Tellus
SERIES A DYNAMIC METEOROLOGY AND OCEANOGRAPHY
PUBLISHED BY THE INTERNATIONAL METEOROLOGICAL INSTITUTE IN STOCKHOLM

Tellus (2010), 62A, 62–70

TELLUS

The eigenvalue equations of equatorial waves on a sphere

By YAIR DE-LEON, CARYNELISA ERLICK and NATHAN PALDOR*, *Department of Atmospheric Sciences, Institute of Earth Sciences, The Hebrew University of Jerusalem, Jerusalem 91904, Israel*

(Manuscript received 13 July 2009; in final form 19 October 2009)

ABSTRACT

Zonally propagating wave solutions of the linearized shallow water equations (LSWE) in a zonal channel on the rotating spherical earth are constructed from numerical solutions of eigenvalue equations that yield the meridional variation of the waves' amplitudes and the phase speeds of these waves. An approximate Schrödinger equation, whose potential depends on one parameter only, is derived, and this equation yields analytic expressions for the dispersion relations and for the meridional structure of the waves' amplitudes in two asymptotic cases. These analytic solutions validate the accuracy of the numerical solutions of the exact eigenvalue equation. Our results show the existence of Kelvin, Poincaré and Rossby waves that are harmonic for large radius of deformation. For small radius of deformation, the latter two waves vary as Hermite functions. In addition, our results show that the mixed mode of the planar theory (a meridional wavenumber zero mode that behaves as a Rossby wave for large zonal wavenumbers and as a Poincaré wave for small ones) does not exist on a sphere; instead, the first Rossby mode and the first westward propagating Poincaré mode are separated by the anti-Kelvin mode for all values of the zonal wavenumber.

1. Introduction

An analytic solution in Cartesian coordinates for linear waves on the unbounded equatorial β-plane, in which the boundary conditions are applied at infinity, was found by Matsuno (1966). The eigenvalue problem was formulated as a Schrödinger equation for a simple harmonic oscillator, whose solutions are well known from quantum mechanics. The quantized energy levels (eigenvalues) are (up to some scaling) odd integers, and the eigenfunctions are Hermite polynomials multiplied by a Gaussian function. Dispersion relations of the different waves are derived from the definition of the energy levels in terms of the phase speed (C) and zonal wavenumber (k). Both Poincaré and Rossby waves found in this analysis are trapped around the equator, but for the first meridional mode ($n = 0$), there is no clear distinction between the two wave types.

A numerical solution in Cartesian coordinates to the eigenvalue problem for linear waves in a channel on the equatorial β-plane (the bounded equatorial β-plane) was presented by Cane and Sarachik (1979), who also considered an asymmetric channel. Analytic solutions in Cartesian coordinates to the eigenvalue problem of linear waves in a symmetric channel on the equatorial β-plane were found by Erlick et al. (2007, hereafter EPZ).

An analytical solution of Laplace's tidal equation (a.k.a. the linearized shallow water equations; LSWE) in spherical coordinates on the entire sphere was presented by Longuet-Higgins (1968), who expanded the variables in spherical harmonic basis functions and solved the algebraic equations for the coefficients of the wave-like perturbations. A numerical solution for the case of a zonal channel situated in the mid-latitudes on a sphere was found by De-Leon and Paldor (2009), who also derived an analytical expression for the dispersion relations by approximating the meridional velocity's eigenfunction by an Airy function.

An analytic solution to the eigenvalue problem for linear waves in a zonal channel situated about the equator on a sphere was further presented by EPZ, but only for harmonic eigenfunctions and only for barotropic values of the radius of deformation. Based on a zeroth-order Taylor expansion about the equator, they derived a second-order eigenvalue equation with constant coefficients and a fifth-order polynomial that relates the eigenvalues to the phase speeds of the relevant waves. Unfortunately, the roots of this approximate polynomial do not match the eigenvalues calculated by direct numerical solution of the eigenvalue equation even though the roots yield all five known wave types: positive Kelvin, negative Kelvin, and sets of modes of positive Poincaré, negative Poincaré, and Rossby waves.

*Corresponding author.
e-mail: nathan.paldor@huji.ac.il
DOI: 10.1111/j.1600-0870.2009.00420.x

To complement the analytic solution, numerical solutions to the exact eigenvalue problem for linear waves in a zonal channel situated about the equator on a sphere were also presented by EPZ, and the existence of non-harmonic solutions for sufficiently wide channels was shown. However, no analytic solution was presented for such cases. Thus, to date, a complete analysis of linear waves in a symmetric channel about the equator on a sphere has not been achieved even though the same problem was completely solved for a channel in the mid-latitudes.

This study is aimed at providing an analytical solution for the missing case of an equatorial channel on a sphere, to complete the picture established by the previously analysed cases of the unbounded equatorial β-plane (Matsuno, 1966), the bounded equatorial β-plane (EPZ), the entire sphere (Longuet-Higgins, 1968), and the mid-latitude channel on a sphere (De-Leon and Paldor, 2009). We formulate for the first time an eigenvalue equation for linear waves in a channel on the equator in spherical coordinates that has the same form as that on the bounded equatorial β-plane and that yields non-harmonic zonally propagating solutions in addition to harmonic ones. This formulation accounts for all of the metric terms and coefficients associated with spherical geometry that are of the same order as those retained in the β-plane model, where the Coriolis frequency is the only latitude dependent parameter (i.e., order y^2 on the plane and order ϕ^2 on the sphere). The second aim of this study is to settle the inherent inconsistency of the equatorial β-plane model, which accounts for some of the y^2 terms (those arising from the variation in the Coriolis force) but neglects other terms of the same order (the tan(latitude) and 1/cos(latitude) associated with the spherical geometry).

The paper is organized as follows. In Section 2, we derive the shallow water equations (SWE) on a rotating sphere. In Section 3, we introduce our analytical approach for solving the eigenvalue equations for linear waves on a sphere. In Section 4, we compare our approximate analytical solutions with exact numerical ones. We conclude with a discussion in Section 5.

