Tellus SERIES A DYNAMIC METEOROLOGY AND OCEANOGRAPHY
PUBLISHED BY THE INTERNATIONAL METEOROLOGICAL INSTITUTE IN STOCKHOLM

Tellus (2009), 61A, 631–634

TELLUS

Bayesian perspective of the unconventional approach for assimilating aliased radar radial velocities

By QIN XU*, NOAA/National Severe Storms Laboratory, Norman, OK 73072, USA

(Manuscript received 21 May 2009; in final form 12 August 2009)

ABSTRACT

The global minimization problem for directly assimilating aliased radial velocities is derived in terms of Bayesian estimation by folding the domain of the original Gaussian non-aliased observation probability density function (pdf) into the Nyquist interval. By truncating the folded tails of the observation pdf, the observation term in the costfunction recovers the aliased observation term formulated previously by an unconventional approach. This establishes the theoretical basis for the unconventional approach and quantifies the involved approximation. The alias-robust radar wind analysis developed based on the unconventional approach is also revisited from the Bayesian perspective.

1. Introduction

An aliasing operator was introduced in Xu et al. (2009, henceforth referred to as X09) to mimic the effect of aliasing (that causes discontinuities in radial-velocity observations) and to modify the observation term in the costfunction with an unconventional approach for direct assimilation of aliased radar radial velocities into numerical models. By applying the aliasing operator to the entire analysis-minus-observation term at each observation point in the observation term of the costfunction, the unconventional approach of X09 can make the costfunction smooth and convex in the vicinity of the global minimum. Although the global minimization problem is difficult in general due to multiple local minima caused by the zigzag-discontinuities of the aliasing operator, the problem can be solved efficiently for some practical applications in which the number of the control variables reduces drastically to three or less, such as the recently developed alias-robust velocity azimuth display (VAD) analysis (to be reported elsewhere) based on the unconventional approach of X09. The alias-robust VAD analysis can improve the environmental-wind information for dealiasing (Eilts and Smith, 1990). It may also replace the modified VAD (Tabary et al., 2001; Gao et al., 2004) to improve the VAD-based dealiasing (Gong et al., 2003) and variational dealiasing (Gao and Droegemeier, 2004).

The modified observation term in (6) of X09 is locally quadratic in the vicinity of the global minimum. When the associated minimization problem is placed into the general framework of Bayesian estimation (Jazwinski, 1970; Chapter 5), the locally quadratic form of the observation term implies that the observation probability density function (pdf) is assumed to be locally Gaussian in the vicinity of its global maximum. The theoretical basis and involved approximation of this assumption have not been studied yet and will be examined in this paper. The next section will review briefly the Bayesian perspective of the conventional minimization problem in radar radial-velocity data assimilation. Section 3 will revisit the global minimization problem presented in X09 in terms of Bayesian estimation. Section 4 will compare the alias-robust VAD wind analysis with the Bayesian VAD wind analysis of Tabary and Petitdidier (2002). Conclusions follow in section 5.

2. Bayesian perspective of conventional minimization problem

We denote by $\mathbf{v}$ the state vector for the gridded wind field to be analysed, by $p(\mathbf{v})$ the prior background pdf of $\mathbf{v}$, and by $p^{(n)}(\mathbf{v})$ the posterior analysis pdf of $\mathbf{v}$ updated by the radar observed radial velocities ($v_r{^\circ}_i$ for $i = 1, 2, \ldots n$) around the analysis time. According to the Bayesian rule, $p^{(n)}(\mathbf{v})$ is given by

$$\begin{aligned} p^{(n)}(\mathbf{v}) &= p(\mathbf{v}|v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n) \\ &= p(\mathbf{v})p(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n|\mathbf{v})/p(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n) \\ &\propto p(\mathbf{v})p(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n|\mathbf{v}), \end{aligned} \quad (1)$$

where $p(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n|\mathbf{v})$ is the pdf of $(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n)$ conditional by $\mathbf{v}$, and $p(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n) = \int d\mathbf{v}p(\mathbf{v})p(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n|\mathbf{v})$ is the normalization factor (independent of $\mathbf{v}$) for $p^{(n)}(\mathbf{v})$. The pdf can be updated by the observations either

