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Zonally propagating wave solutions of Laplace Tidal Equations in a baroclinic ocean of an aqua-planet

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ABSTRACT

Despite the accurate formulation of Laplace's Tidal Equations (LTE) nearly 250 years ago, analytic solutions of these equations on a spherical planet that yield explicit expressions for the dispersion relations of wave solutions have been found only for slowly rotating planets, so these solutions are of no relevance to Earth. Analytic solutions of the LTE in a symmetric equatorial channel on a rotating sphere were recently obtained by approximating the LTE by a Schrödinger equation whose energy levels yield the dispersion relations of zonally propagating waves and whose eigenfunctions determine the meridional structure of the amplitude of these waves. A similar approximation of the LTE on a sphere (with no channel walls) by a Schrödinger equation yields accurate analytic solutions for zonally propagating waves in the parameter range relevant to a baroclinic ocean, where the ratio between the radius of deformation and Earth's radius is small. For sufficiently low (meridional) modes the amplitudes of the solutions vanish at some extra-tropical latitudes but this is not assumed abinitio. These newly found solutions do not restrict the value of the zonal wavenumber to be smaller than the meridional wavenumber as is the case in previous theories on a slowly rotating sphere.

1. Introduction

The theory of tides was properly formulated in 1776 by Pierre-Simon Laplace as a two-dimensional flow of a thin layer of fluid on a rotating sphere subject to the external force associated with a gravitational potential (see the short historical review in chapter 3 of Chapman and Lindzen, 1970). Laplace's formulation followed Galileo Galilei's erroneous attribution of tides to the motion of Earth around the Sun in 1616 (Finocchiaro, 1989). A detailed account of the history of tidal theory, the evolution of its practical application and its extension to thermal tides in the atmosphere can be found in Cartwright (1999). The common application of the tidal theory, which is the numerical predictions of the times of high and low tides at any specific port is of primary importance since it determines the times when ships of given keel depth at maximum cargo can enter a port of a given water depth. Though of lesser practical importance, tides have also been observed with respect to barometric pressure changes in the atmosphere but in contrast to the ocean, atmospheric tides owe their origin to the daily cycle of heating and not to the excess of gravitational attraction by the moon and the sun.

The set of equations proposed by Laplace accounts for the conservation of momentum and mass and is known today as Laplace's Tidal Equations (LTE). In three-dimensional modelling of these equations the forcing appears in the radial coordinate so when the dynamics in this direction is separated from the horizontal dynamics (by the method of separation of variables) the separation constant turns out to be the mean height of the Shallow Water Equations that are in fact the force free counterpart of LTE (see e.g. Chapman and Lindzen, 1970). In the end of the 19th century the horizontal dynamics of the free LTE (i.e. the Shallow Water Equations; SWE, hereafter) was described in terms of two types of wave solutions given by Hough Functions (Margules, 1893; Hough, 1898). However, the solutions in terms of Hough Functions are too complicated to yield explicit expressions for the dispersion relations of these waves. In the middle of the 20th century solutions of the free LTE on the rotating Earth were calculated numerically by expanding the velocity and height variables in spherical harmonic basis functions (Longuet-Higgins, 1968) but these numerical solutions, too, have not yielded explicit expressions for the waves' dispersion relations.

In contrast to the complexity of the problem on a sphere, it was shown in Matsuno (1966) that in Cartesian coordinates on the unbounded equatorial β-plane it is possible to formulate the LTE as a Schrödinger equation (the term Schrödinger equation denotes hereafter the time-independent Schrödinger equation in one-dimension), which yields analytic solutions in which the latitudinal variation of the variables is given by Hermite Functions and explicit expressions of the dispersion relations of the various

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waves are given by the roots of a cubic relationship between the phase speeds and the energy levels of the Schrödinger equation. This analytic theory on the unbounded equatorial β-plane was extended numerically to asymmetric equatorial channels (Cane and Sarachik, 1979).

