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Horizontal convection with a non-linear equation of state: generalization of a theorem of Paparella and Young

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ABSTRACT

An upper bound is derived for the total dissipation rate in an ocean forced exclusively by surface fluxes of heat and freshwater, assuming a non-linear equation of state. This generalizes the upper bound found by Paparella and Young, which is valid for a flow forced by an imposed temperature distribution at the surface and a linear equation of state. Like this previous result, the present one shows that the dissipation rate vanishes in the limit of vanishing molecular diffusivity of temperature and salinity, if the range of temperatures and salinities occurring in the fluid is regarded as given. A numerical evaluation for realistic ocean parameters shows that the upper bound is two orders of magnitude smaller than present estimates of the energy transformation involved in the deep ocean circulation. This supports the conclusion that mechanical forcing by winds and tides is necessary to sustain the deep ocean circulation.

1. Introduction

One century ago, Sandström (1908) published a classical paper describing experiments he carried out in order to understand how the ocean circulation is forced. His results have since then been discussed widely, particularly during the last decade. A review of this discussion, as well as an English translation of the original paper, was recently given by Kuhlbrodt (2008).

Sandström concluded from his experiments that heating and cooling at the surface alone would not be able to excite a circulation in the interior of the ocean. At the same time, he pointed out that the depth of the Gulf Stream is more than 500 m. He proposed that the penetration of heat to such depths could be explained by what is now known as salt fingering, caused by evaporation in the tropics that tends to destabilize the water column. He also inferred from the observations of living animals in the deep ocean that a circulation must exist at great depths to ventilate the deep water, and tentatively suggested that the geothermal heat might play some role for this.

Sandström's conclusion that the surface heat flux cannot excite a circulation in the interior has subsequently been called 'Sandström's theorem', although he did not claim to prove a theorem, but rather reported on experimental observations. It was probably understood already by Sandström that his statement neglected the weak flow that is due to molecular diffusion, as later pointed out by Jeffreys (1925).

Later attempts to give a more rigorous formulation of 'Sandström's theorem' have followed two lines. Along the first line, Jeffreys (1925), Defant (1961) and Marchal (2007) have considered the circulation around a closed circuit in a stationary flow. They found that in order to balance friction, the direction of the flow on such a circuit must be such that the thermodynamic work is positive: fluid elements moving towards lighter regions must be at higher pressures than the those moving towards denser regions.

A limitation of this result is that the particle trajectories in a three-dimensional fluid are in general not closed, even if the flow is exactly stationary. In fact, the typical situation is that a single trajectory will eventually fill the whole fluid volume. A way to circumvent this problem is provided by the overturning stream function in depth-density coordinates introduced by Nycander et al. (2007). They showed that for a stationary flow driven exclusively by the surface heat flux, the integral of this stream function over the entire fluid must be positive. This means that in an average sense, fluid moving towards lighter regions must be at higher pressures than those moving towards denser regions. This result is valid even if there are no closed circuits.

All these results concern the direction of the overturning circulation, but they say nothing about its magnitude. Along the second line of work, a theorem that does this has been proved by Paparella and Young (2002). Considering a Boussinesq fluid with a linear equation of state, forced by differential heating at the surface, they derived an upper bound for the total rate

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of viscous dissipation. This upper bound is proportional to the molecular heat diffusivity, and to the temperature range ΔT (i.e. the difference between the maximum temperature and the minimum temperature occurring anywhere in the fluid).

If the surface temperature is prescribed (the Dirichlet boundary condition), ΔT is known a priori, and the dissipation rate therefore vanishes in the limit of vanishing diffusivity. This result was called the 'anti-turbulence theorem' by Paparella and Young. If instead the heat flux through the surface is prescribed (the Neumann condition), ΔT might conceivably grow indefinitely in the limit of vanishing diffusivity. The anti-turbulence theorem therefore is not valid with a flux boundary condition.

On the other hand, the upper bound provides a useful estimate of the dissipation rate, and hence the rate of energy transformation, if one simply uses the observed value of ΔT in the real ocean, as noted by Winters and Young (2009). The estimate obtained in this way is two orders of magnitude smaller than recent estimates of the rate of energy transformation based on the observed overturning circulation. This indicates that this circulation is not driven by the surface fluxes.

However, this conclusion remains somewhat uncertain, since both Paparella and Young (2002) and Winters and Young (2009) used a linearized equation of state, and neglected salinity. The purpose of this paper is to remove this uncertainty by generalizing their results to a case with surface fluxes of both heat and freshwater, and a non-linear equation of state.