*Corresponding author.
e-mail: Qin.Xu@noaa.gov
DOI: 10.1111/j.1600-0870.2009.00413.x

simultaneously by using (1) or serially by using

$$p^{(i)}(\mathbf{v}) = p^{(i-1)}(\mathbf{v})p(v_r{^\circ}_i|\mathbf{v}, v_r{^\circ}_1, \ldots v_r{^\circ}_{i-1})/p(v_r{^\circ}_i|v_r{^\circ}_1, \ldots v_r{^\circ}_{i-1}) \propto p^{(i-1)}(\mathbf{v})p(v_r{^\circ}_i|\mathbf{v}, v_r{^\circ}_1, \ldots v_r{^\circ}_{i-1}). \quad (2)$$

This equation shows how $p^{(i-1)}(\mathbf{v})$ is updated to $p^{(i)}(\mathbf{v})$ by the ith observation according to the Bayesian rule. When $i = n$, the serially updated pdf in (2) is equivalent to the simultaneously updated pdf in (1). This equivalence can be verified by substituting $p(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n|\mathbf{v}) = p(v_r{^\circ}_1|\mathbf{v})p(v_r{^\circ}_2|\mathbf{v}, v_r{^\circ}_1)\ldots p(v_r{^\circ}_n|\mathbf{v}, v_r{^\circ}_1, \ldots v_r{^\circ}_{n-1})$ and $p(v_r{^\circ}_1, v_r{^\circ}_2, \ldots v_r{^\circ}_n) = p(v_r{^\circ}_1)p(v_r{^\circ}_2|v_r{^\circ}_1)\ldots p(v_r{^\circ}_n|v_r{^\circ}_1, \ldots v_r{^\circ}_{n-1})$ into (1) and comparing the resulting form with the serial expansion of (2) for $i = n$.

The conventional minimization problem in radar wind data assimilation can be derived as a Bayesian estimate based on (1) with the following two assumptions: (i) The background pdf is Gaussian and thus has the following form:

$$p(\mathbf{v}) = [(2\pi)^m \mathrm{Det}(\mathbf{B})]^{-1/2} \exp\left[-(\mathbf{v}-\mathbf{v}^b)^T\mathbf{B}^{-1}(\mathbf{v}-\mathbf{v}^b)/2\right], \quad (3)$$

where m is the state vector dimension, $\mathbf{v}^b$ is the background mean, $\mathbf{B}$ is the background covariance matrix, and $()^T$ denotes the transpose of (). (ii) The observation error is Gaussian random, homogeneous and uncorrelated between different observation points, so the observation pdf is given by

$$p(v_r{^\circ}_1, \ldots v_r{^\circ}_n|\mathbf{v}) = \Pi_i p(v_r{^\circ}_i|\mathbf{v}) \quad (4a)$$

and

$$p(v_r{^\circ}_i|\mathbf{v}) = \left(2\pi\,\sigma_o{^2}\right)^{-1/2} \exp\left[-(v_r{^\circ} - \mathbf{Hv})_i^2/\left(2\sigma_o{^2}\right)\right], \quad (4b)$$

where $\prod_i$ denotes the multiplication over i from 1 to n, $()_i$ denotes the value of () at the ith observation point, $\mathbf{H}$ is the observation operator that projects and interpolates $\mathbf{v}$ onto the radial direction along the radar beam at each observation point, and $\sigma_o{^2}$ is the observation error variance. Substituting (3)-(4) into -ln(1) gives

