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Eulerian and Lagrangian observability of point vortex flows

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ABSTRACT

We study the observability of one and two-point vortex flow from one or two Eulerian or Lagrangian observations. By observability, we mean the ability to determine the locations and strengths of the vortices from the time history of the observations. An Eulerian observation is a measurement of the velocity of the flow at a fixed point in the domain of the flow. A Lagrangian observation is the measurement of the position of a particle moving with the fluid. To determine observability, we introduce the observability and the strong observability rank conditions and compute them for the various vortex configurations and observations in this idealized setting. We find that vortex flows with Lagrangian observations tend to be more observable then the same flows with Eulerian observations. We also simulate extended Kalman filters for the various vortex configurations and observations and find that they perform poorly when the ORC or the strong observability rank condition fails to hold.

1. Introduction

A fundamental problem of Meteorology or Oceanography is to estimate the current and future state of the atmosphere or ocean from current and past measurements. The incorporation of new observations into a model of the atmosphere or ocean is called data assimilation, and there are a variety of ways of doing so, many based on variational methods.

Is it possible to determine the state of the atmosphere or ocean from measurements? This is a difficult question for a complex flow; so, we shall address it in a simpler setting—the two-dimensional flow induced by one or two point vortices in the plane and one or two noise free Eulerian or Lagrangian observations. Admittedly these are ideal situations, but we can't hope to understand the observability of more realistic flows unless we understand the observability of simpler ones. A flow is observable under a set of measurements if the time history of the measurements uniquely determines the flow. We are following earlier work on filtering of vortex flows found in Ide and Ghil (1997a,b).

Consider m point vortices in the plane. Corresponding to the j^{th} vortex, there are three parameters $x = (x_{j1}, x_{j2}, x_{j3})$ that completely determine it. The first two are the coordinates of its centre and the third is its strength. The velocity field at $(\xi_1, \xi_2) \in I\!R^2$ induced by this vortex is

$$u(x_j, \xi) = \frac{x_{j3}}{r_j^2}\begin{bmatrix} x_{j2} - \xi_2 \\ \xi_1 - x_{j1} \end{bmatrix}, \qquad (1)$$

where $r_j^2 = (\xi_1 - x_{j1})^2 + (\xi_2 - x_{j2})^2$. This is an inviscid, incompressible and irrotational flow with a singularity at $\xi = (x_{j1}, x_{j2})$.

The flow induced by all m vortices is

$$u(x, \xi) = \sum_{j=1}^{m} u_j(x_j, \xi). \qquad (2)$$

This is also inviscid, incompressible and irrotational with m singularities at the centres of the vortices. The centre of the k^{th} vortex moves with the flow induced by the remaining $m - 1$ vortices and its strength does not change

$$\begin{bmatrix} \dot{x}_{k1} \\ \dot{x}_{k2} \\ \dot{x}_{k3} \end{bmatrix} = f_k(x) = \sum_{j \neq k} \begin{bmatrix} u_j(x_j, (x_{k1}, x_{k2})) \\ 0 \end{bmatrix}. \qquad (3)$$

There is rich literature on vortex flow. For an introduction we refer the reader to the text Acheson (1990). For further information, the review article Aref (1983) is excellent.

We shall study the observability of vortex flows under two types of measurements. An Eulerian observation is a measurement of the velocity (2) of the flow at a fixed point $\xi^i \in I\!R^2$. A Lagrangian observation is a measurement of the location $\xi^i(t)$ of a particle moving with the flow. We may have more than one

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observation. A flow is said to be observable if the observations uniquely determine it.

Here is an outline of the rest of the paper. In Section 2, we study the observability of a single vortex under one Eulerian or Lagrangian observation. In Section 3, we introduce the observability rank condition (ORC), a test of the observability of an observed dynamical system. In Section 4, this is used to test the observability of a single vortex under either an Eulerian or a Lagrangian observation. The next section introduces the reader to two vortex flow. In Section 6, we study its observability under one or two Eulerian observations. In Section 7, we study the observability of two vortex flow under one or two Lagrangian observations. Section 9 contains the results of some numerical experiments, we study the abilities of continuous time extended Kalman filters to estimate the centres and strengths of two vortices from an Eulerian or a Lagrangian observation.

2. Observability of one vortex flow

Consider a single point vortex of unknown position and strength. There are three state variables, the location of the centre and the strength of the vortex. The dynamics is trivial as none of these variables change. Suppose, we have an Eulerian observation, that is, we measure the velocity of the flow at some point in the plane. With this observation, we cannot determine all three state variables. We know that the centre lies on a line perpendicular to the observed velocity, but we don't know where on the line it is because we don't know its strength.

If we have Eulerian observations of the velocity at two points in the plane and if these points are not collinear with the centre of the vortex, then we know the centre is at the intersection of the perpendiculars to the velocities. Once we know where the centre is, we can determine the strength of the vortex from an observed velocity.

If the two observation points are collinear with the centre of the vortex, then the two observations are parallel. It is still observable, but a bit of analysis is needed. Without loss of generality, we can assume the first observation point at the origin, the second observation at $(r, 0)$ and the vortex at $(s, 0)$. The number r is known to us but s is not. Suppose the velocity observed at the origin is $(0, v)$, the velocity observed at the other observation point is $(0, w)$ and the unknown strength of the vortex is c. Then we have the relations

$$v = \frac{c}{s},$$
$$w = \frac{c}{r-s}.$$

We can solve the first equation for c and plug this into the second to get

$$w(r-s) = vs.$$

From this we conclude

$$s = \frac{wr}{v+w},$$

$$c = \frac{vwr}{v+w}.$$

Now consider the flow of a single vortex with a single Lagrangian observation of position. A Lagrangian observation is the position of a fluid particle moving with the flow. From the history of the position, we can obtain the velocity of our Lagrangian particle. If we take perpendiculars to the velocities at two different times, then they will intersect at the centre of the vortex and the strength of the velocities determines its strength. Hence one vortex flow with one Lagrangian observation is always observable, whereas one vortex flow with one Eulerian observation is never observable. Two vortex flow with one Eulerian or Lagrangian observation is not always observable as we shall see below.

3. Observability rank condition

We next discuss a sufficient condition for the short-time local observability of a dynamical system with observations. Consider an observed dynamics

$$\dot{x} = f(x), \tag{4}$$

$$y = h(x), \tag{5}$$

$$x(0) = x^0. \tag{6}$$

The state $x \in \mathbb{R}^n$ or are local coordinates on a manifold locally diffeomorphic to $\mathbb{R}^n$. For simplicity of exposition, we shall assume the former, but all our results readily generalize to the latter. We shall also assume that f and g are sufficiently smooth functions. The state is not observed directly but the output $y \in \mathbb{R}^p$ is. The system is observable if the map from initial state to output history,

$$x^0 \mapsto y(0:\infty),$$

is one to one. The symbol $y(0:T)$ denotes the trajectory

$$t \mapsto y(t), \quad 0 \le t < T.$$

Observability is an idealized concept; in most practical problems there is noise in both the system dynamics and the observations and these noises make estimation of the state more difficult. If the system is not observable, then there is no hope of an accurate estimate.