Recently, the Schrödinger equation formulation of the LTE, including the derivation of a cubic relation between the waves' phase speeds and the energy levels of the Schrödinger equation was also applied to bounded domains (channels). In Cartesian coordinates the formulation was applied to the mid-latitude β-plane by Paldor et al. (2007) and Paldor and Sigalov (2008), and to the equatorial β-plane by Erlick et al. (2007). In spherical coordinates the formulation was applied to the equator by Erlick et al. (2007) and De-Leon et al. (2010) and to the mid-latitudes by De-Leon and Paldor (2009). However, the wave theory of LTE on the entire sphere (the unbounded case in spherical coordinates), that is, in the ocean of an aqua-planet, has never previously been formulated in a way that enables the derivation of explicit expressions for the dispersion relations, which is the reason why such expressions on the entire sphere presently exist only for a slowly rotating sphere (Longuet-Higgins, 1968). In this study, we attempt to apply the Schrödinger equation formulation of the LTE once again, this time to the entire sphere.

2. Approximate solutions of the Schrödinger equation associated with LTE

In vectorial form the free LTE system (which, as explained above, is the SWE with gH replaced by a separation constant) is given by

$$\begin{aligned} &\frac{\partial \mathbf{V}}{\partial t} + f\,\mathbf{k}\times\mathbf{V} = -g'\nabla\eta \\ &\frac{\partial \eta}{\partial t} = -H'\,\nabla\cdot\mathbf{V}, \end{aligned} \tag{1}$$

where $\mathbf{V}$ is the horizontal velocity vector whose components in the zonal (λ, longitude) and meridional (ϕ, latitude) directions are u and v, respectively, f is the latitude dependent Coriolis frequency ($=2\Omega\sin\phi$, where Ω is earth's rotation frequency), H' is the reduced thickness of the baroclinic fluid (air in the atmosphere and water in the ocean), η is the deviation of total height h from H' (so $h = H'+\eta$), g' is the gravitational constant (or reduced gravity in the case of a two-layer fluid) and $\mathbf{k}$ is a unit vector in the radial (vertical) direction.

System (1) can be nondimensionalized using $(2\Omega)^{-1}$ for the time scale, H' for the height (η) scale and $2\Omega a$ (where a is Earth's radius) for the velocity scale. Writing the scalar version of system (1) in spherical coordinates and assuming zonally propagating wave solution of the form $[u(\phi),v(\phi),\eta(\phi)]e^{ik(\lambda-Ct)}$, (where k is the zonal wavenumber, C is the zonal phase speed and $\omega = Ck$ is the frequency) yields a 3×3 matrix. Defining $V \equiv iv/k$ one obtains the algebraic-like (it does contain the differential operator, $\partial/\partial\phi$) eigenvalue system

$$\begin{bmatrix} 0 & \sin\phi & \dfrac{\alpha}{\cos\phi} \\ \dfrac{\sin\phi}{k^2} & 0 & \dfrac{\alpha}{k^2}\dfrac{\partial}{\partial\phi} \\ \dfrac{1}{\cos\phi} & \tan\phi - \dfrac{\partial}{\partial\phi} & 0 \end{bmatrix} \begin{bmatrix} u \\ V \\ \eta \end{bmatrix} = C \begin{bmatrix} u \\ V \\ \eta \end{bmatrix}, \tag{2}$$

where the parameter $\alpha = g'H'\,(2\Omega a)^{-2}$ is the inverse of Lamb's number. In a baroclinic ocean, where the speed of gravity waves $[=(g'H')^{1/2}]$ is 2–3 m s^{-1}, α is about 5×10^{-6} whereas in a baroclinic atmosphere, where $(g'H')^{1/2}$ is about 20–30 m s^{-1}, α is about 5×10^{-4}. In a barotropic ocean/atmosphere the values of α fall in the range of 0.05–0.1.