2. Linearized shallow water equations on a rotating sphere

The vectorial form of the LSWE with rotation (frequency Ω) is given by

$$\frac{\partial \mathbf{V}}{\partial t} + f\,\mathbf{k}\times\mathbf{V} = -g'\nabla\eta$$
$$\frac{\partial \eta}{\partial t} = -H'\,\nabla\cdot\mathbf{V}, \qquad (1)$$

where $\mathbf{V}$ is the horizontal velocity vector, f is the latitude dependent Coriolis frequency (=$2\Omega\sin\phi$, where ϕ is the latitude), H' is the mean thickness of the layer of fluid (air in the atmosphere and water in the ocean), η is the deviation of total height h from H' (so $h = H'+\eta$), g' is the gravitational constant (or reduced gravity in the case of a two-layer fluid) and $\mathbf{k}$ is a unit vector in the vertical (radial) direction perpendicular to $\mathbf{V}$.

In spherical coordinates, where λ is longitude and (u, v) are the velocity components in the (λ, ϕ) directions, respectively, we scale the equations using a (the Earth's radius) for the (horizontal) length scale, $(2\Omega)^{-1}$ for the time scale (so that $2\Omega a$ is the velocity scale), and H' for the height scale. Thus, the non-dimensional scalar form of system (1) is

$$\frac{\partial u}{\partial t} - v\sin\phi = -\frac{\alpha}{\cos\phi}\frac{\partial\eta}{\partial\lambda}$$
$$\frac{\partial v}{\partial t} + u\sin\phi = -\alpha\frac{\partial\eta}{\partial\phi}$$
$$\frac{\partial\eta}{\partial t} + \frac{1}{\cos\phi}\left\{\frac{\partial u}{\partial\lambda} + \frac{\partial\,(v\cos\phi)}{\partial\phi}\right\} = 0, \qquad (2)$$

where $\alpha = g'\mathrm{H}'/(2\Omega a)^2$ is the only non-dimensional parameter of the differential system and includes within it the four dimensional parameters, g', Ω, a, and H'. This parameter, α, is the inverse of Lamb's parameter, ε, known from the tidal literature, i.e., $\alpha = 1/\varepsilon$.

We now assume a zonally travelling wave solution, $[u(\phi), v(\phi), \eta(\phi)]e^{ik(\lambda - Ct)}$, where k is the zonal wavenumber, C is the zonal phase speed, and $\omega = Ck$ is the frequency. Note that in spherical coordinates, the spatial variables, λ and ϕ, are dimensionless angles, so the zonal wavenumber, k (=$2\pi/\Delta\lambda$, where $\Delta\lambda$ is the wavelength expressed as angle of longitude), is dimensionless even in the dimensional equations (not shown), and the unit of the (dimensional) phase speed, C, is frequency (and not speed as it is in planar theory).

For such zonally travelling wave solutions, $\frac{\partial}{\partial\lambda} = ik$, and $\frac{\partial}{\partial t} = -ikC$, so system (2) can be written in matrix form as

$$\begin{bmatrix} 0 & \sin\phi & \dfrac{\alpha}{\cos\phi} \\ \dfrac{\sin\phi}{k^2} & 0 & \dfrac{\alpha}{k^2}\dfrac{\partial}{\partial\phi} \\ \dfrac{1}{\cos\phi} & \tan\phi - \dfrac{\partial}{\partial\phi} & 0 \end{bmatrix}\begin{bmatrix} u \\ V \\ \eta \end{bmatrix} = C\begin{bmatrix} u \\ V \\ \eta \end{bmatrix}, \qquad (3)$$

where $V \equiv iv/k$. In this formulation, the phase speed, C, is the eigenvalue of the differential operator on the left-hand side.

Expressing $u(\phi)$ as a linear (algebraic) combination of $V(\phi)$ and $\eta(\phi)$ [which is possible because no ϕ-derivatives are present in the first row of system (3)], substituting the resulting expression for $u(\phi)$ in the other equations, and rearranging, we get the second-order set

$$\frac{\partial}{\partial\phi}\begin{bmatrix} V\cos\phi \\ \eta \end{bmatrix} = \frac{1}{C\cos\phi} \times \begin{bmatrix} \sin\phi & \left(\alpha - C^2\cos^2\phi\right) \\ \left(\frac{\omega^2 - \sin^2\phi}{\alpha}\right) & -\sin\phi \end{bmatrix}\begin{bmatrix} V\cos\phi \\ \eta \end{bmatrix}. \tag{4}$$

Eliminating $\eta(\phi)$ from system (4) by differentiating the first equation with respect to ϕ, and substituting the expressions for $\partial\eta/\partial\phi$ from the second equation and that for η from the first equation yields a single, second-order, equation for $V(\phi)\cos\phi$

$$\frac{\mathrm{d}^2(V\cos\phi)}{\mathrm{d}\phi^2} - \left(\tan\phi\frac{\alpha + C^2\cos^2\phi}{\alpha - C^2\cos^2\phi}\right)\frac{\mathrm{d}(V\cos\phi)}{\mathrm{d}\phi} + \left[\frac{\omega^2}{\alpha} - \frac{1}{C} - \frac{\sin^2\phi}{\alpha} - \frac{k^2}{\cos^2\phi} + \tan^2\phi\left(\frac{\alpha + 2C\cos^2\phi}{\alpha - C^2\cos^2\phi}\right)\right] \times (V\cos\phi) = 0. \tag{5}$$

This equation can be transformed into a Schrödinger-like equation by defining

$$V(\phi)\cdot\cos\phi = \psi(\phi)\cdot\sqrt{\left(\frac{\alpha}{C\cos\phi} - C\cos\phi\right)},$$

which yields (after additional algebraic manipulation; see De-Leon and Paldor, 2009)

$$\alpha\frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} + [E - U_1(\phi) - \alpha U_2(\phi)]\,\psi = 0, \tag{6}$$

where

$$E = k^2C^2 - \frac{\alpha}{C} = \omega^2 - \frac{\alpha}{C},$$

$$U_1(\phi) = \sin^2\phi + \frac{\alpha k^2}{\cos^2\phi},$$

$$U_2(\phi) = \frac{3}{4}\tan^2\phi\left(\frac{\alpha + C^2\cos^2\phi}{\alpha - C^2\cos^2\phi}\right)^2 - \frac{1}{2}\left(\frac{\alpha + C^2\cos^2\phi}{\alpha - C^2\cos^2\phi}\right) - \tan^2\phi\left(\frac{\alpha + 2C\cos^2\phi}{\alpha - C^2\cos^2\phi}\right).$$

In the above, E is the energy level (eigenvalue; independent of ϕ), and $U_1(\phi)+\alpha U_2(\phi)$ is the potential. Comparing the above equation with eq. (3) in EPZ reveals that only E and U_1 have planar counterparts, while the term $\alpha U_2(\phi)$ is unique to the spherical geometry and is singular at the point(s) where the transformation from V to ψ yields a singular $\psi(\phi)$ even for regular $V(\phi)$.