In other words, the observed system (4, 5) is observable if the output time trajectory uniquely determines the initial state. The system is locally observable if this map is locally one to one. In other words, neighbouring initial states lead to different output trajectories. Local observability is weaker than observability, but for a complex nonlinear system, it might be all that is verifiable.

The system is short time observable if the map

$$x^0 \mapsto y(0:T)$$

is one to one for every $T > 0$. In other words, an output trajectory immediately distinguishes its initial state. Short time observability is a stronger property than observability, and it is essential to real time state estimation. The system is short time, locally observable if this map is locally one to one.

We can extend these definitions to time varying systems

$$\dot{x} = f(t, x),$$
$$y = h(t, x),$$
$$x(t^0) = x^0,$$

simply by introducing time as an extra state and output variable

$$x_0 = t - t_0,$$
$$y_0 = t - t_0,$$
$$\dot{x}_0 = 1.$$

There is a sufficient condition for short time, local observability. First, let us discuss some notation. The exterior derivative of the function h is the one-form

$$dh(x) = \frac{\partial h}{\partial x_j}(x) dx_j,$$

with the summation convention on repeated indices understood. If h is column vector valued, then dh is a column of one-forms.

The Lie derivative of the function h by the vector field f is the function

$$L_f(h)(x) = \frac{\partial h}{\partial x_j}(x) f_j(x).$$

If h is column vector valued, then so is $L_f(h)$.

We can iterate this operation

$$L^0_f(h)(x) = h(x),$$
$$L^r_f(h)(x) = \frac{\partial L^{r-1} h}{\partial x_j}(x) f_j(x),$$
$$\vdots$$

for $r = 1, 2, \ldots$

Definition 1. The observed system (4, 5) satisfies the ORC at x if

$$\left\{ dL^r_f(h)(x) : r = 0, 1, 2, \ldots \right\} \tag{7}$$

contains n linearly independent covectors. The observed system (4, 5) satisfies the ORC if it satisfies the ORC at every $x \in I\!R^n$.

Let $\lceil n/p \rceil$ denote the smallest integer greater than or equal to n/p. The observed system satisfies the strong observability rank condition (SORC) at x if the covectors

$$\left\{ dL^r_f(h)(x) : r = 0, 1, 2, \ldots, \lceil n/p \rceil - 1 \right\} \tag{8}$$

are linearly independent and

$$\left\{ dL^r_f(h)(x) : r = 0, 1, 2, \ldots, \lceil n/p \rceil \right\} \tag{9}$$

contains n linearly independent covectors. The observed system satisfies the SORC if it satisfies the SORC at every $x \in I\!R^n$.

Theorem 1. Suppose the observed system (4, 5) satisfies the ORC, then it is short time, locally observable.

Proof. Consider the output and its derivatives at $t = 0$ from the initial condition x^0. A simple calculation shows that

$$\begin{aligned} y(0) &= h(x^0), \\ \dot{y}(0) &= L_f(h)(x^0), \\ \ddot{y}(0) &= L^2_f(h)(x^0), \\ &\vdots \\ \frac{d^r y}{dt^r}(0) &= L^r_f(h)(x^0), \\ &\vdots \end{aligned} \tag{10}$$

Clearly, if the functions $h, L_f(h), \ldots, L^r_f(h), \ldots$ locally distinguish points then the system is short time, locally observable. These functions will locally distinguish points if the ORC is satisfied.

This ORC is almost necessary for short-time local observability. If the ORC is violated on an open subset of $I\!R^n$ then (4, 5) is not short time, locally observable. For the proof, see theorem 3.9 of Hermann and Krener (1977).

If a system satisfies the SORC, then we need the minimum number of time differentiations (10) to distinguish points. The SORC is a sufficient condition for the local convergence of an extended Kalman filter Krener (2002).

4. ORC for finite dimensional flows

A finite dimensional flow in a domain $\Omega \subset I\!R^d$, $d = 2$ or 3, is a finite dimensional dynamics (4) whose state $x(t)$ determines the flow field on Ω at time t. Let ξ be coordinates on Ω. Corresponding to $x \in I\!R^n$, the velocity of the fluid at ξ is $u(x, \xi) \in I\!R^d$.

As an example, consider m point vortices in the plane. The state x lives in $I\!R^n$, where $n = 3m$ and consists of the locations and strengths of the m vortices. Once we know x, we know the flow field (2) induced by these vortices.

Here is another example. Suppose we discretize the Navier–Stokes equations in space by a finite element or finite difference scheme and leave time continuous (the method of lines). The state x of (4) are the parameters of discretization.

We consider two types of observations of the fluid. A set of m Eulerian observations are measurements of the flow velocity at m fixed, distinct locations. It takes the form

$$y^i = h^i(x) = u(x, \xi^i), \tag{11}$$

for some fixed $\xi^i \in \Omega$ and $i = 1, \ldots, m$. Now $y^i \in I\!R^d$ and the total number of observations is $p = dm$.

A set of m Lagrangian observations are the positions of m distinct particles moving with the flow. To model it, we use the technique of Ide et al. (2002) and Kuznetsov et al. (2003). We define an augmented system with state

$$z = \begin{bmatrix} x \\ \xi^1 \\ \vdots \\ \xi^m \end{bmatrix},$$

with $\xi^i \in \Omega \subset I\!R^d$ and dynamics

$$\dot{z} = g(z) = \begin{bmatrix} f(x) \\ u(x, \xi^1) \\ \vdots \\ u(x, \xi^m) \end{bmatrix}, \tag{12}$$

where $\xi^i(t)$ is the location of the i^{th} Lagrangian observation at time t. The Lagrangian observations are

$$w^i = k^i(z) = \xi^i, \tag{13}$$

for $i = 1, \ldots, m$. The observations $w^i \in I\!R^d$ and the total number of observed variables is $p = dm$.

Note that

$$L_g(k^i)(z) = u(x, \xi^i),$$

so, one might expect that the ORC for the system (4) with m Eulerian observations (11) is closely related to the ORC for the augmented system (12) with m Lagrangian observations (13). This is true up to a point.

We calculate the first few terms of (7) for the two systems. For the fluid (4) with m Eulerian observations (11) we have,

$$dh^i(x) = du(x, \xi^i) = \frac{\partial u}{\partial x_j}(x, \xi^i) dx_j, \tag{14}$$

$$\begin{aligned} dL_f(h^i)(x) &= dL_f(u)(x, \xi^i) \\ &= \left(\frac{\partial^2 u}{\partial x_j \partial x_l}(x) f_l(x) + \frac{\partial u}{\partial x_l}(x) \frac{\partial f_l}{\partial x_j}(x) \right) dx_j. \end{aligned} \tag{15}$$

We should note that h^i is d vector valued, so, dh^i is a d column vector whose components are one-forms and so is $dL_f(h^i)$.

