. Section 5 presents a summary and conclusions.

2. Rossby waves in a shallow water β-plane model

We begin with the linearised shallow water equations in a mid-latitude β plane (Gill, 1982):

$$\frac{\partial u}{\partial t} = (f_0 + \beta y)\, v - g\frac{\partial h}{\partial x} \tag{1a}$$

$$\frac{\partial v}{\partial t} = -(f_0 + \beta y)\, u - g\frac{\partial h}{\partial y} \tag{1b}$$

$$\frac{\partial h}{\partial t} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{1c}$$

We non-dimensionalise the equations by choosing a horizontal length scale L (for which the meridional variation of the Coriolis parameter is small), a vertical scale H and $1/f_0$ as the time scale, yielding:

$$\frac{\partial u}{\partial t} = (1 + \tilde{\beta} y)\, v - Bu\,\frac{\partial h}{\partial x} \tag{2a}$$

$$\frac{\partial v}{\partial t} = -(1 + \tilde{\beta} y)\, u - Bu\,\frac{\partial h}{\partial y} \tag{2b}$$

$$\frac{\partial h}{\partial t} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{2c}$$

where $Bu = (L_d/L)^2$ is the Burger number, and $L_d = \sqrt{gH}/f_0$ is the deformation radius.

Following Heifetz et al. (2007, hereafter H07), we perform an asymptotic expansion with respect to $\tilde{\beta}$. Since eqs. (2a–2c) are homogeneous in time, we can assume solutions with time dependence $e^{-i\omega t}$, and we will also expand the frequency ω:

$$u(x, y, t) \;=\; e^{-i(\omega_0 + \tilde{\beta}\omega_1 + \ldots)t}\,(u_0 + \tilde{\beta} u_1 + \ldots) \tag{3}$$

with analogous expressions for v and h. Substituting in eqs. (2a–2c) and gathering zero-order terms, we obtain

$$-i\omega_0 u_0 = v_0 - Bu\,\frac{\partial h_0}{\partial x} \tag{4a}$$

$$-i\omega_0 v_0 = -u_0 - Bu\,\frac{\partial h_0}{\partial y} \tag{4b}$$

$$-i\omega_0 h_0 = -\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right), \tag{4c}$$

which are simply the linearised shallow water equations on a f plane. As is well known (e.g. H07) these equations have plane-wave solutions $e^{i(kx + \ell y - \omega_0 t)}$ with zonal and meridional wavenumbers k and l, yielding a cubic dispersion relation with two non-zero roots – corresponding to Poincare waves, which we will not consider further here – and a third root $\omega_0 = 0$ corresponding to stationary, non-divergent geostrophic balance:

$$\mathbf{u}_0 = Bu\,\mathbf{z} \times \nabla h_0, \tag{5}$$

where $\mathbf{z}$ is the unit vertical vector.

As in Rossby et al. (1939), we focus on the simplest form of the wave, in which the phase lines are oriented north-south so that the zero-order structure is independent of y:

$$u_0 = 0 \tag{6a}$$

$$v_0 = Bu\, ikh_0 \tag{6b}$$

$$h_0 = \hat{h}_0 e^{i(kx - \omega t)} \tag{6c}$$

where $\hat{h}_0$ is a constant. The infinite meridional extent is physically inconsistent on a sphere but as discussed in Appendix A, the basic understanding obtained from this simple solution can be extended to narrow channels.

Rossby wave motion is obtained by considering the first-order terms in the expansion of eqs. (2a–2c) and setting $\omega_0 = 0$:

$$-i\omega_1 u_0 = y v_0 + v_1 - Bu\,\frac{\partial h_1}{\partial x} \tag{7a}$$

$$-i\omega_1 v_0 = -y u_0 - u_1 - Bu\,\frac{\partial h_1}{\partial y} \tag{7b}$$

$$-i\omega_1 h_0 = -\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right). \tag{7c}$$