While eq. (6) is in Schrödinger-like form, C cannot be determined from its eigenvalue (E) analytically or numerically, since it appears in both the energy, E, and in the potential, $U_1(\phi)+\alpha U_2(\phi)$. However, C can be determined from eq. (6) if $\alpha U_2(\phi)$ is neglected compared to $U_1(\phi)$, that is, in the limit $\alpha \ll 1$. Actually if $k \gg 1$, neglecting $\alpha U_2(\phi)$ compared to $U_1(\phi)$ is valid even when α is not much smaller than 1, since in such cases $U_1(\phi)$ dominates the potential via its second term, $\alpha k^2/\cos^2\phi$. It should be reiterated, however, that neglecting $\alpha U_2(\phi)$ is justified only when $\alpha - C^2\cos^2\phi$ does not vanish anywhere inside of the domain, though it can vanish at the boundaries. For values of α and k for which $\alpha U_2(\phi)$ can be neglected and under the above restriction for $\alpha - C^2\cos^2\phi$, eq. (6) simplifies to the following Schrödinger equation, where C only appears in the energy:

$$\alpha\frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} + [E - U_1(\phi)]\,\psi = 0. \tag{7}$$

Note that although the spherical geometry term, $\alpha U_2(\phi)$, was neglected in the derivation of eq. (7), eq. (7) is not identical to its planar counterpart (eq. 3 in EPZ), since in the planar theory the k^2 term is constant and can be included in the energy term, while in our spherical theory the term $\alpha k^2/\cos^2\phi$ depends on ϕ and therefore is an inherent part of the potential.

As a Schrödinger equation, with proper boundary conditions, eq. (7) has infinitely many energy levels, E_n, all of which are positive, since $U_1(\phi) = \sin^2\phi + \alpha k^2/\cos^2\phi$ is positive. These eigenvalues, E_n, yield the corresponding phase speeds, C_n, of three waves as the roots of the cubic equation $k^2C_n^3 - E_nC_n - \alpha = 0$.

Even though eq. (7) is a valid approximation of system (6) for small α and/or large k, it should be noted that Kelvin waves have been filtered out of eq. (7). Such waves are a special case of eq. (6), where the spherical terms are singular (i.e. $\alpha - C^2\cos^2\phi = 0$ at some ϕ) and where $\alpha U_2(\phi)$ dominates the potential near such points.

As a second-order differential equation, eq. (7) has two linearly independent solutions for any combination of its parameters, C, k, and α. The differential eigenvalue problem is closed by requiring that the general solution, $\psi(\phi)$, vanishes at two walls; for a zonal channel symmetric about the equator, a southern wall is placed at $\phi = -\Delta\phi$, and a northern wall is placed at $\phi = +\Delta\phi$. Imposing these two boundary conditions (and insisting that $\psi(\phi)$ does not vanish identically) yields the dispersion relation $C(k)$ for given values of α and $\Delta\phi$.

3. Analytical consideration of the spherical LSWE near the equator

The simplification of eq. (6) to eq. (7) allows one to deduce some analytical properties of the solutions. Expanding the potential $U_1(\phi) = \sin^2\phi + \alpha k^2/\cos^2\phi$ in eq. (7) about the equator to second-order in ϕ and dividing the equation through by α, one obtains

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} + \left[E^{\mathrm{int}} - k^2 - \left(\frac{1+\alpha k^2}{\alpha}\right)\cdot\phi^2\right]\psi = 0, \tag{8}$$

where $E^{\mathrm{int}} = E/\alpha = k^2C^2/\alpha - 1/C$. Before turning to the analytical implications of eq. (8), we verify its accuracy by

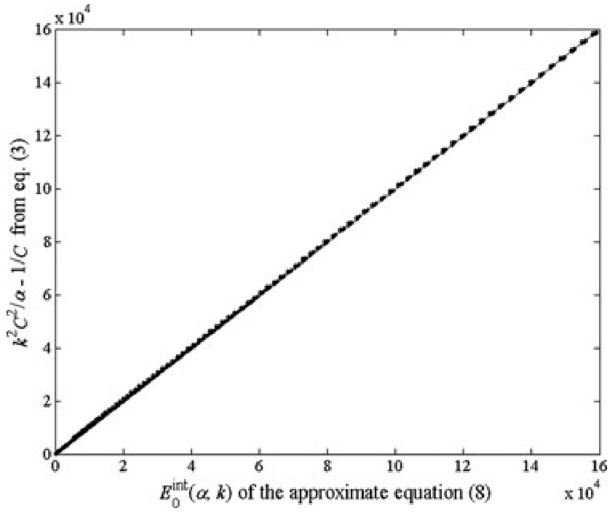

Fig. 1. A comparison between several thousand values of $k^2C^2/\alpha - 1/C$ calculated from the eigenvalues of wave like solutions of the exact system (3) and the corresponding values of $E^{int}{}_0(\alpha, k)$ calculated from the approximate equation (8).

comparing its solutions to solutions of the exact set (3). We calculate numerically (see Section 4 for a detailed description of the numerical method) the phase speed of the $n = 0$ (where n is the meridional mode number) mode from system (3), including the positive Poincaré, negative Poincaré, and Rossby waves, when they can be confidently distinguished from the Kelvin waves and other spurious modes, for four values of α ranging from 10^{-3} to 10^{-6}, for k-values that range from 1 to several hundred, and for two channel widths, $\Delta\phi = 0.05$ and 0.4. For each of the several thousand values of $C(\alpha,k)$ calculated from eq. (3), we generate the combination $k^2C^2/\alpha - 1/C$ (the y-axis of Fig. 1) and calculate the eigenvalue $E^{int}{}_0(\alpha, k)$ from eq. (8) for the same values of α, k and $\Delta\phi$ (the x-axis of Fig. 1). As is clearly evident, the several thousand values of $k^2C^2/\alpha - 1/C$ calculated for the waves of eq. (3) align very tightly with the $E^{int}{}_0(\alpha, k)$ values from eq. (8).