Let d_z be the exterior differentiation operator in the z variables, that is,

$$d_z k^i(z) = \frac{\partial k^i}{\partial x_j}(x, \xi^1, \ldots, \xi^k) dx_j + \frac{\partial k^i}{\partial \xi^i}(x, \xi^1, \ldots, \xi^k) d\xi^i.$$

If ξ lives in $I\!R^d$, then k^i takes values in $I\!R^d$ and $\frac{\partial k^i}{\partial \xi^i}(x, \xi^1, \ldots, \xi^m)$ is a $d \times d$ matrix and $d\xi^i$ is a column of d one-forms on $I\!R^d$.

For the augmented system (12), with m Lagangian observations (13), we have

$$d_z k^i(z) = d\xi^i, \tag{16}$$

$$d_z L_g(k^i)(z) = d_z u(x, \xi^i) = \frac{\partial u}{\partial x_j}(x, \xi^i) dx_j, \tag{17}$$

$$\mod \left\{ d\xi^1, \ldots, d\xi^m \right\},$$

$$\begin{aligned} d_z L_g^2(k^i)(z) &= \left(\frac{\partial^2 u}{\partial x_l \partial x_j}(x) f_l(x) + \frac{\partial u}{\partial x_l}(x) \frac{\partial f_l}{\partial x_j}(x) \right) dx_j \\ &\quad + \frac{\partial^2 u}{\partial x_j \partial \xi}(x, \xi^i) u(x, \xi^i)\, dx_j, \\ &\quad \mod \left\{ d\xi^1, \ldots, d\xi^m, dL_g(\xi^1), \ldots, dL_g(\xi^m) \right\}. \end{aligned} \tag{19}$$

Note that (16) span the extra dimensions of the augmented system. Modulo (16), the one-forms (17) span the same dx_j dimensions as (14). So far so good, but in general (19) does not span the same dx_j dimensions as (15) modulo the one-forms (16), (17) because (19) has the extra term

$$\frac{\partial^2 u}{\partial x_j \partial \xi}(x, \xi^i) u(x, \xi^i) dx_j. \tag{20}$$

The flow of one point vortex, with one observation discussed above, illustrates this point. Let x_1, x_2 denote the centre of the vortex and x_3 its strength. The dynamics (4) is trivial,

$$\dot{x} = f(x) = 0$$

The flow field in $I\!R^2$ corresponding to x is

$$u(x, \xi) = \frac{x_3}{r^2} \begin{bmatrix} \xi_2 - x_2 \\ x_1 - \xi_1 \end{bmatrix}, \tag{21}$$

where $r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2$.

Suppose there is one Eulerian observation, without loss of generality, at the origin $\xi^1 = (0, 0)$, then (11) becomes

$$y = h^1(x) = u(x, \xi^1) = \frac{x_3}{r^2} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

and

$$\begin{aligned} &dh^1(x) \\ &\quad = \frac{1}{r^4} \begin{bmatrix} 2x_1 x_2 x_3 dx_1 + \left(x_2^2 - x_1^2\right) x_3 dx_2 - x_2 r^2 dx_3 \\ \left(x_2^2 - x_1^2\right) x_3 dx_1 - 2x_1 x_2 x_3 dx_2 + x_1 r^2 dx_3 \end{bmatrix}, \\ &\quad dL_f^r(h^1)(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad r = 1, 2, \ldots. \end{aligned}$$

Without loss of generality, we can assume that the centre of the vortex lies on the x_1 axis, it is not at the origin and it has nonzero strength; so, $x_1 \neq 0$, $x_2 = 0$ and $x_3 \neq 0$. Then

$$dh^1(x) = \begin{bmatrix} -\frac{x_3}{x_1^2} dx_2 \\ \\ -\frac{x_3}{x_1^2} dx_1 + \frac{1}{x_1} dx_3 \end{bmatrix},$$

so, the rank of (7) is 2.

We have two independent one-forms, so, the three-dimensional Eulerian observed system does not satisfy the ORC.

Note that a vector that annihilates dh^1 is

$$v = \begin{bmatrix} 1 \\ 0 \\ \frac{x_3}{x_1} \end{bmatrix},$$

so, the observation is insensitive to the vortex moving farther from it while increasing its strength. If we knew the strength, x_3, of the vortex, the one vortex system with one Eulerian observation is observable.

If we have one Lagrangian observation, then the augmented dynamics is

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{\xi}_1^1 \\ \dot{\xi}_2^1 \end{bmatrix} = g(z) = \begin{bmatrix} 0 \\ \frac{x_3}{r^2}\left(\xi_2^1 - x_2\right) \\ \frac{x_3}{r^2}\left(x_1 - \xi_1^1\right) \end{bmatrix},$$

and the observation is

$$\begin{bmatrix} y_1^1 \\ y_2^1 \end{bmatrix} = k^1(z) = \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \end{bmatrix}.$$

If we assume, without loss of generality, that at some t, the Lagrangian observation is made at the origin, $\xi^1(t) = 0$, and $x_1 \neq 0$, $x_2 = 0$ and $x_3 \neq 0$,

$$d_z k^1(z) = \begin{bmatrix} d\xi_1^1 \\ d\xi_2^1 \end{bmatrix}, \tag{22}$$

$$d_z L_g(k^1)(z) = \begin{bmatrix} -\frac{x_3}{x_1^2} dx_2 \\ -\frac{x_3}{x_1^2} dx_1 + \frac{1}{x_1} dx_3 \end{bmatrix}. \tag{23}$$

We need to compute (20). We start with (21) and compute

$$\frac{\partial u}{\partial \xi}(x, \xi) = \frac{x_3}{r^4} \begin{bmatrix} -2(x_1-\xi_1)(x_2-\xi_2) & (x_1-\xi_1)^2 - (x_2-\xi_2)^2 \\ (x_2-\xi_2)^2 - (x_1-\xi_1)^2 & 2(x_1-\xi_1)(x_2-\xi_2) \end{bmatrix}.$$