System (2) can be transformed (after some tedious algebraic manipulations, see De-Leon and Paldor, 2009) to the following equation, which is the starting point for the extension of the Schrödinger equation formulation of LTE to the entire sphere

$$\alpha\frac{d^2\psi}{d\phi^2} + \left[\underbrace{k^2C^2 - \frac{\alpha}{C}}_{E} - \underbrace{\left(\sin^2\phi + \frac{\alpha k^2}{\cos^2\phi}\right)}_{U_1(\phi)} - \alpha U_2(\phi)\right]\psi = 0, \tag{3}$$

where

$$\begin{aligned} U_2(\phi) = {}& \frac{3}{4}\tan^2\phi\left(\frac{\alpha + C^2\cos^2\phi}{\alpha - C^2\cos^2\phi}\right)^2 - \frac{1}{2}\left(\frac{\alpha + C^2\cos^2\phi}{\alpha - C^2\cos^2\phi}\right) \\ & - \tan^2\phi\left(\frac{\alpha + 2C\cos^2\phi}{\alpha - C^2\cos^2\phi}\right), \end{aligned}$$

and $\psi(\phi) = V(\phi)\cos\phi/\sqrt{(\frac{\alpha}{C\cos\phi} - C\cos\phi)}$. In these expressions E is the energy and $U_1(\phi)+\alpha U_2(\phi)$ is the potential of the Schrödinger equation [where $U_1(\phi)$ has a counterpart in the β-plane theory while $U_2(\phi)$ is unique to the sphere]. The boundary conditions associated with eq. (3) in the case of the entire sphere are $\psi(\phi = \pm\pi/2) = 0$, which ensures the regularity of the meridional velocity component at the poles. Similar second order equations were derived by Chapman and Lindzen (1970) and by Longuet-Higgins (1968) but in these equations the sought frequency (that yields the dispersion relation) appears in the coefficients of the differential operator (and not in the eigenvalue, α^{-1} – the free parameter of the problem) so the following simplification of eq. (3) is not possible.

Since $U_1(\phi)$ and $U_2(\phi)$ in eq. (3) are both of order 1, for $\alpha \ll 1$ the term $\alpha U_2(\phi)$ is much smaller than $U_1(\phi)$, so that the potential of the Schrödinger eq. (3) can be accurately approximated by $U_1(\phi)$ only. The same approximation holds for α of order 1 provided $k \gg 1$ since in this case the contribution of $\alpha U_2(\phi)$ can be neglected compared to the $\alpha k^2/\cos^2\phi$ term in $U_1(\phi)$. These approximations were shown to yield accurate solutions in a zonal channel straddling the equator (De-Leon et al., 2010) and the change of domain to the entire sphere should not affect the considerations regarding the neglect of $\alpha U_2(\phi)$ compared to $U_1(\phi)$, at least in latitudes well below the poles.

An expansion of $U_1(\phi)$ to second-order in ϕ yields the potential: $U_{\text{parab}}(\phi) = \alpha k^2 + (1+\alpha k^2)\phi^2$, and this parabolic potential approximates $U_1(\phi)$ for $\phi \ll 1$ regardless of the values of α, k and C. In this limit, eq. (3) is approximated by

$$\alpha \frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} + \left[E - \alpha k^2 - (1+\alpha k^2)\phi^2\right]\psi = 0. \quad (4)$$

A quadratic potential appears in the Schrödinger equation of a harmonic oscillator in quantum mechanics (e.g. Sakurai, 1994) and the well known solutions of the latter problem provide approximate solutions to eq. (3) in terms of the permissible discrete values of the energy, E_n, as well as the corresponding functions $\psi_n(\phi)$ (where n is the mode number). Clearly, the accuracy of these approximate solutions has yet to be determined. Following the definition of $E = k^2C^2 - \alpha/C$, each value of E_n yields three values of the phase speed C via the roots of the cubic $k^2C^3 - E_nC - \alpha = 0$. An approximate expression for the two large (in absolute value) roots of this cubic equation are obtained by considering α to be negligible compared to the other two terms of the cubic, which yields $C^2 = E_n/k^2$. These fast moving waves are the well-known Inertia-Gravity (also known as Poincaré) waves. An approximate expression for the small (in absolute value) root is obtained by neglecting the k^2C^3 term in the cubic, which yields $C = -\alpha/E_n$. This third root is the phase speed of the Planetary (also known as Rossby) wave.

