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TELLUS

LETTER TO THE EDITOR

Comment on "Computational periodicity as observed in a simple system," by Edward N. Lorenz (2006a)

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ABSTRACT

Systems of ordinary differential equations that exhibit chaotic responses have yet to be correctly integrated. So far no 'convergent' computational results have been determined for chaotic differential equations. Various computed numbers are not solutions of the continuous differential equations; all chaotic responses are simply numerical noise and have nothing to do with the solutions of differential equations. It would be an exciting contribution if a convergent computed chaotic solution for a Lorenz model could be obtained.

1. Introduction

Due to the continuous improvements in computers over the last one-half century, numerical simulations have developed into important tools, if not the only ones, for solving differential equations related to real problems from atomic to astronomic scales. Consequently, the number of engineers and scientists engaged in the use of such tools has increased dramatically. Computation has become, in many ways, the modern version of 'mathematical analysis,' and the kinds of problems analysed by numerical methods continues to grow. Most of the truly challenging problems, such as those related to chaos or turbulence, are routinely approached in this way.

Numerical methods convert continuous differential equations to a set of algebraic equations to be solved by computers. Derivatives in the continuous equations are replaced by corresponding discretized forms. For example, $\frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$, where $\frac{\Delta x}{\Delta t}$ equals $\frac{dx}{dt}$ in the limit as Δt approaches zero. Instead of taking the limit, Δt is kept as a finite but 'small' value in the discretizing procedure such that $\frac{\Delta x}{\Delta t}$ sufficiently approximates $\frac{dx}{dt}$ when Δt is 'small.' Von Neumann established that discretized algebraic equations must be *consistent* with the differential equations, and must be *stable* in order to obtain *convergent* numerical solutions for the given differential equations. This can be easily checked by ensuring that the difference between computed results for successively reduced time-step size is acceptably small. A successful check can be taken as an indication that $\frac{\Delta x}{\Delta t}$ is indeed a good approximation of $\frac{dx}{dt}$ and that the solutions of the discrete approximations have *converged* to the solution of the continuous equations. 'Any computed results that fail this check are numerical errors and have no mathematical meaning.' This is the most important fundamental rule in solving differential equations numerically by computers. Unfortunately, this rule has been consistently ignored in computational chaos, and has often been argued irrelevant by researchers in numerical chaos. One wonders how a set of discretized algebraic equations can be related to differential equations if convergence cannot be assured. This major point is the motivation for this comment on the paper by Lorenz (2006a).

Lorenz (2006a) studied numerically the following set of nonlinear differential equations:

$$\frac{dX}{dt} = -Y^2 - Z^2 - \frac{1}{4}(X - 8) \ , \tag{1a}$$

$$\frac{dY}{dt} = XY - 4XZ - Y + 1 \, , \tag{1b}$$

$$\frac{dZ}{dt} = 4XY + XZ - Z \, . \tag{1c}$$

The system of equations in (1) corresponds to the 'winter' system with time-independent source terms (Lorenz, 1990), referred to below as Lorenz II to distinguish it from his earlier system of equations, Lorenz I (Lorenz, 1963), which was discussed in Yao (2005).

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Lorenz (2006a) was led to investigate the characteristics of numerical solutions of system (1) by the comments of a reviewer of a companion paper in the same issue of the journal (Lorenz, 2006b). The reviewer had used a numerical method to integrate the system and had not been successful in reproducing the results in Lorenz (2006b). A similar situation occurred for the original Lorenz I system. Young (1966) showed that the standard fourth-order Runge–Kutta method gives numerical results different from those due to Lorenz (1963). The method used by Lorenz to integrate the Lorenz I equation system is a second-order method. While Young showed the different numerical results were due to the different numerical methods, neither author investigated the convergence of numerical methods.

Lorenz (2006a) used several numerical methods to demonstrate that computed results can be spurious if integration time steps are not small enough. He classified spurious solutions according to their properties as computational instability (CI), computational chaos (CC), and computational periodicity (CP). All spurious solutions are incorrect results of the discrete approximations and do not represent solutions of the ordinary differential equations (ODE). Lorenz also pointed out that the Courant–Friedrichs–Levy (CFL) condition, originally put forward for partial differential equations, is equally true for ordinary differential equations.

Cloutman (1996, 1998) has shown that the 'computational' classifications obtained by Lorenz can also be obtained from a single differential equation, and simple ODE systems, and are thus classifications of numerical errors and cannot be considered as mathematical descriptions of chaotic responses. It is well known from the Poincare–Bendixson Theorem that chaos cannot exist for a system whose dimension is smaller than three. For equations that cannot exhibit chaotic response, computational chaos can sometimes be removed by refining the step size. This is a classic example to show that numerical errors are frequently incorrectly interpreted as chaos.