Then

$$\frac{\partial^2 u}{\partial x_1 \partial \xi}(x, \xi) = \frac{x_3}{r^4} \begin{bmatrix} -2(x_2-\xi_2) & 2(x_1-\xi_1) \\ -2(x_1-\xi_1) & 2(x_2-\xi_2) \end{bmatrix} - \frac{4x_3(x_1-\xi_1)}{r^6} \begin{bmatrix} -2(x_1-\xi_1)(x_2-\xi_2) & (x_1-\xi_1)^2 - (x_2-\xi_2)^2 \\ (x_2-\xi_2)^2 - (x_1-\xi_1)^2 & 2(x_1-\xi_1)(x_2-\xi_2) \end{bmatrix}$$

$$= \frac{x_3}{r^6} \begin{bmatrix} -(x_2-\xi_2)\left(6(x_1-\xi_1)^2 - 2(x_2-\xi_2)^2\right) & -(x_1-\xi_1)\left(2(x_1-\xi_1)^2 - 6(x_2-\xi_2)^2\right) \\ (x_1-\xi_1)\left(2(x_1-\xi_1)^2 - 6(x_2-\xi_2)^2\right) & (x_2-\xi_2)\left(6(x_1-\xi_1)^2 - 2(x_2-\xi_2)^2\right) \end{bmatrix},$$

$$\frac{\partial^2 u}{\partial x_2 \partial \xi}(x, \xi) = \frac{x_3}{r^4} \begin{bmatrix} -2(x_1-\xi_1) & -2(x_2-\xi_2) \\ 2(x_2-\xi_2) & -2(x_1-\xi_1) \end{bmatrix} - \frac{4x_3(x_2-\xi_2)}{r^6} \begin{bmatrix} -2(x_1-\xi_1)(x_2-\xi_2) & (x_1-\xi_1)^2 - (x_2-\xi_2)^2 \\ (x_2-\xi_2)^2 - (x_1-\xi_1)^2 & 2(x_1-\xi_1)(x_2-\xi_2) \end{bmatrix}$$

$$= \frac{x_3}{r^6} \begin{bmatrix} -(x_1-\xi_1)\left(2(x_1-\xi_1)^2 - 6(x_2-\xi_2)^2\right) & -(x_2-\xi_2)\left(6(x_1-\xi_1)^2 - 2(x_2-\xi_2)^2\right) \\ (x_2-\xi_2)\left(6(x_1-\xi_1)^2 - 2(x_2-\xi_2)^2\right) & (x_1-\xi_1)\left(2(x_1-\xi_1)^2 - 6(x_2-\xi_2)^2\right) \end{bmatrix}$$

and

$$\frac{\partial^2 u}{\partial x_3 \partial \xi}(x, \xi) = \frac{1}{r^4} \begin{bmatrix} -2(x_1-\xi_1)(x_2-\xi_2) & (x_1-\xi_1)^2 - (x_2-\xi_2)^2 \\ (x_2-\xi_2)^2 - (x_1-\xi_1)^2 & 2(x_1-\xi_1)(x_2-\xi_2) \end{bmatrix}.$$

Before we compute (20), we can make some simplifying assumptions because we are not going to differentiate further. We assume we are at a time t where $\xi^1(t) = 0$ and $x_1 \neq 0, x_2 = 0$, $x_3 \neq 0$. Then at this t

$$\frac{\partial^2 u}{\partial x_1 \partial \xi}(x, \xi) = \frac{x_3}{x_1^3} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},$$

$$\frac{\partial^2 u}{\partial x_2 \partial \xi}(x, \xi) = \frac{x_3}{x_1^3} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

$$\frac{\partial^2 u}{\partial x_3 \partial \xi}(x, \xi) = \frac{1}{x_1^2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and

$$u = \frac{x_3}{x_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so the extra term (20) is

$$\begin{bmatrix} -\frac{2x_3^2}{x_1^4} dx_1 + \frac{x_3}{x_1^3} dx_3 \\ -\frac{2x_3^2}{x_1^4} dx_2 \end{bmatrix} \tag{24}$$

The ORC is satisfied if there are three linearly independent one-form among (23) and (24). To determine this, we evaluate the determinant of

$$\begin{bmatrix} -\dfrac{x_3}{x_1^2} & \dfrac{1}{x_1} \\ -\dfrac{2x_3^2}{x_1^4} & \dfrac{x_3}{x_1^3} \end{bmatrix}$$

and see that it is

$$\frac{x_3^2}{x_1^5},$$

which is not zero. Hence the SORC holds for the flow of one vortex with one Lagrangian observation. As we have seen, the ORC does not hold for the flow of one vortex with one Eulerian observation.

5. Two vortex flow

Two vortex flow can be quite complicated but the motion of the centres of the vortices is relativity simple. The centres move on a circle, on a pair of concentric circles or along parallel straight lines.

The system is six-dimensional, the centre of one vortex is at x_{11}, x_{12} and its strength is x_{13} while the centre of the other vortex is at x_{21}, x_{22} and its strength is x_{23}. The dynamics is

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_{21} \\ \dot{x}_{22} \\ \dot{x}_{23} \end{bmatrix} = f(x) = \begin{bmatrix} \frac{x_{23}}{r^2}\,(x_{22} - x_{12}) \\ \frac{x_{23}}{r^2}\,(x_{11} - x_{21}) \\ 0 \\ \frac{x_{13}}{r^2}\,(x_{12} - x_{22}) \\ \frac{x_{13}}{r^2}\,(x_{21} - x_{11}) \\ 0 \end{bmatrix} \tag{25}$$

where $r^2 = (x_{11} - x_{21})^2 + (x_{12} - x_{22})^2$. The distance r between the centres remains constant because each centre moves perpendicular to the line between them.

If the strengths are equal, $x_{13} = x_{23}$, then the centres will rotate around a single circle staying as far away as possible, see Fig. 1.

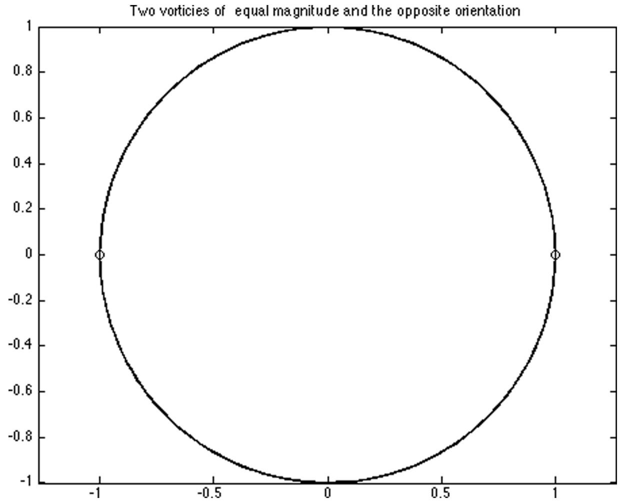

Fig. 1. The motion of the centres of two vortices of equal strengths and the same orientation.

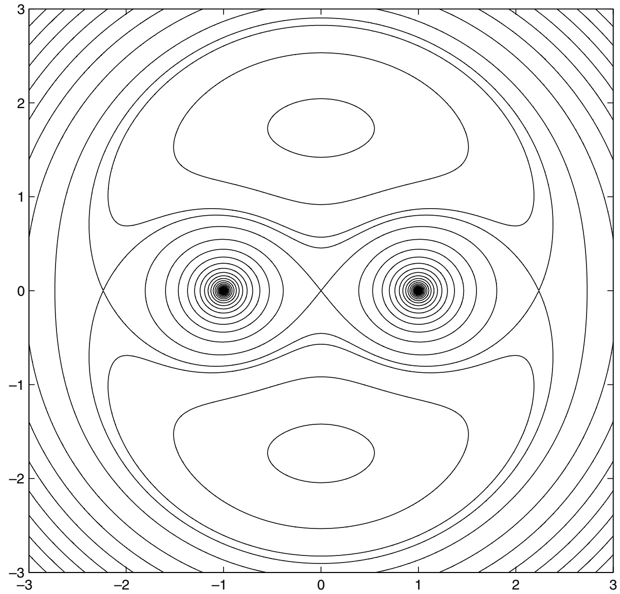

Fig. 2. The streamlines in a corotating frame off two vortices of equal strengths and the same orientation.