In addition to determining the dominance of $U_1(\phi)$ in the potential of eq. (3) α also determines the rate at which the solution $\psi(\phi)$ varies with ϕ. This is a direct result of its presence as a multiplier of the second derivative term (this becomes obvious when one considers the simple case of a uniform potential). Thus, for small α the range of ϕ over which $\psi(\phi)$ varies [i.e. the wavelength in the case of an oscillating $\psi(\phi)$ and the length scale of decay/growth in the case of a decaying/growing $\psi(\phi)$] gets smaller as α decreases.

In general, given a particular potential $U(\phi)$ the energy levels, E_n, that solve the Schrödinger equation are related to the structure of the corresponding solution functions $\psi_n(\phi)$ as follows: $\psi_n(\phi)$ is oscillatory in the range of ϕ where $E_n - U(\phi) > 0$ and decays/grows (faster than exponential) in the range of ϕ where $E_n - U(\phi) < 0$ [and the latitude, ϕ_{turn}, where $E_n = U(\phi_{\text{turn}})$ is the turning latitude]. These general characteristics are easily verified by considering the case when $E_n - U(\phi)$ is constant. For sufficiently small α the decay of $\psi_n(\phi)$ in the latitudes where $E_n - U(\phi) < 0$ occurs fast enough so that the function $\psi_n(\phi)$ vanishes at those latitudes [save perhaps for a tiny layer of width $\alpha^{1/2}$ near the latitude where $E_n - U(\phi) = 0$]. The crucial point is that the latitudes where $E_n - U(\phi) < 0$ are irrelevant to the solution and the lowest latitude where this inequality is satisfied increases with both α (for fixed n) and n (for fixed α). The fast decay of $\psi_n(\phi)$ at small α corresponds to a small energy gap between adjacent energy levels, which guarantees that the condition $E_n - U(\phi) > 0$ is satisfied even for large values of n.

Applying the known solutions of a quantum harmonic oscillator to eq. (4) implies that the eigenfunctions $\psi_n(\phi)$ vary as Hermite Functions (i.e. Hermite polynomials multiplied by a Gaussian) and $E_n = \alpha k^2 + (2n+1)(\alpha + \alpha^2k^2)^{1/2}$ [where αk^2 is the minimum of $U_{\text{parab}}(\phi)$ and $2(\alpha + \alpha^2k^2)^{1/2}$ is the energy gap between adjacent levels]. Thus, for small α the energy levels are stacked very close to one another while for large values of α the separation between neighbouring energy levels is large.

The analytical conclusions discussed above are confirmed by the following numerical calculations. First, the three potentials, $U_1(\phi) + \alpha U_2(\phi)$, $U_1(\phi)$ and $U_{\text{parab}}(\phi)$ are shown in Fig. 1 for $k = 10$ and for (α, C) values of (10^{-2},10^{-2}), (10^{-4},10^{-2}) and (10^{-6}, 10^{-3}). As expected, since the values of α are all sufficiently small, the approximation of $U_1(\phi) + \alpha U_2(\phi)$ by $U_1(\phi)$ is valid at latitudes below about 1.1 Radian (i.e. 60°). However, the overall approximation of $U_1(\phi) + \alpha U_2(\phi)$ by $U_{\text{parab}}(\phi)$ [in which $U_1(\phi)$ is used in the intermediate stage] is accurate up to latitudes of about 0.6 Radian (i.e. over 30°). For $\alpha \geq 10^{-2}$ the turning latitude, ϕ_{turn}, is below 0.6 only for $n \leq 2$ while for smaller values of α that are relevant to a baroclinic ocean (and baroclinic atmosphere) ϕ_{turn} is below 0.6 even for $n = 10$ (for $\alpha = 10^{-4}$) or even $n = 60$ (for $\alpha = 10^{-6}$). Calculations of the three potentials with other (α, C) values yielded similar matches provided the values of α are small (less than 0.01) and $\alpha - C^2\cos^2\phi \neq 0$.