If we decompose the first-order velocity field into geostrophic and ageostrophic components[1]

$$\mathbf{u}_1 = \mathbf{u}_1^g + \mathbf{u}_1^a \text{ with } \mathbf{u}_1^g = Bu\mathbf{z} \times \nabla h_1 \tag{8}$$

and substitute (4a–4c) and (8) into (5), we find the solutions

$$u_1^a = i\omega_1 v_0 \tag{9a}$$

$$v_1^a = -y v_0 \tag{9b}$$

$$\omega_1 = \frac{-k}{k^2 + Bu^{-1}}. \tag{9c}$$

In particular, eq. (9c) implies that the frequency, up to first-order accuracy, in dimensional form is

$$\omega_{Ro} = \frac{-\beta k}{k^2 + L_d^{-2}}, \tag{10}$$

which is the classical Rossby wave dispersion relation. Hence, it is the interaction between the zero-order geostrophic solution and the first-order ageostrophic component

[1]In order to avoid confusion in definitions with H07 we hereafter refer to the 'geostrophic' component when is f evaluated by f_0.

that determines the Rossby wave propagation. Note that the second-order terms (together with boundary conditions) are required to determine the first-order height field h_1 and its associated geostrophic velocity components u_1^g, v_1^g. However, these first-order terms affect only on the third-order correction to the dispersion relation (see H07) and are beyond the scope of this note.

Solution (6) is sketched in Fig. 1 (which is equivalent to Figure 12.2 Gill, 1982) to explain how the zero-order fields propagate to the west in concert. The ageostrophic zonal correction u_1^a added in phase with h_0 enables the residual Coriolis force to translate v_0 a quarter of a wavelength to the west. Nonetheless, it is evident from Fig. 1 and (7c) that the contribution of u_1^a to the divergence acts to shift the height field h_0 *eastward*. Hence, such a structure cannot propagate coherently on the f plane. This is where β must

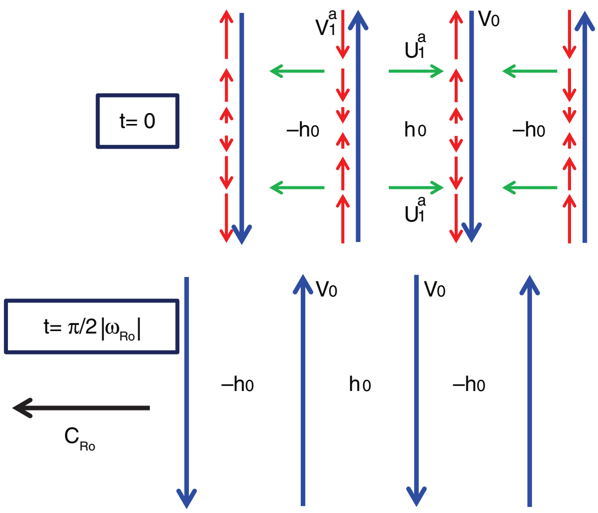

Fig. 1. Schematic illustration of the Rossby wave propagation mechanism on a β-plane for the case where the zero-order geostrophic balance is independent of latitude [eqs. (6a–6c)]. Long vertical arrows represent the zero-order geostrophic meridional velocity v_0, whereas short vertical arrows represent their first-order ageostrophic correction v_1^a. The first-order ageostrophic zonal component u_1^a, represented by the short horizontal arrows, is in phase with the zero-order height (pressure) anomaly h_0. The interaction between the zero-order geostrophic field and the first-order ageostrophic correction explains the westward propagation of the zero-order field (illustrated by being shifted by a quarter of a wavelength, after a quarter of a period of $T/4 = \pi/(2|\omega_{Ro}|)$. The residual Coriolis force acting on u_1^a shifts v_0 westward, but its contribution $\frac{\partial u_1^a}{\partial x}$ to the divergence field, acts to shift h_0 *eastward.* v_1^a is in (out of) phase with v_0 at low (high) latitudes. This keeps the Coriolis force constant on the β-plane so it can be balanced by the constant PGF. Its contribution to the divergence field $\frac{\partial v_1^a}{\partial y}$, opposes and overwhelms $\frac{\partial u_1^a}{\partial x}$, therefore allows h_0 to propagate westward coherently with the zero-order momentum field.