When the vortices rotate on a circle or on a pair of concentric circles, it is informative to consider the flow in the frame that corotates with the vortices. A corotating point is one where the flow appears stationary in this corotating frame. For reasons that will become apparent later, we are particularly interested in corotating points that are collinear with the centres of the vortices.

Figure 2 shows the streamlines of the flow of two equal vortices in a corotating frame. Note there are five corotating points. Three are collinear with the vortex centres at the crossing streamlines. The other two corotating points are at the bottoms of the basins above and below the centres.

If the strengths are unequal, $x_{13} \neq x_{23}$, but of the same orientation, $x_{13}x_{23} > 0$, the centres of the two vortices move on two concentric circles in the plane, staying as far away as possible, see Fig. 3. The streamlines in a corotating frame are shown in Fig. 4. Note there are still five corotating points and three are collinear with the centres.

If the vortices are of opposite orientations, $x_{13}\ x_{23} < 0$, and the strengths are not negatives of each other, $x_{13} \neq -x_{23}$, again the centres of the two vortices move on two concentric circles in the plane but now staying as close as possible, see Fig. 5. The streamlines in a corotating frame are shown in Fig. 6. There are three corotating points. One is at the bottom of the basin collinear with the vortices. The other two are above and below the line between the vortex centres at the saddle points in Fig. 6.

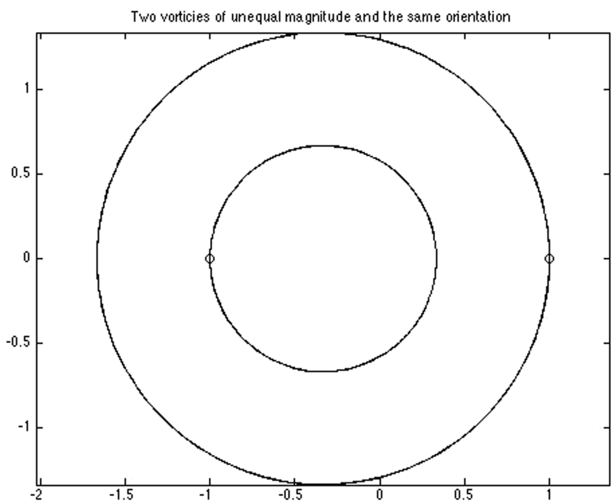

Fig. 3. The motion of the centres of two vortices of unequal strengths but the same orientation.

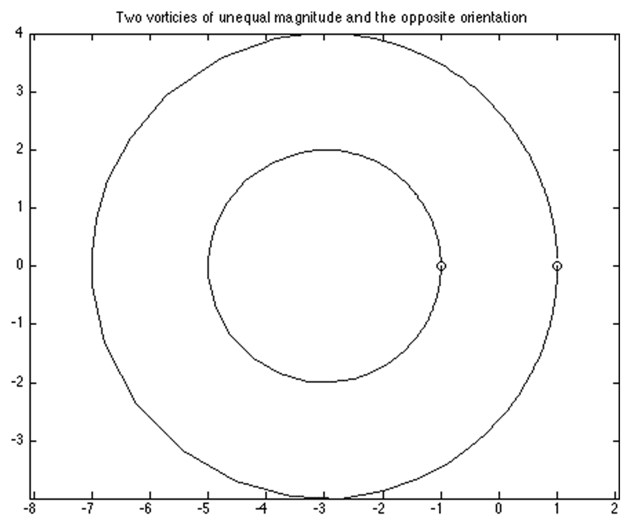

Fig. 5. The motion of the centres of two vortices of unequal strengths and opposite orientation.

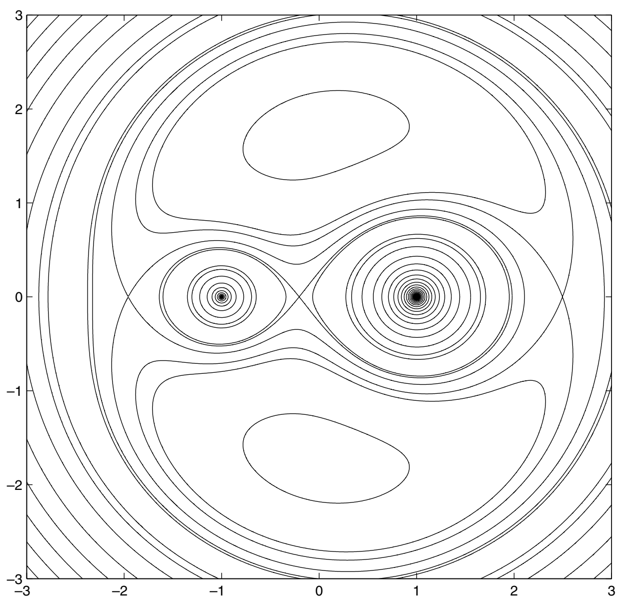

Fig. 4. The streamlines in a corotating frame off two vortices of unequal strengths but the same orientation.

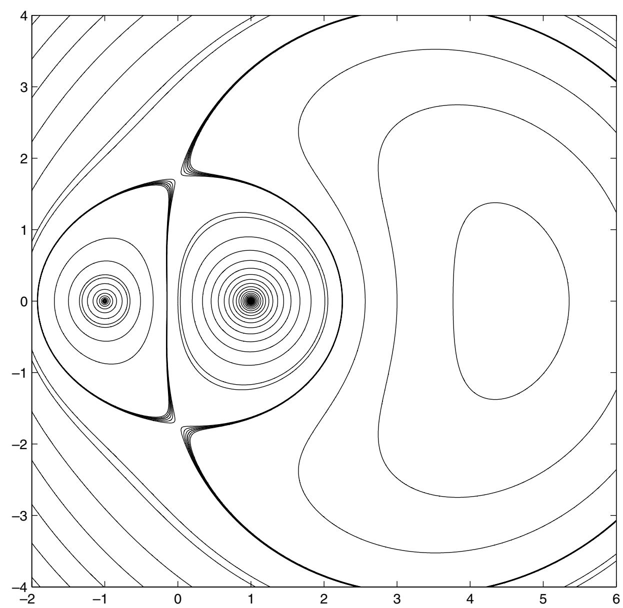

Fig. 6. The streamlines in a corotating frame off two vortices of unequal strengths and opposite orientations.

If the strengths of the vortices are negatives of each other, $x_{13} = -x_{23}$, then the two centres will fly off to infinity along two parallel lines, see Fig. 7. The streamlines in a co-translating frame are shown in Fig. 8.