Inserting the expressions for the energy levels of $U_{\text{parab}}(\phi)$, $E_n = \alpha k^2 + (2n+1)(\alpha + \alpha^2k^2)^{1/2}$, into the general cubic $k^2C^3 - EC - \alpha = 0$ yields

$$k^2C_n^3 - \left[\alpha k^2 + (2n+1)\sqrt{\alpha(1+\alpha k^2)}\right]C_n - \alpha = 0. \quad (5)$$

In Fig. 2, the resulting dispersion relations of two wave types obtained by the negative roots of eq. (5) (continuous curves) are compared to the exact ones (markers) calculated by solving system (2) numerically using the numerical procedure outlined in De-Leon and Paldor (2009). The results are shown for α values of 10^{-6} and 10^{-4} (relevant to a baroclinic ocean and a baroclinic atmosphere, respectively) and the modes shown are n = 1–8 for (negative) Poincaré waves and n = 1, 3, 7 for Rossby waves (since their dispersion curves are much closer to one another than those of Poincaré curves). From Fig. 2 it is clear that the agreement between the exact and approximate analytical estimates is excellent whereas for $\alpha = 10^{-2}$ and small k (results not shown), the agreement is quite poor. Positive Poincaré modes are very well approximated by dropping the sign of the negative modes, so they are not shown here.

The role that k^2 plays in the solutions presented above requires some additional analysis. From the expressions given above for E_n and $U_{\text{parab}}(\phi)$ it can be easily verified that the latitude where $E_n - U_{\text{parab}}(\phi)$ changes sign (known as the turning latitude, ϕ_{turn})