Having established the accuracy of (8), we now turn to analysing the properties of its solutions. Defining a new energy,

$$\tilde{E} = \frac{k^2C^2}{\alpha} - \frac{1}{C} - k^2 \tag{9}$$

and a quadratic potential,

$$\tilde{U} = \left(\frac{1 + \alpha k^2}{\alpha}\right) \cdot \phi^2, \tag{10}$$

and transforming the independent variable ϕ to $Z = \phi/\Delta\phi$ (where $\Delta\phi$ is half the channel width), so that the parameters of the boundaries appear in the equation, we get

$$\frac{d^2\psi}{dZ^2} + (E^* - \delta^2 Z^2)\psi = 0, \tag{11}$$

where

$$E^* = \tilde{E}\,(\Delta\phi)^2 = \left(\frac{k^2C^2}{\alpha} - \frac{1}{C} - k^2\right) \cdot (\Delta\phi)^2\,, \tag{12}$$

and

$$\delta = \left(\frac{1 + \alpha k^2}{\alpha}\right)^{\frac{1}{2}} (\Delta\phi)^2\,. \tag{13}$$

An analogous equation that has a form similar to (11) was derived in EPZ for the β-plane with slightly different definitions of δ (no αk^2 term in its numerator) and the same E^*. However, no such equation was ever derived before in spherical coordinates.

The boundary condition, $\psi = 0$, is now applied at $Z = \pm 1$ for all values of $\Delta\phi$, and this boundary condition quantizes the energy levels of the Schrödinger equation (11). Analytical solutions of this equation exist in two asymptotes of its potential, $\delta^2 Z^2$: $\delta \to \infty$ and $\delta \to 0$.

In the limit $\delta \to \infty$, eq. (11) becomes the Schrödinger equation for a harmonic oscillator from quantum mechanics, and its eigenfunctions are given as (see Abramowitz and Stegun, 1972; chapter 22.6)

$$\psi_n(Z) = Ae^{-\delta Z^2/2} \cdot H_n\left(\delta^{1/2} Z\right), \tag{14}$$

where H_n is the Hermite polynomial of order n, and A is an undetermined constant. In order to determine the smallest value of δ for which (14) provides a valid solution of the problem, we require that $\delta^{1/2} Z \geq 2.5$ at the wall ($Z = 1$) for the $n = 0$ mode, since when this requirement is satisfied, the value of the eigenfunction at the wall is $\approx e^{-3} \approx 4 \times 10^{-2}$, which is sufficiently small to be considered 0. This requirement implies that

$$\delta^{1/2} = \left(\frac{1 + \alpha k^2}{\alpha}\right)^{\frac{1}{4}} \cdot \Delta\phi \geq 2.5 \quad \text{or} \quad 1 + \alpha k^2 \geq O\left[\frac{40\alpha}{(\Delta\phi)^4}\right]. \tag{15}$$

The eigenvalues associated with the eigenfunctions of (14), including the expression for the energy levels in eq. (12) are

$$\begin{aligned} E_n^* &= (2n+1)\,\delta = (2n+1)\left(\frac{1 + \alpha k^2}{\alpha}\right)^{\frac{1}{2}} (\Delta\phi)^2 \\ &= \left(\frac{k^2C_n^2}{\alpha} - \frac{1}{C_n} - k^2\right)(\Delta\phi)^2\,, \quad (n = 0, 1, 2, \ldots). \end{aligned} \tag{16}$$

Equation (16) yields a cubic equation for the phase speed

$$k^2C_n^3 - \left[\alpha k^2 + (2n+1) \cdot \sqrt{\alpha(1 + \alpha k^2)}\right] C_n - \alpha = 0. \tag{17}$$

The small root of this equation (approximated by neglecting the $C_n{}^3$ term) is the phase speed of the Rossby wave

$$C_n = -\frac{\alpha}{\left[\alpha k^2 + (2n+1) \cdot \sqrt{\alpha(1 + \alpha k^2)}\right]}, \tag{18}$$

while the other two large roots [obtained by neglecting α in eq. (17) and dividing it through by C_n] approximate the frequency of the two Poincaré waves

$$\omega_n^2 = \left[\alpha k^2 + (2n+1) \cdot \sqrt{\alpha(1 + \alpha k^2)}\right]. \tag{19}$$

For $n = 0$, these approximate expressions for the frequencies of the Rossby mode and the negative Poincaré mode nearly coalesce (and as expected, our numerical results for $\alpha = 10^{-4}$ and $k = 7$ confirm the coalescence to $-\alpha^{\frac{1}{2}}k$) at sufficiently small αk^2, which is a fingerprint of the emergence of a mode on the sphere similar to the mixed mode on the β-plane. (See Matsuno, 1966 for the mixed mode on the β-plane, fig. 3 in EPZ for numerical results regarding the mixed mode on the sphere, and Section 5 for further discussion.)