Suppose that the strengths are not negatives of each other, $x_{13} \neq -x_{23}$, and, without loss of generality, the vortices start at $(x_{11}(0), x_{12}(0)) = (1, 0)$ and $(x_{21}(0), x_{22}(0)) = (-1, 0)$ then the two vortices will rotate around the point

$$\xi^c = \left(\xi_1^c, \xi_2^c\right) = \left(\frac{x_{13} - x_{23}}{x_{13} + x_{23}}, 0\right),$$

with angular velocity

$$\omega = \frac{x_{13} + x_{23}}{4}.$$

The induced flow will be momentarily stagnant at

$$\xi^s = \left(\frac{x_{23} - x_{13}}{x_{13} + x_{23}}, 0\right) = -\xi^c,$$

but generally this stagnation point will rotate with the vortices remaining on the line between their centres. The one exception is when the strengths are equal, $x_{13} = x_{23}$, for then the stagnation point is the centre of rotation at (0, 0) and remains there.

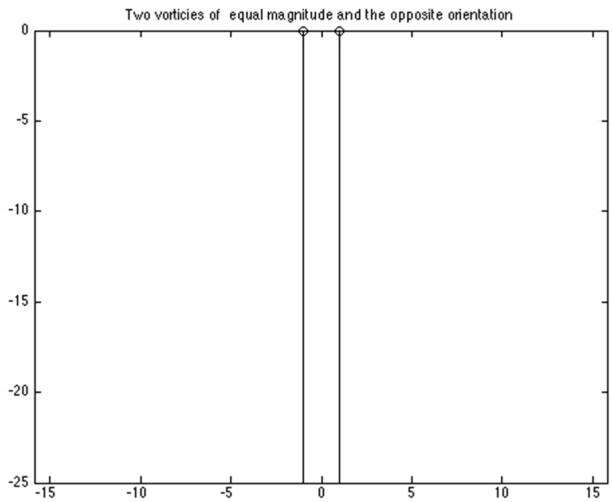

Fig. 7. The motion of the centres of two vortices of equal strengths and the opposite orientation.

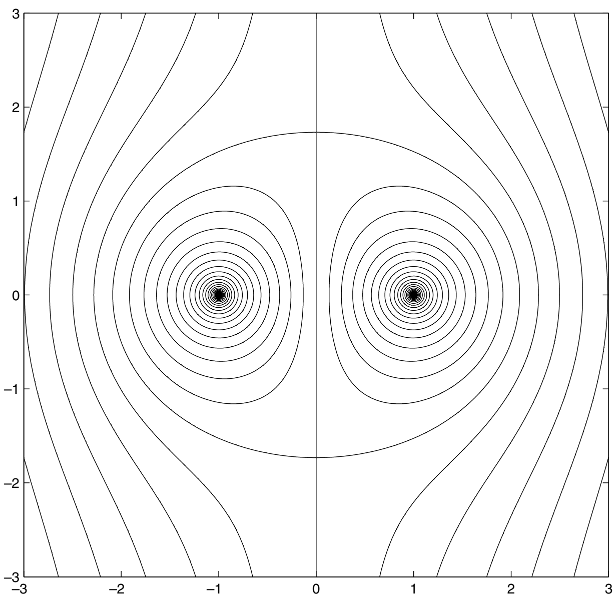

Fig. 8. The streamlines in a cotranslating frame off two vortices of equal strengths and opposite orientations.

Suppose that the strengths are not opposite $x_{13} \neq -x_{23}$ and, without loss of generality, the vortices are momentarily at $(x_{11}(0), x_{12}(0)) = (1, 0)$ and $(x_{21}(0), x_{22}(0)) = (-1, 0)$. Then at this moment, the collinear, corotating points are at $(\xi_1, 0)$, where ξ_1 is a root of the cubic

$$\omega(\xi_1 - \xi_1^c)(\xi_1^2 - 1) = x_{13}(\xi_1 + 1) + x_{23}(\xi_1 - 1).$$

When the orientations of the vortices are the same, $x_{13}\, x_{23} > 0$, there are always three corotating points that are collinear with the vortex centres. One lies between the centres and the other two lie to either side of the centres.

When the orientations of the vortices are opposite, $x_{13}\, x_{23} < 0$, there is only one co-rotating point that is collinear with the vortex centres. It lies outside the centres in the direction of the stronger vortex.

6. Eulerian observability of two vortex flow

Suppose there are two vortices of unknown positions and strengths, and there is one Eulerian observation, the velocity at a fixed point, without loss of generality, the origin. The dynamics is (25) and the observation is

$$y = \begin{bmatrix} \dfrac{x_{12}x_{13}}{r_1^2} + \dfrac{x_{22}x_{23}}{r_2^2} \\ -\dfrac{x_{11}x_{13}}{r_1^2} - \dfrac{x_{12}x_{23}}{r_2^2} \end{bmatrix}, \tag{26}$$

where $r_i^2 = x_{i1}^2 + x_{i2}^2$.

Is this system always observable? The answer is, clearly, no. We can interchange the vortices and not change the output trajectory. The state vectors

$$\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} \equiv \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \\ x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} \tag{27}$$

are equivalent in that they generate the same flow and output trajectory. But clearly, this is a problem with the way we are modelling things. We should take as state space not $\mathbb{R}^6$ but $\mathbb{R}^6$ mod this equivalence relation.

More precisely, we must exclude from the state space the possibility that the centres of vortices coincide, $(x_{11}, x_{12}) \neq (x_{21}, x_{22})$, and that one of the strengths is zero $x_{13} \neq 0$, $x_{23} \neq 0$. We also assume that the Eulerian observation at the origin does not coincide with a centre of a vortex $(x_{11}, x_{12}) \neq (0, 0)$, $(x_{21}, x_{22}) \neq (0, 0)$. This defines an open subset of $\mathbb{R}^6$, and we identify points of this subset that satisfy the equivalence relation (27). The resulting state space is a six-dimensional manifold.

Even if we redefine the state space in this fashion, the system may be unobservable. Consider two equal vortices symmetrically placed with respect to the observer. Without loss of generality, we can assume the observer is at the origin. A symmetric configuration is one satisfying

$$\begin{aligned} x_{11} &= -x_{21}, \\ x_{12} &= -x_{22}, \\ x_{13} &= x_{23}. \end{aligned} \tag{28}$$

The vortices will rotate in a circle around the origin where the observed velocity is identically zero, see Fig. 1. Therefore, we cannot infer anything about their locations and strengths except that the configuration is symmetric.

But this is a very special configuration of the vortices and the observer; perhaps most configurations are observable. To test this we, wrote software to compute the SORC for two vortices and one Eulerian or Lagrangian observation.

For one Eulerian observation, the dimension of the state space is six and the dimension of the output is two. The computed rank of

$$\begin{bmatrix} dh(x) \\ dL_f(h)(x) \\ dL_f^2(h)(x) \end{bmatrix} \tag{29}$$

is six if the observation is not collinear with the centres of the vortices. Hence, the SORC is satisfied at almost all x.