$$-\ln p^{(n)}(\mathbf{v}) = J_b(\mathbf{v}) + J_o(\mathbf{v}) + c, \quad (5)$$

where $J_b(\mathbf{v}) = (\mathbf{v} - \mathbf{v}^b)^T\mathbf{B}^{-1}(\mathbf{v} - \mathbf{v}^b)/2$, $J_o(\mathbf{v}) = \sum_i (v_r{^\circ} - \mathbf{Hv})_i{^2}/(2\sigma_o{^2})$, and $\sum_i$ denotes the summation over i from 1 to n, and c is a constant independent of $\mathbf{v}$. Note that $J_b + J_o$ recovers the convention form of the costfunction used in radar wind data assimilation, which is a special case of the conventional form of the costfunction used in operational variational data assimilation [e.g. (2.1) of Parrish and Derber, 1992]. Thus, according to (5), minimizing $J_b + J_o$ is equivalent to maximizing $\ln p^{(n)}(\mathbf{v})$ or $p^{(n)}(\mathbf{v})$. This places the conventional minimization into the perspective of Bayesian estimation. For radial and tangential velocity analyses, $\mathbf{B}$ can be constructed by properly derived analytical covariance functions (Xu and Gong, 2003; Xu et al., 2006). If the state vector includes not only the velocity field but also the mass and/or thermodynamic fields, then the cross-covariances between these fields need to be considered in their joint pdf and included in the background term $J_b(\mathbf{v})$ in (5).

3. Global minimization problem revisited

In the presence of aliasing, the domain of the observation pdf in (4b) is folded from (-∞, ∞) into the Nyquist interval ($|v_r{^\circ} - \mathbf{Hv}| \le v_N$), so the aliased observation pdf is

$$\begin{aligned} p(v_r{^\circ}_i|\mathbf{v}) &= \left(2\pi\,\sigma_o{^2}\right)^{-1/2} \times \sum\nolimits_k \exp\left\{-[Z[v_r{^\circ} - \mathbf{Hv}, v_N]i - 2kv_N]^2/\left(2\sigma_o{^2}\right)\right\} \\ &= \left(2\pi\,\sigma_o{^2}\right)^{-1/2}[1 + \mu_i(\mathbf{v})] \times \exp\left\{-Z[v_r{^\circ} - \mathbf{Hv}, v_N]_i^2/\left(2\sigma_o{^2}\right)\right\}, \end{aligned} \quad (6)$$

where $\sum_k$ denotes the summation over integer k from -∞ to ∞, $Z[(), v_N] = () - 2v_N N[(), v_N]$ is the aliasing operator, v_N is the Nyquist velocity, $N[(), v_N] = \mathrm{Int}[()/(2v_N)]$ is the Nyquist number of (), Int[()] represents the nearest integer of (), and $\mu_i(\mathbf{v})$ is the sum of $\exp\{-2(k^2 - kZ[v_r{^\circ} - \mathbf{Hv}, v_N]_i/v_N)(v_N/\sigma_o)^2]$ over k from ±1 to ±∞ (with $k = 0$ excluded). Substituting (3)-(4a) with (6) into −ln(1) gives

$$-\ln p^{(n)}(\mathbf{v}) = J_b(\mathbf{v}) + J_{o2}(\mathbf{v}) + \sum\nolimits_i \ln[1 + \mu_i(\mathrm{v})] + c, \quad (7)$$

where $J_b(\mathbf{v})$ is the same as in (5), and $J_{o2}(\mathbf{v}) = \sum_i Z[v_r{^\circ} - \mathbf{Hv}, v_N]_i{^2}/(2\sigma_o{^2})$ recovers the modified observation term in (6) of X09. As shown in X09, $J_{o2}(\mathbf{v})$ is quadratic in the vicinity of the global minimum although its global structure is complicated by multiple local minima caused by the zigzag-discontinuities of the aliasing operator.