come into play: on the β plane, an ageostrophic meridional correction v_1^a must be added to (subtracted from) v_0 at lower (higher) latitudes [Fig. 1 and (7a)] in order to keep the Coriolis force constant in y, so it can be balanced by the zero-order zonal pressure gradient force (PGF) which is also meridionally uniform, thus keeping $u_0 = 0$. Quite remarkably, this very same structure generates the required divergence component $\partial v_1^a/\partial y$, to oppose and overwhelm $\partial u_1^a/\partial x$, and thus enable the westward propagation of h_0.

This combined momentum–height (pressure/mass) description of how the β-effect ensures the coherent propagation of the zero-order geostrophic fields may be regarded as an alternative perspective to the classical PV one. From the PV perspective, it is clear how the planetary vorticity gradient breaks the symmetry and prevents an eastward propagation to the right of it. This is also clear from the momentum–height one – flipping the arrows of $\mathbf{u}_1^a$ in Fig. 1 yields a coherent eastward propagation of v_0 and h_0, but it also generates a non-vanishing u_0 field.

The momentum–height description explains why Rossby waves are relatively 'slow': the mechanisms that translate both the momentum and height fields are residual. For the momentum, it is the residual imbalance between the Coriolis force and the PGF, whereas for the height it is the residual imbalance between the divergence components along and perpendicular to the direction in which the Coriolis parameter varies. This stands in contrast with the 'fast' Poincare waves propagation mechanism on the f plane. There, both the PGF and the Coriolis force and the two components of the divergence act constructively to translate the momentum and the height fields.

It also explains why non-divergent Rossby waves are faster than divergent Rossby waves, or more generally why the frequency increases as L_d and thus Bu increase. For a given zonal wavenumber, the larger the amplitude of u_1^a the faster is the meridional velocity propagation and hence the propagation of the wave as a whole. When the flow is divergent, the zonal component of the divergence $\partial u_1^a/\partial x$ must be smaller in magnitude than the meridional component $\partial v_1^a/\partial y$ in order to yield a non-zero total divergence and allow h to change. However, the meridional convergence is fixed by the β effect, and it is therefore the zonal component that is flexible. In fact, it can be seen from eq. (6) that

$$\frac{\partial u_1^a/\partial x}{\partial v_1^a/\partial y} = \frac{-k^2}{k^2 + Bu^{-1}} = k\omega_1 \qquad (11)$$

so that $\partial u_1^a/\partial x$ (and therefore u_1^a) increases with Bu, reaching its maximum value in the limit $Bu \to \infty$ when the flow is non-divergent.

3. Topographic Rossby waves

On an f plane with a meridionally sloping bottom such that $H(y) = H_0 - \gamma y$, the linearised equations of motion become:

$$\frac{\partial u}{\partial t} = f_0 v - g\frac{\partial h}{\partial x} \tag{12a}$$

$$\frac{\partial v}{\partial t} = -f_0 u - g\frac{\partial h}{\partial y} \tag{12b}$$

$$\frac{\partial h}{\partial t} = -(H_0 - \gamma y)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \gamma v \tag{12c}$$

Note that there is now an additional term in the continuity eq. (7c), representing the advection of the background height gradient; as shown below, this is the term responsible for Rossby wave motion in the present case, playing the role of the β terms in eq. (1).

Non-dimensionalising as in Section 2, we obtain

$$\frac{\partial u}{\partial t} = v - Bu\frac{\partial h}{\partial x} \tag{13a}$$

$$\frac{\partial v}{\partial t} = -u - Bu\frac{\partial h}{\partial y} \tag{13b}$$

$$\frac{\partial h}{\partial t} = -(1 - \tilde{\gamma} y)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \tilde{\gamma} v \tag{13c}$$