Assume that the configuration of the vortices and the observer is not one of the symmetric ones discussed above (28). If the Eulerian observation is collinear with the centres of the vortices then the computed rank of (29) is five, and the SORC does not hold. Unless the observation is at the centre of rotation of the vortices, the Eulerian observer does not stay collinear with their centres and the computed rank increases to six. So, the ORC is satisfied for such configurations.

If the observation is at the centre of rotation, then the Eulerian observer stays collinear with their centres; so, the computed rank of (29) remains five and SORC remains unsatisfied. The direction that is orthogonal to the one-forms (29) is that of moving the vortices away from the observer while maintaining collinearity and increasing their strength. The line between the centres is rotating, so, the rank of (7) is six and the ORC is satisfied.

Consider a symmetric configuration (28) with the Eulerian observation at the origin. The two vortices will rotate around the origin maintaining these relations. The computed rank of (29) is three as expected, and so, the SORC does not hold. One expects the rank to be three as there are three directions to change the six-dimensional state, still satisfying the relations (28). Moreover, the rank of (7) is also three; so, the ORC also does not hold also.

Now consider two vortex flow with two Eulerian observations at different locations. The observation y is four-dimensional. If the vortices are of different strengths, $x_{13} \neq x_{23}$, then the computed rank of

$$\begin{bmatrix} dh^1(x) \\ dh^2(x) \end{bmatrix} \tag{30}$$

is four, and the computed rank of

$$\begin{bmatrix} dh^1(x) \\ dh^2(x) \\ dL_f(h^1)(x) \\ dL_f(h^2)(x) \end{bmatrix} \tag{31}$$

is six; so, the SORC holds.

If the vortices are of same strength, $x_{13} = x_{23}$, but at least one of the observations is not collinear with the centres of the vortices and the other observation is not half way between them, then the rank of (30) is four and the rank of (31) is six; so, the SORC holds.

If the vortices are of same strength, $x_{13} = x_{23}$, and both observations are collinear with the centres of the vortices then the rank of (30) is four, but the rank of (31) is five; so, the SORC does not hold. However, the ORC does hold.

If the vortices are of same strength, $x_{13} = x_{23}$, and one observation is halfway between the centres of the vortices, then the rank of (30) is four but the rank of (31) is five; so, the SORC does not hold. However, the ORC does hold.

So, two vortex flow with two Eulerian observations is always short time locally observable.

7. Lagrangian observability of two vortex flow

For two vortices and one Lagrangian observation, the augmented state space is eight-dimensional. If the observation is not collinear with the centres of the vortices, then the computed rank of

$$\begin{bmatrix} dk(x) \\ dL_g(k)(x) \\ dL_g^2(k)(x) \\ dL_g^3(k)(x) \end{bmatrix} \tag{32}$$

is eight; so, the SORC holds.

If the Lagrangian observer is collinear with the centres of the vortices but not half way between two vortices of equal strength, then the rank is seven; so, the SORC is not satisfied. If the Lagrangian observer is not at a corotating, collinear point then the rank of (32) immediately becomes eight; so, the ORC is satisfied. If the observer is at a co-rotating, collinear point then the rank of (32) remains seven; so, the SORC remains unsatisfied. But the computed rank of (7) is eight; so, the ORC is satisfied.

Now consider a Lagrangian observer halfway between two vortices of the same strength, (28), Fig. 3. Then the observation is made at a stagnation point of the flow, and so, it remains there. The computed rank of (32) is five, as expected. There are three directions to change the vortices, which leave the observer halfway between two vortices of the same strength.

The augmented state of two vortex flow with two Lagrangian observations is ten-dimensional and the observation is four-dimensional. It always satisfies the SORC, the computed rank of

$$\begin{bmatrix} dk(x) \\ dL_g(k)(x) \end{bmatrix} \tag{33}$$

Table 1. Flow observability

	1 V	2	2 S V
1 EO	NO	ORC	NO
1 LO	STO	ORC	NO
2 EO	STO	SORC	ORC
2 LO	STO	SORC	SORC

is eight and the computed rank of

$$\begin{bmatrix} dk(x) \\ dL_g(k)(x) \\ dL_g^2(k)(x) \end{bmatrix} \tag{34}$$

is 10. Hence, the system is short time locally observable.

8. Summary

Table 1 summarizes the above.

The entries have the following meanings,

1 V: one vortex;

2 V: two vortices, not symmetric with respect to an observation;

2 SV: two equal vortices, symmetric with respect to an observation;

1 EO: one Eulerian observation;

1 LO: one Lagrangian observation;

2 EO: two Eulerian observations;

2 LO: two Lagrangian observations;

NO: not observable;

STO: short time observable;

ORC: ORC holds always and SORC holds if at least one observation is not collinear with the vortex centres, hence short time locally observable;

SORC: SORC always holds, hence short time locally observable.

9. Extended Kalman filtering of two vortex flow

In this section, we study the performance of continuous time extended Kalman filters (EKF) for two vortex flow with one Eulerian or Lagrangian observation. We are particularly interested in its convergence or divergence when the initial state of the system does not satisfy the ORC or SORC.

We assume that the observations are available continuously. In most atmospheric and oceanic prediction situations, the observations are only available at discrete times. We chose the continuous observations so that we don't have to introduce a time step. Admittedly, this is an ideal situation and one expects better performance with continuous observations. If a continuous observation EKF fails to converge, then one expects that a discrete observation EKF will also fail to converge.

First, we describe an EKF in general for a general observed system (4, 5). The EKF has two state variables, $\hat{x}(t) \in I\!R^n$ and $P(t) \in I\!R^{n\times n}$. In the derivation of the EKF, it is assumed that the true state $x(t)$ is approximately Gaussian distributed, with mean $\hat{x}(t)$ and covariance $P(t)$. These variables evolve according to the differential equations

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + P(t)H'(t)R^{-1}(t)\,[y(t) - h(\hat{x}(t),\,]$$

$$\dot{P}(t) = F(t)P(t) + P(t)F'(t) + Q(t) \\ \quad - P(t)H'(t)R^{-1}(t)H(t)P(t),$$

where

$$F(t) = \frac{\partial f}{\partial x}(\hat{x}(t)),$$
$$H(t) = \frac{\partial h}{\partial x}(\hat{x}(t)).$$

The EKF is derived by adding driving and observation noises to the observed system (4). The term $y(t) - h(\hat{x}(t)$ is called the innovation; it is the additional information supplied by the observation at time t. The EKF is a copy of the systems dynamics driven by the weighted innovation. The weight depends on the covariance $P(t)$, which evolves according to the linearized dynamics around the estimated trajectory. For more details on the EKF, we refer the reader to Gelb (1974).