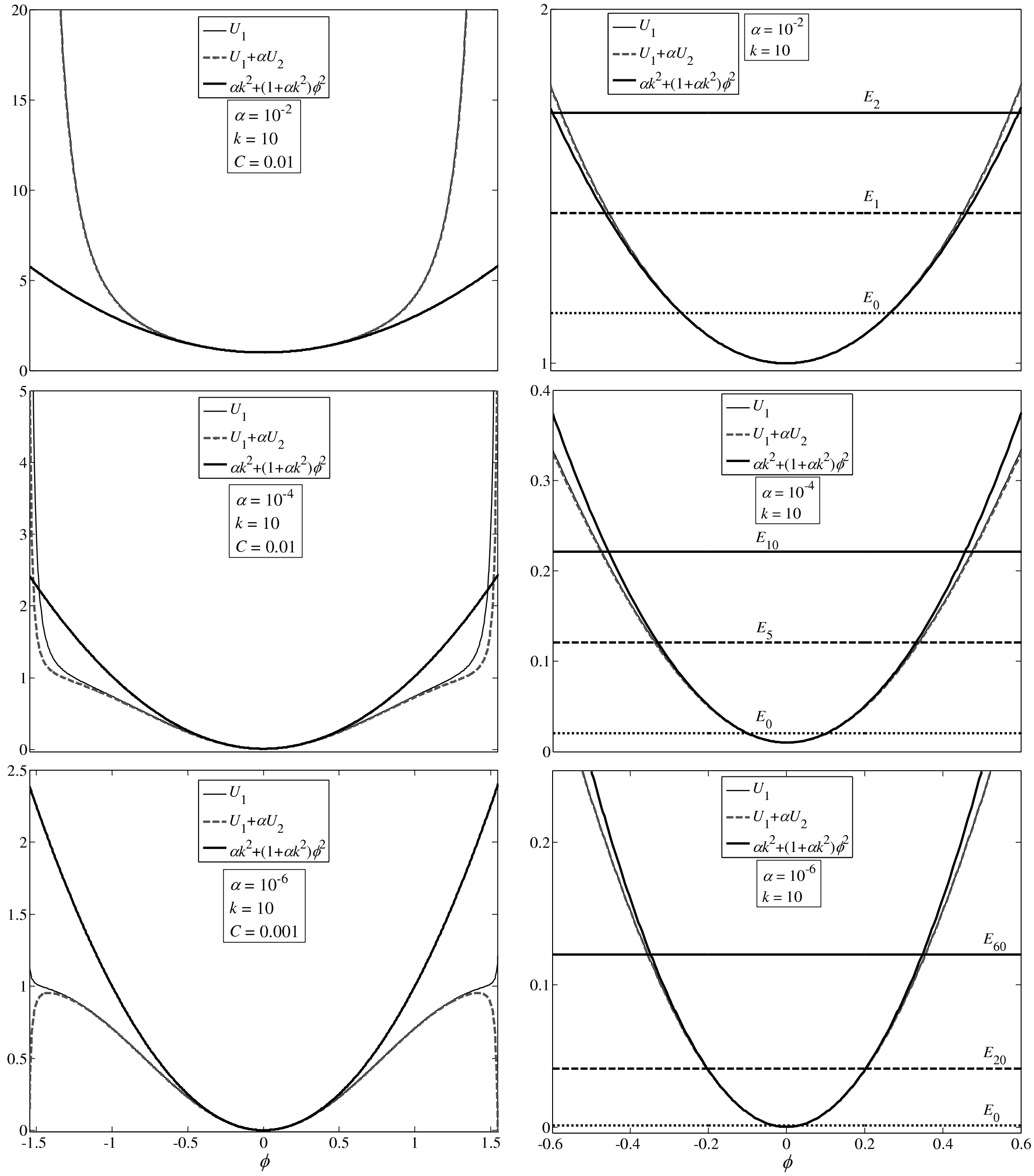

Fig. 1. The exact and approximate potentials and some of the energy levels. The approximations of $U_1(\phi)+\alpha U_2(\phi)$ by $U_1(\phi)$ and that of $U_1(\phi)$ by $U_{\mathrm{parab}}(\phi) = \alpha k^2+(1+\alpha k^2)\phi^2$ over the entire sphere are both accurate up to latitudes of about 0.6 [in fact $U_1(\phi)$ and $U_1(\phi)+\alpha U_2(\phi)$ can hardly be distinguished from one another outside the polar regions]. Zoom in on these latitudes (i.e. the vicinity of minima of the potentials and associated energy levels) is shown in the right panels. As α decreases the number of energy levels whose turning latitudes, ϕ_{turn}, are smaller than 0.6 increases. Thus, for $\alpha = 10^{-2}$ $\phi_{\mathrm{turn}} < 0.6$ only for $n \leq 2$ (upper panels), while for $\alpha = 10^{-4}$ $\phi_{\mathrm{turn}} < 0.6$ for $n < 10$ (middle panels) and for $\alpha = 10^{-6}$ $\phi_{\mathrm{turn}} < 0.6$ even for $n = 60$ (lower panels).

occurs at

$$\phi_{\mathrm{turn}} = \sqrt{2n+1}\left(\frac{\alpha}{1+\alpha k^2}\right)^{\frac{1}{4}}. \tag{6}$$

For a fixed $\phi_{\mathrm{turn}} < 0.6$, where $U_1(\phi)$ can be accurately approximated by $U_{\mathrm{parab}}(\phi)$, the maximal value of n at which eq. (6) is still satisfied, increases with k (for fixed α, provided $\alpha k^2 > 1$) so one can expect the error in the above calculations of the analytical and numerical dispersion relations, $(C_{\mathrm{exact}} - C_{\mathrm{analytic}})/C_{\mathrm{exact}}$, to decrease when k increases above 10 for $\alpha = 10^{-2}$. These straightforward analytical inferences were confirmed numerically (results not shown). Equation (6) is also a quantitative confirmation of the qualitative assessment discussed earlier (see second paragraph after eq. 4) regarding the role of α in determining the accuracy of our approximation for fixed k and n and the range where $\psi(\phi)$ oscillates (see also Fig. 1).