After the submission of the present paper, I learned that the same result has been obtained by McIntyre (2009) in simultaneous and independent work in collaboration with F. Paparella and W. R. Young.

2. Proof

We start from the primitive equations in the Boussinesq approximation as given by Vallis (2006, p. 72). In this model, the equation of state defines bouyancy as a function of potential temperature, salinity and depth, rather than potential temperature, salinity and pressure. By this slight modification, an energy conserving model is obtained. (In his book, Vallis acknowledges this device as a personal communication from W. R. Young.) For reasons explained by Young (2009), we will also replace the customary potential temperature θ by the conservative temperature Θ. The model equations are then

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + f\hat{z} \times \mathbf{v} = -\frac{1}{\rho_0}\nabla p + b\hat{z} + \frac{\mathbf{F}}{\rho_0}, \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \nabla \cdot (\kappa_T \nabla\Theta), \tag{3}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \nabla \cdot (\kappa_S \nabla S), \tag{4}$$

$$b = b(\Theta, S, z), \tag{5}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$

Here p is pressure, ρ_0 a constant reference density and ρ_1 the deviation from this density, $b = -g\rho_1/\rho_0$ is buoyancy, $\mathbf{F}$ friction, Θ conservative temperature, S salinity and κ_T and κ_S represent molecular diffusivity of heat and salt, respectively. [The diffusive terms are of a conventional and simplified form; more complicated and accurate expressions for the diabatic molecular fluxes are given by Davis (1994).] We assume that the surface is covered by a rigid lid at $z = 0$, and that there is no wind stress. The flow is driven by surface fluxes of heat and freshwater, the latter being represented as a salinity flux because of the Boussinesq approximation. The fluxes through the bottom and the sidewalls are assumed to vanish.

Taking the scalar product between $\rho_0\mathbf{v}$ and eq. (1), and integrating over the entire fluid volume, we obtain the budget equation for kinetic energy

$$\frac{\mathrm{d}K}{\mathrm{d}t} = C - D, \tag{6}$$

where the kinetic energy is defined by

$$K = \int \rho_0 \frac{|\mathbf{v}|^2}{2}\, \mathrm{d}V, \tag{7}$$

the conversion between potential and kinetic energy by

$$C = \int \rho_0 b w\, \mathrm{d}V, \tag{8}$$

and the rate of viscous energy dissipation by

$$D = -\int \mathbf{v} \cdot \mathbf{F}\, \mathrm{d}V. \tag{9}$$

We then define the effective potential energy per unit mass h:

$$h(\Theta, S, z) = -\int_0^z b(\Theta, S, z')\, \mathrm{d}z'. \tag{10}$$

This can be interpreted as the energy required to move a fluid parcel adiabatically (i.e. conserving Θ and S) from the surface to its actual position, taking into account the fact that the buoyancy force varies with depth. It is the same as the Boussinesq dynamic enthalpy defined by Young (2009), but different from the gravitational potential energy (Nycander, 2009, to be published). Equation (10) gives

$$b = -\left(\frac{\partial h}{\partial z}\right)_{\Theta,S} \tag{11}$$

and

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \left(\frac{\partial h}{\partial \Theta}\right)_{S,z} \frac{\mathrm{d}\Theta}{\mathrm{d}t} + \left(\frac{\partial h}{\partial S}\right)_{\Theta,z} \frac{\mathrm{d}S}{\mathrm{d}t} - bw. \tag{12}$$

The effective potential energy of the whole fluid is

$$U = \int \rho_0 h\, \mathrm{d}V. \tag{13}$$

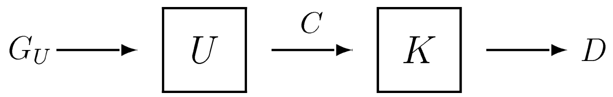

Fig. 1. Illustration of the energy budget equations (6) and (14). Here U is the effective potential energy, K the kinetic energy, C the conversion between potential and kinetic energy, D the viscous dissipation and G_U the generation of potential energy.

Using eq. (12) we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}t} = G_U - C, \tag{14}$$

where

$$G_U = \int \rho_0 \left[\left(\frac{\partial h}{\partial \Theta} \right)_{S,z} \nabla \cdot (\kappa_T \nabla \Theta) + \left(\frac{\partial h}{\partial S} \right)_{\Theta,z} \nabla \cdot (\kappa_S \nabla S) \right] \mathrm{d}V. \tag{15}$$

The budget eqs. (6) and (14) are illustrated by the box diagramme in Fig. 1. Evidently, in a statistical steady state we have

$$D = C = G_U \tag{16}$$

Thus, an upper bound on G_U also provides an upper bound on the total dissipation rate D. We therefore proceed to derive an upper bound on G_U.