where

$$\tilde{\gamma} = \frac{\gamma L}{H_0} = -\frac{\Delta H}{H_0} \tag{14}$$

is the small parameter of the problem. As before, we expand the variables in powers of $\tilde{\gamma}$. At zero order, we re-obtain eq. (3), and focus again on the zonally-propagating plane-wave solution (4). At first order, the equations become

$$-i\omega_1 u_0 = v_1^a \tag{15a}$$

$$-i\omega_1 v_0 = -u_1^a \tag{15b}$$

$$-i\omega_1 h_0 = -\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) + v_0 \tag{15c}$$

whose solutions are

$$u_1^a = i\omega_1 v_0 \tag{16a}$$

$$v_1^a = 0 \tag{16b}$$

$$\omega_1 = \frac{-k}{k^2 + Bu^{-1}}. \tag{16c}$$

Comparing with the dynamics on the β-plane (Fig. 1) the picture is the same but $v_1^a = 0$. The ageotrophic zonal wind u_1^a is responsible for translating v_0 westward, but tends to translate h_0 eastward. The latter tendency is overwhelmed by the meridional advection at the zero order of deep water from the south and shallow water from the north, instead of convergence due to ageostrophic meridional wind as in the β plane.

4. Discussion

The aim of this note is to provide a simple yet rigorous mathematical derivation and associated physical interpretation of Rossby waves directly from the momentum and continuity equations. To appreciate what is gained by taking this perspective, it is useful to contrast with the conventional approach – which dates back to the original work by Rossby et al. (1939) – in which the equations are combined to form expressions for the evolution of PV and divergence. In the β-plane case, these are

$$\frac{\partial q}{\partial t} = -\tilde{\beta} v \tag{17a}$$

$$\frac{\partial \delta}{\partial t} = (1 + \tilde{\beta} y)\,\zeta - \tilde{\beta} u - Bu\nabla^2 h, \tag{17b}$$

where ζ and δ are respectively vorticity and divergence while $q = \zeta - (1 + \tilde{\beta} y)h$ is the linearised PV. In the sloping-bottom f plane case, we have instead

$$\frac{\partial q}{\partial t} = -\tilde{\gamma} v \tag{18a}$$

$$\frac{\partial \delta}{\partial t} = \zeta - Bu\nabla^2 h, \tag{18b}$$

where now $q = (1 - \tilde{\gamma} y)\zeta - h$. Expanding in powers of the small parameter as previously, to zero order (11b) and (12b) both give $\zeta_0 = Bu\nabla^2 h_0$, which expresses non-divergent geostrophic balance, while to first order (11a) and (12a) both give $-i\omega_1(\zeta_0 - h_0) = -v_0$, from which the Rossby wave dispersion relation follows.

Thus, the classical Rossby wave dispersion relation solution in the PV approach only involves structures of the zero-order terms, which makes the β-plane and sloping-bottom cases completely indistinguishable. The physical interpretation, which was discussed in the Introduction, is also identical for both cases and involves only the meridional advection of the mean PV gradient – whether that gradient is set up by changes in planetary vorticity or mean depth is inconsequential. The price to be paid for the simplicity and unity of this viewpoint is a loss of information: the PV approach yields no insight into the role played by the ageostrophic perturbation in shifting the wave westward, which instead are fundamental in our alternative approach. Note also that the ageostrophic terms are quite different in the β plane and sloping-bottom cases, so that the two cases are not isomorphic from the momentum–pressure perspective.

Information about the ageostrophic terms is of course not truly 'lost' in the PV perspective, since the vorticity and divergence contain the same information as the raw wind-field. The first-order ageostrophic terms can in fact be derived from the first-order expansion of the divergence equation, but this is cumbersome as it requires an inverse

Helmholtz decomposition. The ageostrophic terms are much more easily and intuitively accessible through the momentum equations, as we have shown here.

It is important to note that the above analysis only explains how the zero-order structure propagates with the Rossby phase speed due to its interaction with the first-order ageostrophic correction. In order to explain how the latter propagates, one must explore its interaction with the second-order correction. This interaction adds a third-order correction to the dispersion relation and is beyond the scope of this note. Furthermore, in order to find the complete first-order correction fields (the missing geostrophic component which is irrelevant to the discussed mechanism), one should incorporate the suitable boundary conditions on the channel walls (see H07).