Note that $|Z[v_r{^\circ} - \mathbf{Hv}, v_N]_i/v_N| \le 1$ and $(v_N/\sigma_o)^2 \gg 1$ (because $v_N > 10\ \mathrm{m\ s^{-1}} \gg \sigma_o$ for the operational WSR-88D radars). Using these conditions, one can verify that $\mu_i(\mathbf{v}) \ll 1$ and $|\sum_i \ln[1 + \mu_i(\mathbf{v})]| \approx |\sum_i \mu_i(\mathbf{v})| \ll J_{o2}(\mathbf{v})$. The global minimizer of $J_b(\mathbf{v}) + J_{o2}(\mathbf{v})$ is thus a valid approximation of the global maximizer of $\ln p^{(n)}(\mathbf{v})$ or $p^{(n)}(\mathbf{v})$ in (7). This establishes the theoretical basis for the modified observation term J_{o2} originally formulated with the unconventional approach in section 2.2 of X09. The approximation involved in the original formulation of J_{o2} is thus quantified by the small term $\sum_i \ln[1 + \mu_i(\mathbf{v})]$ truncated from (7). As shown in X09, J_{o2} has multiple local minima and so does $J_b + J_{o2}$. This implies that the non-Gaussian $p^{(n)}(\mathbf{v})$ derived in (7) is multimodal, but in the vicinity of the global maximum it corresponds to the Gaussian pdf in (1) with the observation pdf given by the original non-aliased Gaussian form in (4b). Thus, as a Bayesian (conditional maximum likelihood) estimate with the observation pdf given by the aliased form in (6), the maximizer of $p^{(n)}(\mathbf{v})$ in (7) is identical to the Bayesian estimate with the observation pdf given by the non-aliased Gaussian form in (4b), but the latter requires the radial-velocity observations to be thoroughly dealiased first.

4. Bayesian VAD wind analysis

As shown in Xu et al. (2006; section 5a), the traditional radar VAD wind analysis can be viewed as a Bayesian estimate in the limit of infinitely large background error variance and de-correlation length scale with zero background mean. In this limit, (5) and (7) reduce to

$$-\ln p^{(n)}(\mathbf{v}_0) \propto -\ln[\Pi_i\, p(v_r{}^{\circ}{}_i|\mathbf{v}_0)] = J_o(\mathbf{v}_0) + c \tag{8}$$

and

$$-\ln p^{(n)}(\mathbf{v}_0) \propto -\ln[\Pi_i\, p(v_r{}^{\circ}{}_i|\mathbf{v}_0)] = J_{o2}(\mathbf{v}_0) + \sum\nolimits_i \ln[1+\mu_i(\mathbf{v}_0)] + c \approx J_{o2}(\mathbf{v}_0) + c, \tag{9}$$

respectively, where $\mathbf{v}_0 = (u_0, v_0, w_0)^T$ is the reduced state vector for the horizontally averaged velocity field in the VAD analysis. The radial velocity is thus related to $\mathbf{v}_0$ by $v_r = \mathbf{h}^T\mathbf{v}_0$, where $\mathbf{h}^T = (\sin\theta, \cos\theta \sin\phi, \cos\theta \cos\phi)$ is the reduced observation operator, θ is the elevation angle and ϕ is the azimuthal angle (clockwise with respect to the *y*-coordinate). Note that $J_o(\mathbf{v}_0) \propto \sum_i (v_r{}^{\circ} - \mathbf{h}^T\mathbf{v}_0)_i{}^2$ in (8) is the costfunction used by the traditional VAD analysis. Via (8), the traditional VAD analysis can be reviewed as a Bayesian estimate, as mentioned above. Likewise, $J_{o2}(\mathbf{v}_0) \propto \sum_i Z[v_r{}^{\circ} - \mathbf{h}^T\mathbf{v}_0, v_N]_i{}^2$ in (9) is the costfunction formulated in (6) of X09 for the alias-robust VAD analysis, so the alias-robust VAD analysis can be viewed via (9) as a Bayesian estimate while the observation pdf is given by the truncated Gaussian function in (6) with $\mu_i(\mathbf{v}) = 0$, as indicated by the approximation in the last step of (9).

Tabary and Petitdidier (2002) developed a Bayesian VAD wind analysis technique to retrieve wind profiles from raw (not dealiased) radial-velocity observations. In their technique, the most likely VAD wind was estimated from a discrete posterior pdf on a coarse grid of the horizontal VAD wind speed and direction. The discrete pdf was updated serially by radial-velocity observations with the observation pdf specified by a cosine-type function over the Nyquist interval [see their Eq. (14)]. If their cosine-type observation pdf is replaced by the truncated Gaussian pdf in (6) with $\mu_i(\mathbf{v}) = 0$, then their Bayesian VAD wind analysis can be simplified. In particular, the posterior pdf can be formulated analytically and updated implicitly by using the observations simultaneously without actually computing the pdf, because the analysis only needs to estimate the state vector by updating the global maximum of the pdf (or, equivalently, the global minimum of the costfunction) rather than the entire pdf. In principle, as shown by (1) and (2) in section 2, the updating should lead the same final result regardless whether the observations are used simultaneously or serially. For practical applications, however, it is often convenient and efficient to update the global minimum of the costfunction by using the observations simultaneously, and this is demonstrated by the recently developed alias-robust VAD analysis (to be reported elsewhere) based on the unconventional approach of X09.