For system (1), Lorenz showed that, for sufficiently small integration time steps, the largest computed Lyapunov exponent is positive and its values are nearly constant with small fluctuations; consequently, he concluded that the equations of system (1) have *exact* chaotic (numerical) solutions. This would be true if the numerical solutions were convergent. In the absence of convergence the Lyapunov exponents are properties of the discrete approximations and not properties of the continuous differential equations. We have repeated some of the numerical computations with system (1) in order to check for convergence and will discuss our findings below.

2. Some mathematical properties of Lorenz II

Before presenting our computational results, we will briefly discuss two properties of the non-linear differential equations in system (1). These equations have a single stationary point at (7.9963, −0.0065, 0.0298). The associated eigenvalues at the stationary point are −0.2498 and (6 .9962 ± i 31 .9852); thus, it is a reverse spiral point that is same as the two singular points of Lorenz I, which generate two wings of the butterfly. There is no virtual separatrix for a reverse spiral point; thus, no explosive amplification of truncation errors can occur for system (1) (Yao, 2005).

The divergence of the flow

$$\frac{\mathrm{d}}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}Y}\frac{\mathrm{d}Y}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}Z}{\mathrm{d}t} = 2\left(X - \frac{9}{8}\right) \tag{2}$$

is not always negative, so the system is not uniformly volume-contracting. Rigorously speaking, there is no attractor for Lorenz II (Viana, 2000).

3. Numerical Investigation

We have used several numerical methods to integrate system (1). These include the standard explicit first-order Euler method, a second-order mid-point method, the standard explicit fourth-order Runge–Kutta method, an implicit Euler method, various finite-difference approximations with various levels of implicitness, several off-the-shelve packaged ODE solvers, several of the ODE-solver routines given in Press et al. (1986), and a Taylor-series expansion method (Ascher and Petzold, 1998, 73–74), among other methods. The order of the Taylor-series expansion method ranged up to sixth.

The primary objective of these calculations was to investigate the convergence of discrete approximations to solutions of the continuous equations. By convergence we mean that, as the time-step size is refined, the numerical approximations for the dependent variables approach limiting values for all values of the independent variable. None of the numerical solution methods showed this property. The time-step size ranged from 0.025 to 10^{-7} and the period of integration ranged from about one year to over 30 yr in dimensionless time units. The largest step size is the value used by Lorenz (1990), and Pielke and Zeng (1994) in previous investigations. Neither Lorenz (1990) nor Pielke and Zeng (1994) checked the convergence of their numerical methods. So far as we know no other investigations into the convergence of the numerical solutions have been carried out for system (1).

A straightforward computed trajectory of Lorenz II is shown in Fig. 1. All numerical solution methods with all step sizes gave similar visual results. While the general appearance of the trajectory is similar for all numerical solution methods and step sizes, as will be shown below, the numerical values are different; hence they do not represent solutions of the continuous differential equations. It is clear that the trajectory looks like a loosely wound ball of yarn, and is not periodic. It is not centred at the stationary point; instead, its major span is between approximately $X = 0.2$ and 2.3. Even though there is no attractor, it seems to settle to an open attracting set. It does not have an attractor and is not periodic, leading to the question, what is it? We should

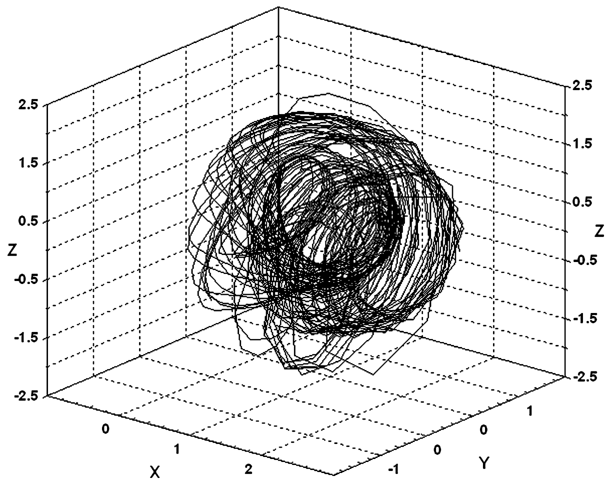

Fig. 1. The trajectory.