In the limit $\delta \to 0$, eq. (11) is a second-order equation with constant coefficients whose eigenfunctions, identical to the eigenfunctions of the potential well of quantum mechanics, are

$$\psi_n(Z) = A \sin\left[\frac{(n+1)\pi(Z+1)}{2}\right], \qquad (n = 0, 1, 2, \ldots), \tag{20}$$

where A is an undetermined constant. The associated eigenvalues are

$$E_n^* = \frac{(n+1)^2\pi^2}{4}, \qquad (n = 0, 1, 2, \ldots), \tag{21}$$

and the cubic equation for the phase speed is therefore

$$k^2 C_n^3 - \left[\alpha k^2 + \frac{\alpha(n+1)^2\pi^2}{4(\Delta\phi)^2}\right] C_n - \alpha = 0. \tag{22}$$

An exact expression for E_n^* valid not just in one of the limits but over the entire range of δ cannot be obtained. However, an approximate expression that is sufficiently accurate over the entire range of δ can be found in a manner similar to EPZ, eq. (9). A Pythagorean sum of the two asymptotic values, $E_n^* = [E_n^*(\delta \to 0)^2 + E_n^*(\delta \to \infty)^2]^{1/2}$, is one such expression, but numerical experiments show that a more accurate, yet sufficiently simple, choice is

$$E_n^* = \left\{\left[\frac{(n+1)^2\pi^2}{4}\right]^3 + [(2n+1)\delta]^3\right\}^{1/3}, \quad (n = 0, 1, 2, \ldots). \tag{23}$$

In Fig. 2a, the numerical solutions for the eigenvalues, E_n^*, of eq. (11) are shown as a function of δ for the first three modes: $n = 0, 1, 2$, and in Fig. 2b, the exact numerical solution of E_0^* from eq. (11) is shown together with the values of E_0^* obtained from eqs (16), (21), and (23). From Fig. 2b, it is clear that (23) provides an accurate analytical approximation to the exact numerical solution for all values of δ.

4. Numerical results for the dispersion relations and eigenfunctions

The numerical results for system (3) and for the approximate second-order ψ-equations (8) and (11) were obtained using the Chebyshev collocation method on a grid, as explained in Trefethen (2000) and in De-Leon and Paldor (2009). This method is based on writing the differential operator of the eigenvalue equation in matrix form and solving the resulting algebraic eigenvalue problem, $AX = EX$, where the matrix A contains

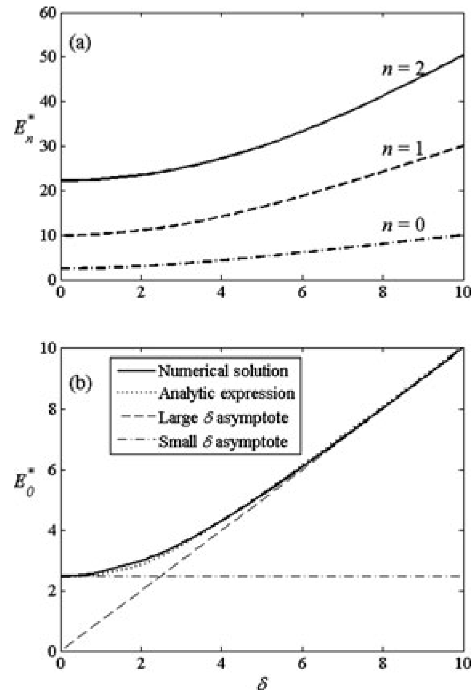

Fig. 2. (a) The eigenvalues of eq. (11), $E_n^*(\delta)$, for $n = 0$, 1 and 2. (b) The exact numerical solution (eq. 11), the approximate (analytic) expression (eq. 23), and the asymptotic expressions at large (eq. 16) and small (eq. 21) δ for $n = 0$.

both the differential operator and the ϕ-dependent 'potential', the eigenvalue is E [which stands for either E^{int}, E^* in the case of (8) and (11), or for C in the case of (3)), and the eigenvector $\boldsymbol{X}$ stands for the $[u(\phi), V(\phi), \eta(\phi)]$-vector in eq. (3) or for $\psi(\phi)$ in eqs. (8) and (11)]. The transformation of the differential operator into a matrix is done at the N collocation points: $x_j = \cos(j\pi/N)$ $(j = 0\ldots N)$ that are the extrema of $T_N(x)$, the Nth Chebyshev polynomial, on the $[-1, 1]$ interval. This transformation of the operator mandates our transformation of the independent variable from ϕ to $Z = \phi/\Delta\phi$ which varies between -1 and 1. We used $N = 120$ and confirmed the accuracy of our calculations by comparing the eigensolutions with those obtained with $N = 320$.

4.1. Eigenvalues and dispersion relations

Following the analysis of Section 3 that relates the frequencies of the various wave types to the value of the energy, E_n, one expects the dispersion relations, $\omega(k) = kC(k)$ of each of the modes to vary with the value of the potential in eq. (11), that is, to vary with δ. Since according to eq. (13), δ is a function of three parameters, α, $\Delta\phi$, and k, many combinations of α, $\Delta\phi$, and k may yield the same value of δ. However, the dependence of δ on k appears only in the combination $(1 + \alpha k^2)^{\frac{1}{2}}$, so for sufficiently small α, the variation of δ over a finite range of