The Rossby dispersion relation can be obtained alternatively from manipulating equation set (1) to obtain a single homogeneous third-order equation for the meridional velocity (e.g. Vallis, 2014). After approximating f by f_0 but leaving β as a constant, the solution for a plane wave yields a cubic equation for the frequency with constant coefficients. The high frequency limit provides the Poincare frequencies and the low limit the Rossby one. For completeness, we show in Appendix B how a rigorous expansion of this equation in $\tilde{\beta}$ provides the Rossby dispersion relation and the associated wave structure.

Paldor et al. (2007) and Paldor and Sigalov (2008) found that non-harmonic trapped Rossby waves may have much faster phase speeds (by a factor of four) than the zero-order harmonic waves discussed here. Recently, Paldor et al. (private communication) analysed oceanic altimetry data and found such fast Rossby waves, trapped to the south Australian shores. It will therefore be interesting to analyse this fast propagation mechanism from the momentum–pressure perspective.

5. Summary and conclusions

We have presented a simple yet rigorous derivation of Rossby waves directly from the momentum and continuity equations based on an asymptotic expansion in terms of the non-dimensional β parameter (in the β-plane case) or mean height gradient (in the f plane case with a sloping bottom). As opposed to the conventional PV explanation, it shows in detail how the Coriolis force acting on the small ageostrophic terms translates the geostrophic leading-order solution to propagate westward in concert. In the β-plane, this mechanism can be summarised as follows: in order for the zero-order harmonic wave to maintain its structure when the Coriolis parameter increases with latitude, a first-order correction to the meridional velocity field is required to keep the Coriolis force harmonic to zero order; this correction in turn generates a cross-isobaric flow which leads to westward propagation of the zero-order height and velocity fields. In the sloping-bottom case, the meridional advection of the background height gradient requires cross-isobaric flow in order to maintain the height field harmonic, again leading to westward propagation.

This physical picture is not entirely new, being already present in Gill (1982) and in somewhat different form in Cai and Huang (2013). However, Gill (1982) only provides a qualitative discussion, while Cai and Huang (2013) provide full mathematical solutions but the treatment is highly complex. Our aim here was to marry the simple physical insight to a correspondingly simple mathematical treatment, and demonstrate that the momentum–pressure perspective is as accessible, intuitive and satisfying as the conventional PV perspective. Whether one considers the PV or the momentum–pressure perspective to be the more fundamental is a question of taste; we have strived here to show that they provide alternative and complementary insight into Rossby waves, and deserve to be treated on an equal footing.

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7. Appendices

A. Bounded Rossby waves

While a channel wave guide is not necessarily a realistic feature in the mid-latitudinal atmosphere, it is consistent with the β-plane approximation which requires a small meridional extension. This motivated H07 to solve analytically the first-order correction to a zero-order harmonic wave on a channel:

$$v_0 = \hat{v}_0 e^{(ikx-\omega t)} \cos \ell y, \ell = \frac{(2n+1)}{L}\frac{\pi}{2} \tag{A1}$$

where the quantisation of the meridional wavenumber (n is an integer) ensures the vanishing of the meridional velocity on the channel 'walls' at the zero order, $v_0(y = \pm L) = 0$. The associated first-order correction to eq. (5) that also satisfies the boundary conditions is somewhat complex [equation set (20) and Figs. 1, 2 in H07]; however, it obeys the following basic principles obtained from the infinite channel width solution. Up to the first order, the Rossby dispersion relation is indeed $\omega_1 = -k/[(k^2 + l^2) + Bu^{-1}]$ and is influenced only by the interaction between the zero-order dynamics and by the ageostrophic component of the first-order correction.[2] The role of the first-order ageostrophic component is double. It first ensures the undistorted propagation of the zero-order momentum of the harmonic wave, despite the meridional linear variation of the Coriolis force. This requires the ageostrophic solution to be composed of a combination of harmonic and non-harmonic structures [equation set (22) in H07]. Second, the harmonic component of the divergence field acts to shift the zero-order height field h_0, *eastward*, however this tendency is overwhelmed by the non-harmonic convergence part that shifts h_0 westward, in concert with the zero-order momentum fields.