5. Conclusions

In this paper, the global minimization problem formulated in X09 for directly assimilating aliased radar radial velocities is revisited in terms of Bayesian estimation. It is shown that the unconventional approach used to modify the observation term in (6) of X09 can be justified and established formally as a probabilistic approach based on the Bayesian estimation theory. In particular, the global minimization problem for directly assimilating aliased radial velocities can be derived in terms of Bayesian estimation by folding the domain of the original Gaussian non-aliased observation pdf into the Nyquist interval [see (6)]. The modified observation term in (6) of X09 can be then derived by truncating the folded tails of the Gaussian observation pdf [that is, neglecting $\mu_i(\mathbf{v})$ in (7)]. The truncation therefore quantifies the approximation involved in the unconventional approach of X09. The alias-robust VAD wind analysis based on the unconventional approach is then compared with the Bayesian VAD wind analysis of Tabary and Petitdidier (2002). The comparison suggests that using the truncated Gaussian observation pdf in place of their cosine-type observation pdf can simplify the analysis and improve the computational efficiency.

6. Acknowledgments

The author is thankful to Dr. David Parrish and Dr. Jidong Gao for their comments and suggestions that improved the presentation of the results. The research work was supported by the ONR Grant N000140410312 and by the NOAA HPCC program and NCEP radar data quality control project.

References

Eilts, M. D. and Smith, S. D. 1990. Efficient dealiasing of Doppler velocities using local environment constraints. *J. Atmos. Oceanic Technol.* **7**, 118–128.

Gao, J. and Droegemeier, K. K. 2004. A variational technique for dealiasing Doppler radial velocity data. *J. Appl. Meteor.* **43**, 934–940.

Gao, J., Droegemeier, K. K., Gong, J. and Xu, Q. 2004. A method for retrieving mean horizontal wind profiles from single-Doppler radar observations contaminated by aliasing. *Mon. Wea. Rev.* **132**, 1399–1409.

Gong, J., Wang, L. and Xu, Q. 2003. A three-step dealiasing method for Doppler velocity data quality control. *J. Atmos. Oceanic Technol.* **20**, 1738–1748.

Jazwinski, A. H. 1970. *Stochastic Processes and Filtering Theory*. Academic Press, New York, 376.

Parrish, D. and Derber, J. 1992. The National Meteorological Center's spectral statistical analysis system. *Mon. Wea. Rev.* **120**, 1747–1763.

Tabary, P., Scialom, G. and Germann, U. 2001. Real-time retrieval of the wind from aliased velocities measured by Doppler radars. *J. Atmos. Oceanic Technol.* **18**, 875–882.

Tabary, P. and Petitdidier, M. 2002. Application of a Bayesian wind profile retrieval technique to radar data collected in the Alpine southern upslope region and comparison with upstream wind profiler measurements. *J. Atmos. Oceanic Technol.* **19**, 875–887.

Xu, Q. and Gong, J. 2003. Background error covariance functions for Doppler radial-wind analysis. *Quart. J. Roy. Meteor. Soc.* **129**, 1703–1720.

Xu, Q., Liu, S. and Xue, M. 2006. Background error covariance functions for vector wind analyses using Doppler radar radial-velocity observations. *Quart. J. Roy. Meteor. Soc.* **132**, 2887–2904.

Xu, Q., Nai, K., Wei, L. and Zhao, Q. 2009. An unconventional approach for assimilating aliased radar radial velocities. *Tellus* **61A**, doi: 10.1111/j.1600-0870.2009.00412.x.