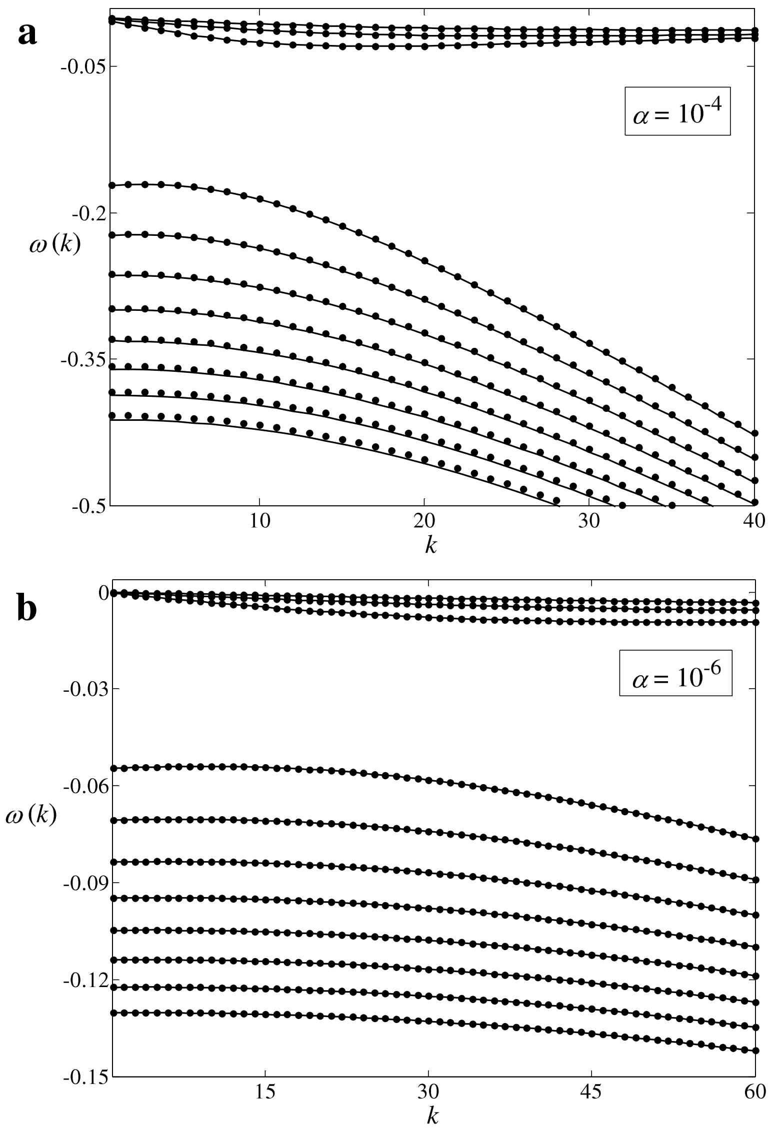

Fig. 2. Dispersion relations of Rossby and negative Poincaré waves. The modes shown are $n = 1$–8 for Poincaré waves (lower parts in the two panels; n increases downwards), and $n = 1,3,7$ for Rossby waves (upper parts of the two panels, n increases upwards). The exact values (markers) were calculated numerically from system (2) and they compare very accurately with the approximate analytical expressions (continuous curves), obtained from the roots of eq. (5). The accuracy for $\alpha = 10^{-6}$ (panel b) is excellent at all shown values of n, while for $\alpha = 10^{-4}$ (panel a), some differences can be detected at the larger n values.