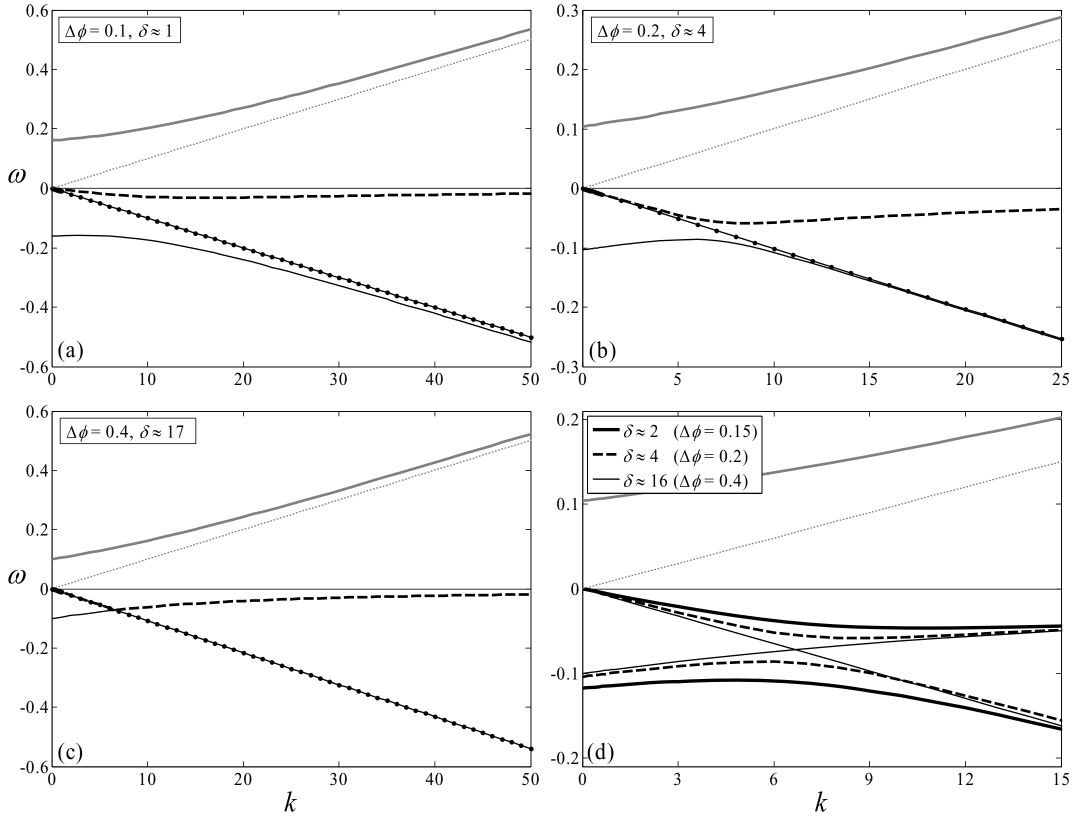

Fig. 3. The dispersion relation of the $n = 0$ mode of the various waves in the three δ-regimes: (a) the harmonic, small δ, regime; (b) the intermediate δ regime; (c) the Hermite, large δ, regime. Panel (d) is a zoom in on the small k, negative ω region of the cases shown in panels a-c, where the increase in δ brings about the development of the mixed mode—Poincaré at small k and Rossby at large k. In all panels, positive modes are plotted in grey and $\alpha = 10^{-4}$.

k-values is small enough to remain in only one of three possible 'regions' of δ: harmonic, intermediate, or Hermite (see Fig. 2b). Accordingly, we present the dispersion relations for exactly these three regions: small, harmonic $\delta(<2)$; large, Hermite-like $\delta(>7$, see eq. 15); and intermediate δ (i.e., $2 < \delta < 7$), where the transition from harmonic functions to Hermite functions takes place. The dispersion relations shown in Fig. 3 are computed for the $n = 0$ mode and for a fixed value of $\alpha = 10^{-4}$, and the transition from one δ-region to the other is achieved mainly by varying $\Delta\phi$.

For small δ values (Fig. 3a), the modes are separated and are similar to those known from mid-latitude theories: Rossby waves (black dashed line), positive Poincaré waves (grey solid line), negative Poincaré waves (thin black solid line), a Kelvin wave with $C = \alpha^{\frac{1}{2}}$ that decays monotonically with distance from the equator in either hemisphere (thin grey dotted line) and anti-Kelvin waves with $C = -\alpha^{\frac{1}{2}}/\cos(\Delta\phi)$ that decay monotonically with distance from either of the two boundaries (dotted line).

For large δ values (Fig. 3c), the first negative Poincaré wave at $k<6$ and the first Rossby wave at $k > 7$ seem to be a continuation of one another, while the first negative Poincaré wave at $k > 7$ and the first Rossby wave at $k < 6$ seem to overlap with the anti-Kelvin mode. In the region of overlap, the overlapping mode and the anti-Kelvin mode have the same $\omega(k)$ relations and the same V-eigenfunctions, which are symmetric or anti-symmetric. The dispersion relations of the positive waves (grey shades) have similar characteristics to those in the small δ limit.

For intermediate δ values (Fig. 3b), as δ increases, the separated modes approach one another, and for sufficiently large δ-values, they yield a mode similar to the planar mixed mode. This approach to the mixed mode with the increase in δ is even clearer in Fig. 3d, which is a zoom in on the three types of dispersion relations near $k = 7$.

4.2. *Eigenfunctions*

The classification into three regions of δ values, described in the previous subsection, is also reflected in the eigenfunctions. Since both $u(\phi)$ and $\eta(\phi)$ of the Rossby and Poincaré waves are determined by linear combinations of the $V(\phi)$-eigenfunction and of $dV/d\phi$, where the coefficients of the linear combinations include the associated eigenvalues, C, we only show the $V(\phi)$

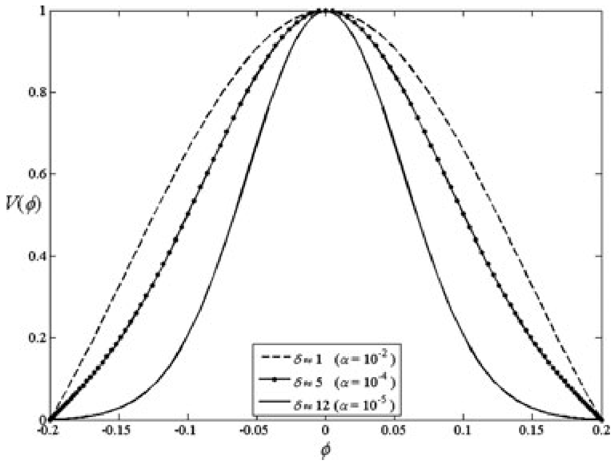

Fig. 4. The $V(\phi)$-eigenfunctions for different values of δ, normalized such that MAX$[V(\phi)] = 1$. Note that for each δ, there are several identical eigenfunctions corresponding to different waves (Poincaré or Rossby) but associated with the same E.

eigenfunctions of these waves. The $V(\phi)$-eigenfunctions shown in Fig. 4 are all scaled such that their maximum value is 1.0. These functions are generated by varying α and k in the three regions of δ values for a fixed channel width, $\Delta\phi = 0.2$. Note that in order to enable a comparison between the eigenfunctions in Fig. 4, α varies between the curves while $\Delta\phi$ is fixed, as opposed to Fig. 3, in which the value of α is fixed and $\Delta\phi$ varies between panels. This difference is also the reason for the slightly different values of δ between Figs. 3 and 4. As expected, for small δ values (dashed line), V is quite harmonic, for large δ values (solid line), V is a Hermite function, and for the intermediate δ values (dotted line), V has a form in between harmonic and Hermite. For each region of δ, there is a good match between the $V(\phi)$-eigenfunctions for Poincaré and Rossby waves. This supports the validity of the approximate differential eigenvalue equation (7), as the different waves are associated with the same eigenfunction and the same eigenvalue, E, but have different $C(E)$ roots.