The effective potential energy h is by definition zero at $z = 0$, implying that $\partial h/\partial\Theta$ and $\partial h/\partial S$ are also zero there. Since the heat flux and the salinity flux through the other boundaries vanish, all the boundary terms after a partial integration of eq. (15) vanish, and we obtain

$$G_U = -\int \rho_0 \left[\kappa_T \nabla\Theta \cdot \nabla \left(\frac{\partial h}{\partial \Theta} \right)_{S,z} + \kappa_S \nabla S \cdot \nabla \left(\frac{\partial h}{\partial S} \right)_{\Theta,z} \right] \mathrm{d}V.$$

This expression can be further rewritten as

$$G_U = -\int \rho_0 \left[\kappa_T \frac{\partial^2 h}{\partial \Theta^2} (\nabla\Theta)^2 + \kappa_S \frac{\partial^2 h}{\partial S^2} (\nabla S)^2 + (\kappa_T + \kappa_S) \frac{\partial^2 h}{\partial\Theta \partial S} \nabla\Theta \cdot \nabla S - \kappa_T \left(\frac{\partial b}{\partial \Theta} \right)_{S,z} \frac{\partial \Theta}{\partial z} - \kappa_S \left(\frac{\partial b}{\partial S} \right)_{\Theta,z} \frac{\partial S}{\partial z} \right] \mathrm{d}V. \tag{17}$$

We now assume that the equation of state has the simple analytic form given by Vallis (2006)

$$b = g \left[\frac{gz}{c^2} + \beta_T (1 - \gamma z)\Theta + \frac{\beta_T^*}{2} \Theta^2 - \beta_S S \right]. \tag{18}$$

Here c is the speed of sound, β_T the value of the thermal expansion coefficient at the surface, and β_S the haline contraction coefficient. The term proportional to γ describes the thermobaric effect, that is, the fact that the thermal expansion coefficient increases with increasing pressure, while the term proportional to β_T^* means that the thermal expansion coefficient increases with increasing temperature, which gives rise to cabbeling. These are the two main non-linear effects of the exact equation of state, and they are both retained in the simplified equation above. The reference temperature where Θ is approximately zero is 283 K. Numerical values of the coefficients in eq. (18) are given by Vallis (2006).

From eqs (10) and (18) we can calculate the effective potential energy

$$h = -g \left[\frac{gz^2}{2c^2} + \beta_T \left(z - \frac{\gamma z^2}{2} \right) \Theta + \frac{\beta_T^*}{2} z \Theta^2 - z \beta_S S \right]. \tag{19}$$

Evaluating the integrand in eq. (17) using eqs (18) and (19) we obtain

$$G_U = \int \rho_0 g \left[\kappa_T z \beta_T^* (\nabla\Theta)^2 + \kappa_T [\beta_T (1 - \gamma z) + \beta_T^* \Theta] \frac{\partial \Theta}{\partial z} - \kappa_S \beta_S \frac{\partial S}{\partial z} \right] \mathrm{d}V. \tag{20}$$

We now want to find an upper bound for the contribution from each term in the integrand above. To this end we define $\Delta\Theta$ as the maximum temperature range, that is, the difference between the maximum and the minimum value of Θ occurring anywhere in the fluid. Analogously, we define ΔS as the maximum salinity range. Since $z < 0$ the first term in the integrand of (20) is negative, and therefore bounded by zero. The other terms can be estimated after partial integration

$$\int \frac{\partial \Theta}{\partial z} \mathrm{d}V = \int [\Theta(x, y, 0) - \Theta(x, y, -H)] \, \mathrm{d}S \leq \int \Delta\Theta \, \mathrm{d}S,$$

and

$$-\int z \frac{\partial \Theta}{\partial z} \mathrm{d}V = \int H [\langle\Theta\rangle - \Theta(x, y, -H)] \, \mathrm{d}S \leq \int H \Delta\Theta \, \mathrm{d}S,$$

where $H(x, y)$ is the depth of the ocean, and $\langle\Theta\rangle$ is the depth average of Θ

$$\langle\Theta\rangle = \frac{1}{H} \int_{-H}^{0} \Theta \, \mathrm{d}z.$$

Furthermore,

$$\int \Theta \frac{\partial \Theta}{\partial z} \mathrm{d}V = \int \frac{\Theta(x, y, 0) + \Theta(x, y, -H)}{2} [\Theta(x, y, 0) - \Theta(x, y, -H)] \, \mathrm{d}S \leq \int \hat{\Theta} \Delta\Theta \, \mathrm{d}S,$$

where $\hat{\Theta}$ is the maximum of Θ, and

$$-\int \frac{\partial S}{\partial z} \mathrm{d}V = -\int [S(x, y, 0) - S(x, y, -H)] \, \mathrm{d}S \leq \int \Delta S \, \mathrm{d}S.$$