The associated eigenfunctions that solve the Schrödinger equation with $U_{\mathrm{parab}}(\phi)$ are

$$\psi_n(\phi) = Ae^{-\delta\phi^2/2} H_n(\delta^{1/2}\phi), \tag{7}$$

where A is an arbitrary constant, H_n is the Hermite polynomial of order n and $\delta = (\frac{1+\alpha k^2}{\alpha})^{1/2}$ [=the inverse of the square of the parameter that multiplies $(2n + 1)$ on the right-hand side of eq. 6]. The physical meaning of δ is the rate at which the eigenfunction decays at latitudes above ϕ_{turn} and the rate of its oscillations at latitudes below it.

3. Discussion

The formulation of LTE as a time independent Schrödinger equation for zonally propagating waves yields explicit expressions for their phase speeds and meridional structures in a baroclinic ocean (and atmosphere) that covers the entire spherical aqua-planet. In contrast, for the fast-rotating Earth the original formulation of these equations by Laplace nearly 250 years ago yielded explicit solutions only on the equatorial β-plane (Matsuno, 1966). The trapping of the lower modes at low latitudes in the present spherical theory results only from the rapid latitudinal increase of the potential and does not employ assumptions that compromise Earth's spherical geometry.

The mathematical consideration leading to the linkage between the zonal wavenumber k, and the meridional wavenumber, n, ($k^2 \le n^2$ in the case of spherical harmonics and spheroidal functions) are common to many physical problems in spherical geometry, including the solutions of Helmholtz equation in a non-rotating sphere or the classical solutions of LTE on a slowly rotating sphere (Longuet-Higgins, 1968). In contrast, in the current formulation of the LTE as a Schrödinger equation the value of k is not limited by the value of n, that is, the number of zero crossings of the eigenfuction in the zonal direction is independent of the number of zero crossings in the meridional direction. Our numerical solutions of system (2) support this independence of k on n in the solutions of LTE.

Kelvin waves are filtered out of the foregoing analysis. In the analytic theory these waves correspond to the singular case where $\alpha - C^2 \cos^2 \phi = 0$ at some $-\pi/2 \leq \phi \leq \pi/2$, which is left out by neglecting $U_2(\phi)$. In the numerical solutions of system (2) these modes interfere with the $n = 0$ modes, which are not included in the dispersion relations shown in Fig. 2.

As mentioned above, the general dispersion relation, eq. (5) can be further simplified to yield separate approximate expressions for Rossby and Poincaré waves. For the slowly propagating Rossby waves, one assumes that the $k^2 C_n{}^3$ term is negligible, which yields

$$\text{Rossby waves}: C_n = -\frac{\alpha}{\left[\alpha k^2 + (2n+1)\sqrt{\alpha(1+\alpha k^2)}\right]}. \tag{8}$$

For the fast propagating Poincaré waves, one assumes that the α term in eq. (5) is negligible and when the resulting equation is divided through by C_n one obtains

$$\text{Poincare waves}: C_n^2 = \alpha + \frac{(2n+1)}{k^2}\sqrt{\alpha(1+\alpha k^2)}. \tag{9}$$

These approximate and highly accurate expressions can be used by numerical modelers who develop global scale Ocean General Circulation Models (OGCM) as test cases for the model's accuracy. To do so, one has to initiate the model with (u, v, η) eigenfunctions corresponding to specific values of (k, n) and a particular C-value, which appears in the relationship between ψ and (u, v, η) and compare the frequency of the time evolution to eq. (8) or eq. (9). The same dispersion relations and eigenfunction structure can also be incorporated in an observation study when the technology for estimating the propagation speed and special (meridional and zonal) structure of baroclinic ocean waves becomes sufficiently reliable. In particular, the realization that these waves are trapped to low latitudes even for very high meridional wavenumber (even for $n = 60$, the lower panels of Fig. 1 show that the wave's amplitude decays poleward of $25^\circ \sim 0.4$ Radian) might explain why such waves are not observed at all in high latitudes.

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