For the Kelvin waves, the η-eigenfunctions decrease monotonically with distance from either the equator (Kelvin) or the walls (anti-Kelvin), but the V-eigenfunction does not vanish identically in contrast to the planar theory (see EPZ).

5. Discussion and conclusions

We derived eq. (17), which links the energy levels, E_n, with the three roots of the phase speed, C_n, in the limit $\delta \to \infty$. As is suggested by the numerical results of Section 4.1 (see Fig. 3c) and as is known from the planar theory (e.g. Matsuno, 1966), for $n = 0$, a mixed mode exists that unites the negative Poincaré wave and the Rossby wave. If such a mixed mode were to exist in the present spherical theory for some n, then the value of ω_n at the intersection of the two modes could be estimated analytically by multiplying eq. (17) through by k to get the following cubic equation for ω_n

$$\omega_n^3 - \left[\alpha k^2 + (2n+1)\cdot\sqrt{\alpha(1+\alpha k^2)}\right]\omega_n - \alpha k = 0. \tag{24}$$

This polynomial does not have an $\omega_n{}^2$ term, implying that the sum of the three roots is zero. In addition, the product of the three roots of the polynomial equals αk (the negative of the free term in the cubic equation), so two of the roots must be negative, and one root must be positive. If the two negative modes intersect and their identical negative frequency ω_n is denoted by $-x$, then the third root must equal $+2x$, so the cubic polynomial can be written as

$$(\omega_n - 2x)(\omega_n + x)^2 = \omega_n^3 - 3x^2\omega_n - 2x^3 = 0. \tag{25}$$

Comparing the coefficients of this polynomial with those of (24) yields two expressions for x that can be combined into the following relation between α, k and n

$$\frac{3\left(\frac{\alpha k}{2}\right)^{2/3} - \alpha k^2}{\sqrt{\alpha + \alpha^2 k^2}} = 2n + 1. \tag{26}$$

The value of ω_n at the assumed intersection is then

$$|\omega_n| = x = \left(\frac{\alpha k}{2}\right)^{1/3}. \tag{27}$$

Estimating x from eq. (27) yields a value very close to the numerical results shown in Fig. 3c. For $\alpha = 10^{-4}$ and $k \approx 6.7$, eq. (27) yields a value for x of 0.07, while the value for x in Fig. 3c [based on the exact system (3) and not on the approximate set (11)] is 0.072, demonstrating the validity of (27).

However, the left-hand side of eq. (26), shown in Fig. 5a as contours plotted from several thousand pairs of α, and k values, does not contain a 1.00 contour. The maximal value of the left-hand side of eq. (26) is 0.9997, which is very close to but does not equal 1. This implies that eq. (26) cannot be satisfied with $n > 0$, and that it is only nearly satisfied at $n = 0$, that is, that the modes do not truly intersect; there is no mixed mode.

To support the analytical argument that on the sphere there is no mixed mode, we look more closely at the crossing point between the negative $n = 0$ Poincaré mode and the $n = 0$ Rossby mode. Figure 5b was calculated for $\alpha = 10^{-4}$ with a fine k-resolution of 0.001 near the assumed intersection point of the two waves in Fig. 3c. In Fig. 5b, it is clearly shown that the two modes do not cross each other but rather remain separated, though very close to one another, for all k. From Fig. 3c, it is also clear that the negative Poincaré mode joins the anti-Kelvin mode at large k, and that the Rossby mode joins the anti-Kelvin mode at small k.

Two differences exist between our spherical theory and the β-plane theory of Matsuno (1966) that can account for this difference in the existence of the mixed mode: (1) on an infinite

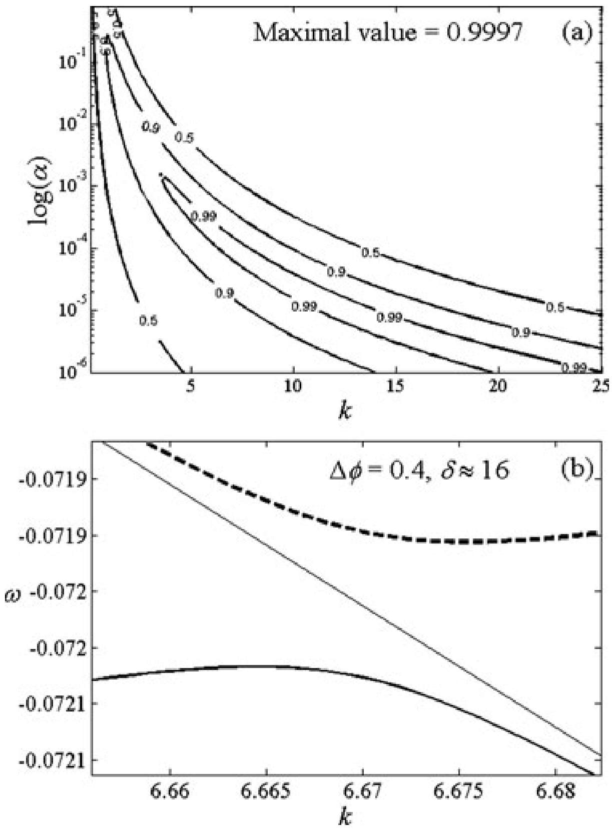

Fig. 5. (a) Contours of the left-hand side of eq. (26). The maximal value in all (several thousand) pairs of α and k is 0.9997; no 1.00 contour exists. (b) A zoom in on the range of low negative ω and small k, where the presumed mixed mode intersects the anti-Kelvin mode (thin solid line). No intersection occurs, but the phase speed of the $n = 0$ Rossby mode at small k (thick dashed line) and the phase speed of the negative $n = 0$ Poincaré mode at large k (thick solid line) are both asymptotic to that of the anti-Kelvin mode.