Thus, eq. (20) gives the estimate

$$G_U < \int \rho_0 g \left[\kappa_T \left(\beta_T (1 + \gamma H) + \beta_T^* \hat{\Theta} \right) \Delta\Theta + \kappa_S \beta_S \Delta S \right] \mathrm{d}S. \tag{21}$$

Denoting the surface area of the ocean by A, and using eq. (16), assuming steady state, we obtain the final estimate for the total dissipation rate

$$D < A\rho_0 g(\kappa_T \hat{\beta}_T \Delta\Theta + \kappa_S \beta_S \Delta S), \quad (22)$$

where $\hat{\beta}_T$ is defined by

$$\hat{\beta}_T = \beta_T(1 + \gamma \hat{H}) + \beta_T^* \hat{\Theta}, \quad (23)$$

and $\hat{H}$ is the maximum depth. We may think of $\hat{\beta}_T$ as the temperature expansion coefficient at the maximum values of the pressure and temperature that occur anywhere in the fluid. (This is larger than the maximum value of the expansion coefficient that actually occurs in the fluid, since the maximum temperature occurs near the surface, while the maximum pressure occurs at the bottom.)

The inequality (22) generalizes the anti-turbulence theorem by Paparella and Young (2002) to a non-linear equation of state. It shows that the dissipation rate vanishes in the limit of vanishing diffusivities, provided that the maximum range of temperature and salinity in the fluid is regarded as given.

As noted by Winters and Young (2009), we may also estimate the right-hand side of eq. (22) from realistic ocean parameters, regardless of the boundary condition. We set $A = 3.61 \times 10^{14}$ m^2, $\rho_0 = 1.0 \times 10^3$ kg m^{-3}, $g = 10$ m s^{-2}, $\kappa_T = 1.4 \times 10^{-7}$ m^2 s^{-1}, $\kappa_S = 1.5 \times 10^{-9}$ m^2 s^{-1}, $\Delta\Theta = 30$ K, $\Delta S = 5$ psu and $\beta_S = 0.78 \times 10^{-3}$ psu^{-1}. Using $\beta_T = 1.67 \times 10^{-4}$ K^{-1}, $\gamma = 1.1 \times 10^{-4}$ m^{-1}, $\beta_T^* = 1.0 \times 10^{-5}$ K^{-2}, $\hat{H} = 6000$ m and $\hat{\Theta} = 20$ K, we obtain $\hat{\beta}_T = 4.8 \times 10^{-4}$ K^{-1}. Using these values we obtain

$$D < 7.2 \times 10^9 \text{ W}$$

This may be compared to the value 2×10^{12} W needed to sustain the deep circulation according to the estimate of Munk and Wunsch (1998).

Note that the contribution from salinity in eq. (22) is negligible. Also, using the linearized equation of state corresponds to replacing $\hat{\beta}_T$ by β_T in eq. (22). Thus, the non-linearity of the equation of state increases the upper bound for the dissipation rate by approximately a factor of three.

3. Conclusion

The main result of this work is the upper bound (22) for the dissipation rate in an ocean forced solely by surface fluxes of heat and freshwater. The equation of state was here assumed to have the non-linear form given by eq. (18). This generalizes a similar bound that was derived by Paparella and Young (2002) for a linear equation of state, and without freshwater flux. Thus, the anti-turbulence theorem of Paparella and Young is also valid with a non-linear equation of state: if the range of temperature and salinity values occurring anywhere in the fluid is regarded as given (i.e. for Dirichlet conditions at the surface), the viscous dissipation rate D vanishes in the limit of vanishing diffusivities.

This theorem is perhaps mainly relevant for turbulence theory. In the real ocean, however, the diffusivities are fixed (but small). A more useful application of the upper bound is therefore to use the observed temperature and salinity ranges in the estimate. In this way one arrives at an upper bound for the dissipation rate that does not depend on the formulation of the surface boundary condition. Evaluating the right-hand side of (22) for realistic oceanic parameters in this way gives an upper bound which is two orders of magnitude smaller than the energy transformation involved in the deep ocean circulation, according to recent estimates. This supports the conclusion of Munk and Wunsch (1998), Paparella and Young (2002), Nycander et al. (2007) and others, that mechanical forcing by winds and tides is necessary to sustain the deep ocean circulation.

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