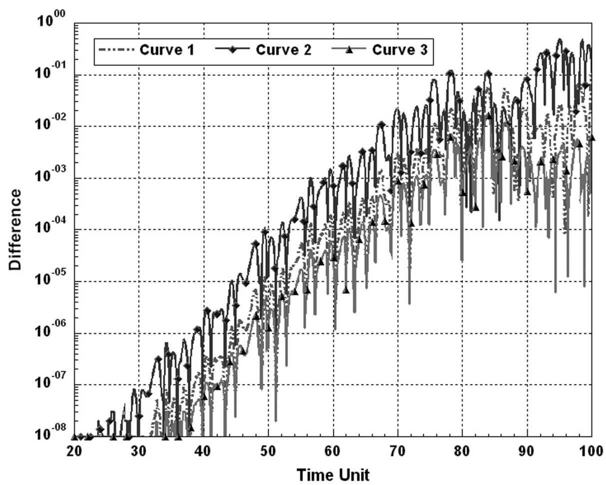

Fig. 2. Error plot.

point out that the computed trajectory shown is *not* a convergent computed result; hence it is not a correct solution of system (1).

Typical results from the numerical investigations into convergence are shown in Fig. 2. The $X(t)$ computed by the fifth-order Taylor-series method with 10^{-6} time step is selected as the reference case. The curves 1 and 2 are the difference of $X(t)$ by the Taylor-series method for time-steps 10^{-7} and 10^{-5} from the referenced case, respective. The curve 3 shows the absolute value of the difference between the explicit forth-order Runge–Kutta method for the time step 10^{-6} and the reference case. All other numerical methods that we have investigated show the same behaviour. These results are in complete agreement with the results presented by Teixeira et al. (2007), who investigated the original Lorenz I system (Lorenz, 1963) and other atmospheric models. As noted by Yao and Hughes (2007) in a comment about the paper of Teixeira et al. (2007), convergence of the numerical solution methods has not yet been shown for the Lorenz I system. The present results indicate that convergence for the Lorenz II system has also not yet been shown.

A close examination of Fig. 2 reveals that the time histories of numerical errors for two different time-steps or between two different methods are very similar in shape. The error amplification in time is not uniform, but occurs in ‘irregular valleys’. This suggests the existence of certain dynamic structures in the phase space, and seems to agree with the *exponential amplification* of errors described in Yao (2005). When two trajectories move along the direction of a stable manifold, the distance between them shrinks; in other words, errors are reduced. The consequence is the formation of valleys in Fig. 2, where the numerical error becomes a local minimum point. When trajectories move along an unstable manifold, errors are amplified. We have attempted to search in the phase space near the local minimum error point, but could not find any additional trajectory shape.

References

Ascher, U. M. and Petzold, L. R., 1998. *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*, Society for Industrial and Applied Mathematics, SIAM, New York.

Cloutman, L. D., 1996. *A Note on the Stability and Accuracy of Finite Difference Approximations to Differential Equations*. Lawrence Livermore National Laboratory, UCRL-ID-125549.

Cloutman, L. D., 1998. *Chaos and Instabilities in Finite Difference Approximations to Nonlinear Differential Equations*. Lawrence Livermore National Laboratory, UCRL-ID-131333.

Lorenz, E. N., 1963. Deterministic nonperiodic flow. *J. Atmos. Sci.* **20**, 130–141.

Lorenz, E. N., 1990. Can chaos and intransitivity lead to interannual variability? *Tellus* **42A**, 378–389.

Lorenz, E. N., 2006a. Computational periodicity as observed in a simple system. *Tellus* **58A**, 549–557.

Lorenz, E. N., 2006b. An attractor embedded in the atmosphere. *Tellus* **58A**, 425–429.

Pielke, R. A. and Zeng, X., 1994. Long-term variability of climate. *J. Atmos. Sci.* **51**(1), 155–159.

Press, W. H., Flannery, B. P., Teukolsky, S. A. and Vettering, W. T., 1986. *Numerical Recipes; The Art of Scientific Computing*, Cambridge University Press, Cambridge.

Teixeira, J., Reynolds, C. A. and Judd, K., 2007. Time step sensitivity of nonlinear atmospheric models: numerical convergence, truncation error growth, and ensemble design. *J. Atmos. Sci.* **64**, 175–189.

Viana M., 2000. What’s new on Lorenz strange attractors? *Math. Intell.* **22**, 6–19.

Yao, L. S., 2005. Computed chaos or numerical errors. online manuscript, http://arxiv.org/abs/nlin.CD/0506045.

Yao, L. S. and Hughes, D., 2007. Comment on, “time step sensitivity of nonlinear atmospheric models: numerical convergence, truncation error growth, and ensemble design”. *J. Atmos. Sci.*, **65**(2), 681–682.

Young, C., 1966. Comment on “deterministic nonperiodic flow.” *J. Atmos. Sci.* **23**, 628–629.