B. Expansion in $\tilde{\beta}$ of the meridional velocity third-order equation

Equation set (1) can be transformed into a single homogeneous third-order partial differential equation of the meridional velocity v_0 (e.g. Vallis, 2014):

$$\left\{\frac{\partial^3}{\partial t^3} + \left[(f_0 + \beta y)^2 - gH\nabla^2\right]\frac{\partial}{\partial t} - \beta\frac{\partial}{\partial x}\right\} v = 0. \qquad \text{(B1)}$$

Expanding the meridional velocity in $\tilde{\beta}$:

$$v(x, y, t) \;=\; e^{-i(\omega_0 + \tilde{\beta}\omega_1 + \ldots)t}\,(\hat{v}_0 + \tilde{\beta}\hat{v}_1 + \ldots)e^{ikx} \qquad \text{(B2)}$$

and substitute in (B1) yields:

$$\left\{(\omega_0 + \tilde{\beta}\omega_1 + \ldots)\left[-(\omega_0 + \tilde{\beta}\omega_1 + \ldots)^2 + f_0^2(1 + \tilde{\beta}\frac{y}{L})^2 - gH\nabla^2\right] + kgH\frac{f_0}{L}\tilde{\beta}\right\}(\hat{v}_0 + \tilde{\beta}\hat{v}_1 + \ldots) = 0 \qquad \text{(B3)}$$

The zero-order expansion yields:

$$\omega_0(-\omega_0^2 + f_0^2 - gH\nabla^2)v_0 = 0 \qquad \text{(B4)}$$

and for a non-trivial v_0 solution (of say a plane wave) we obtain

$$\omega_0 = 0; \omega_0^2 Po = f_0^2 + K^2 gH \qquad \text{(B5)}$$

corresponding to the stationary, non-divergent geostrophic balance and the two non-zero Poincare roots. As pointed out by H07, the symmetry properties of system (1) implies that only a solution with zero even powers of ω ($\omega_0 = \omega_2 = \ldots = 0$) can be expanded in series of positive integer powers of $\tilde{\beta}$. Therefore, Poincare waves are zero-order solution of (B3) on the *f*-plane but require a different type of expansion.

Hence, for the $\omega_0 = 0$ branch the $O(\tilde{\beta})$ of (B3) yields:

$$\left[\omega_1(f_0^2 - gH\nabla^2) + kgH\frac{f_0}{L}\right]v_0 = 0 \qquad \text{(B6)}$$

which for $v_0 = \hat{v}_0 e^{ikx}$ admits:

$$\omega = \tilde{\beta}\omega_1 + O(\tilde{\beta}^3); \tilde{\beta}\omega_1 = -\frac{\beta k}{k^2 + L_d^{-2}} = \omega_{Ro} \qquad \text{(B7)}$$

The $O(\tilde{\beta}^2)$ provides the relation between the first and zero-order structures of v:

$$\frac{\partial^2 \hat{v}_1}{\partial y^2} = \frac{2}{L_d^2}\left(\frac{y}{L}\right)\hat{v}_0 \qquad \text{(B8)}$$

so that for unbounded domain

$$\hat{v}_1 = \left[\frac{1}{3}\left(\frac{y}{L}\right)^2 - 1\right]\left(\frac{y}{L}\right)\hat{v}_0 = \hat{v}_{1\,g} + \hat{v}_{1\,a} \qquad \text{(B9)}$$

[2]The first-order ageostrophic component has a non-vanishing meridional velocity field on the boundaries and therefore must be 'corrected' by the geostrophic first-order component of the solution. However, the latter affects only on the third-order correction to the Rossby dispersion relation. Hence, the only relevant influence of channel walls on ω at the first order is the dictation of a quantised meridional harmonic solution on the zero-order wave structure.

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