plane, the anti-Kelvin mode cannot exist, whereas on the sphere, the presence of two channel walls, at $\phi = +\Delta\phi$ and at $\phi = -\Delta\phi$, allows an anti-Kelvin mode to exist and to act as a separator between the Rossby modes and negative Poincaré modes. (2) On the β-plane, when $\omega = -\alpha^{\frac{1}{2}}k$, the zonal velocity is singular over the entire y-domain (see the discussion following Matsuno, 1966, eq. 13), so the only way to portray the modes when $\omega = -\alpha^{\frac{1}{2}}k$ is by forming a 'mixed mode'. In contrast, on the sphere, the zonal velocity is regular over the entire domain, which enables separate Rossby and negative Poincaré modes to exist simultaneously. The zonal velocity is related to the meridional velocity and to its ϕ-derivative via

$$u(C^2\cos^2\phi - \alpha) = C\sin\phi\cos\phi\,(V\cos\phi) - \alpha\frac{\partial\,(V\cos\phi)}{\partial\phi}, \tag{28}$$

which admits solutions with $C = -\alpha^{\frac{1}{2}}/\cos(\Delta\phi)$, provided that $V = 0 = \mathrm{d}V/\mathrm{d}\phi$ at the wall ($\phi = \Delta\phi$), and admits a solution with $C = +\alpha^{\frac{1}{2}}$, provided that $\mathrm{d}V/\mathrm{d}\phi = 0$ at $\phi = 0$ (but not at all ϕ).

A similar conclusion regarding the non-existence of the mixed mode was reached by Gent and McWilliams (1983), who solved the eigenvalue problem for the meridional velocity on the bounded equatorial β-plane but ignored the ramifications of their solution on the zonal velocity (see their fig. 1).

We should also note that in contrast to Matsuno (1966), who obtained only the Hermite (i.e., large δ) solutions, in the present theory, we obtain harmonic solutions at small δ and intermediate solutions for intermediate values of δ, in addition to the Hermite solutions at large δ. The harmonic solutions are particularly relevant to large values of α corresponding to barotropic cases in the ocean or to baroclinic cases in the atmosphere, while the intermediate and Hermite solutions are relevant to small values of α corresponding to baroclinic cases in the ocean.

Table 1. The contributions and main findings of the various works on the linear waves on the equator of the rotating earth

	Channel (bounded)	Unbounded
β-plane	EPZ Eigenfunctions: Harmonic: small δ; Hermite: large δ Mixed mode exists at large δ	Matsuno (1966) Eigenfunctions: Hermite Mixed mode emerges
Sphere	Present study Harmonic: small δ; Hermite: large δ No mixed mode even for $\delta \gg 1$!	Longuet-Higgins (1968) Eigenfunctions (on the entire sphere): Hough Functions.

Note: The main result of the present work is that the mixed mode found both in the unbounded β-plane and in the large-δ regime of the bounded β-plane (i.e., channel) is a fluke of the planar geometry and does not exist as a single mode on a sphere.

This study deals with a symmetric channel, where the eigenvalue equation for equatorial waves can by solved analytically, and the phase speeds of two anti-Kelvin waves are identical. We conjecture that in an asymmetric channel, the two anti-Kelvin waves will have different phase speeds. At small k, the Rossby mode will asymptote toward the slower anti-Kelvin wave (associated with the wall located at a lower latitude), while at large k, the negative Poincaré mode will asymptote toward the faster anti-Kelvin wave (associated with the wall located at a higher latitude). Thus the separation between the two modes (i.e., the Rossby mode and the negative Poincaré mode) by the two anti-Kelvin waves is expected to be even clearer in the asymmetric channel than in the symmetric channel. Since no zonal walls exist in reality in the ocean, our results relating to the anti-Kelvin mode(s) do not have a true analogue in nature. However, they provide a sound mathematical set-up, a clear analysis of the problem, and a proof of the separation between the Rossby and Poincaré waves on a sphere that is not provided by previous analyses.

A summary of the various works that were published on the general issue of linear waves in domains that include the equator (both planar and spherical) is presented in Table 1, which also highlights the contributions and main findings of each of these works, including the disappearance of the mixed mode on a sphere.

6. Acknowledgments

This study was supported by research grant No. 579/05 of the Israel Science Foundation to the Hebrew University of Jerusalem. Yair De-Leon was supported by a graduate student fellowship provided by the Meirbaum Foundation of the Hebrew University.

References

Abramowitz, M. and Stegun, I.A. 1972. *Handbook of Mathematical Functions*. Dover Publications, Inc., NY, USA, 1043.

Cane, M.A. and Sarachik, E.S. 1979. Forced baroclinic ocean motions III: the linear equatorial basin case. *J. Mar. Res.* **37**, 355–398.

De-Leon Y. and Paldor, N. 2009. Linear waves in mid-latitudes on the rotating spherical Earth. *J. Phys. Oceanogr.* **39**(12), 3204–3215.

Erlick, C., Paldor, N. and Ziv, B. 2007. Linear waves in a symmetric equatorial channel. *Q. J. R. Meteorol. Soc.* **133**(624), 571–577.

Gent, P.R. and McWilliams, J.C. 1983. The equatorial waves of balanced models. *J. Phys. Oceanogr.* **13**, 1179–1192.

Longuet-Higgins, M.S. 1968. The eigenfunctions of Laplace's tidal equations over a sphere. *Phil. Trans. R. Soc. Lond. A.* **262**, 511–607.

Matsuno, T. 1966. Quasi-geostrophic motions in the equatorial area. *J. Meteorol. Soc. Japan.* **44**, 25–43.

Trefethen, L.N. 2000. *Spectral Methods in MATLAB*. SIAM, Philadelphia, PA, 165 pp.