There are four design parameters of the EKF: the initial conditions $\hat{x}(0)$; $P(0)$; the $n \times n$ driving noise covariance $Q(t) \geq 0$ and the $p \times p$ observation noise covariance $R(t) > 0$. By convergence of a filter we mean the following. Suppose we start the system at some initial condition $x(0)$ and the filter at a different initial condition $\hat{x}(0)$. We integrate the system equations with no noise to get state $x(0:\infty)$ and output $y(0:\infty)$ trajectories. We pass the noise free output trajectory to the filter and integrate the filter equations to get a state estimate trajectory $\hat{x}(0:\infty)$. The filter is convergent if the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ goes to zero as $t \to \infty$ for all $x(0)$ and $\hat{x}(0)$. Again this is an ideal situation. In a real situation, there always will be noise. If an EKF fails to converge when there is no noise but only initial estimate errors, then one expects it will perform poorly when noises are present.

In general, convergence is too much to expect for nonlinear systems and their filters. One is forced to settle for local convergence: $\tilde{x}(t) = x(t) - \hat{x}(t)$ goes to zero as $t \to \infty$ if the initial error $\tilde{x}(0) = x(0) - \hat{x}(0)$ is sufficiently small. We have shown Krener (2002) that if the system satisfies the SORC then an extended Kalman filter is locally convergent.

For two vortex flow with one Eulerian observation, the f and h of the EKF are given by (25) and (26). Therefore, the EKF for one Eulerian observation at ξ takes the form

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}) \\ &\quad + P(t)H'(t)R^{-1}(t)\left[u(x(t),\xi) - u(\hat{x}(t),\xi)\right], \\ \dot{P}(t) &= F(t)P(t) + P(t)F'(t) + Q(t) \\ &\quad - P(t)H'(t)R^{-1}(t)H(t)P(t), \end{aligned} \tag{35}$$

where $n = 6$, $p = 2$ and

$$F(t) = \frac{\partial f}{\partial x}(\hat{x}(t)),$$

$$H(t) = \frac{\partial u}{\partial x}(\hat{x}(t),\xi).$$

For two vortex flow with one Lagrangian observation, the f and h of the EKF are the g and k of (12) and (13). Therefore, the EKF for one Lagrangian observation at $\xi(t)$ takes the form

$$\begin{aligned}\dot{\hat{x}}(t) &= f(\hat{x}(t)) \\ &\quad + P_{12}(t)R^{-1}(t)(\xi(t) - \hat{\xi}(t)), \\ \dot{\hat{\xi}}(t) &= u(\hat{x}(t),\hat{\xi}(t)) + P_{22}(t)R^{-1}(t)(\xi(t) - \hat{\xi}(t)), \\ \dot{P}(t) &= G(t)P(t) + P(t)G'(t) + Q(t) \\ &\quad - P(t)K'R^{-1}(t)KP(t), \end{aligned} \tag{36}$$

where $n = 8$, $p = 2$ and

$$G(t) = \begin{bmatrix} \frac{\partial f}{\partial x}(\hat{x}(t)) & 0 \\ \frac{\partial u}{\partial x}(\hat{x}(t),\hat{\xi}(t)) & \frac{\partial u}{\partial \xi}(\hat{x}(t),\hat{\xi}(t)) \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}.$$

The P of (35) is different from the P of (36), P_{11} is the 'error covariance' of x, P_{22} is the 'error covariance' of ξ and $P_{12} = P_{21}^{'}$ is the 'error cross variance' between x and ξ.

For two vortex flow with a Lagrangian observation, we shall also study the performance of a reduced order extended Kalman filter (REKF). This is motivated by the fact that GPS observations of position are essentially noise free, $R(t) = 0$. Therefore, we can take their time derivative, the velocity of the moving observation, as the measured variable.

Here is a REKF for the observed system.

$$\begin{aligned}\dot{z}_1 &= g_1(z_1, z_2), \\ \dot{z}_2 &= g_2(z_1, z_2), \\ y &= k(z_1, z_2) = z_2, \end{aligned}$$

where $z_1 \in I\!R^{n-p}$, $z_2 \in I\!R^{p}$. If there is little or no noise in the observation, then a reduced order EKF takes the form

$$\begin{aligned}\dot{\hat{z}}_1(t) &= g_1(\hat{z}_1(t), y(t)) \\ &\quad + P(t)G_2'(t)Q_2^{-1}(t)\left(\dot{y}(t) - g_2(\hat{z}_1(t), y(t),)\right) \\ \dot{P}(t) &= G_1(t)P(t) + P(t)G_1'(t) + Q_1(t) \\ &\quad - P(t)G_2'(t)Q_2^{-1}(t)G_2(t)P(t), \end{aligned} \tag{37}$$

where $\hat{z}_1 \in I\!R^{n-p}$, $P \in I\!R^{(n-p)\times(n-p)}$ and

$$\begin{aligned} G_1(t) &= \frac{\partial g_1}{\partial z_1}(\hat{z}_1(t), y(t)), \\ G_2(t) &= \frac{\partial g_2}{\partial z_1}(\hat{z}_1(t), y(t)). \end{aligned}$$

Again P is different, P is the 'error covariance' of z_1.

For the augmented system (12) with one Lagrangian observation (13),

$$\begin{aligned} z_1 &= x, \\ z_2 &= \xi, \\ w &= \xi, \\ g_1(z_1, z_2) &= f(x), \\ g_2(z_1, z_2) &= u(x, \xi), \end{aligned}$$

and so, the reduced order EKF take the form

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}) + P(t)G_2'(t)Q_2^{-1}(t) \\ &\quad \times \left(u(x(t),\xi(t)) - u(\hat{x}(t),\xi(t))\right), \\ \dot{P}(t) &= G_1(t)P(t) + P(t)G_1'(t) + Q_1(t) \\ &\quad - P(t)G_2'(t)Q_2^{-1}(t)G_2(t)P(t), \end{aligned} \tag{38}$$

where

$$\begin{aligned} G_1(t) &= \frac{\partial f}{\partial x}(\hat{x}(t)), \\ G_2(t) &= \frac{\partial u}{\partial x}(\hat{x}(t), \xi(t)). \end{aligned}$$

Note the similarity between the REKF for two vortex flow with one Lagrangian observation and the EKF for two vortex flow with one Eulerian observation. Aside from different names for the design parameters, the only difference is that the fixed observation of the flow velocity at ξ is replaced by a moving observation of the flow velocity at $\xi(t)$.

We simulate these extended Kalman filters with the same data against a two vortex system with no driving or observation noises. In all the examples, the Eulerian observation is made at the origin and the Lagrangian observation starts at the origin. The driving and observation noise covariances are taken to be the identity $Q(t) = I, R(t) = I$ as is the initial error covariance $P(0) = I$.

The first example, Fig. 9 is that of unequal vortices that are not collinear with the origin. All three filters are converging, but the one with Eulerian measurements is converging more slowly.

Next in Fig. 10, the vortices are unequal but collinear with the observation at the origin. The SORC does not hold but the ORC does. The Lagrangian observation moves from the origin,

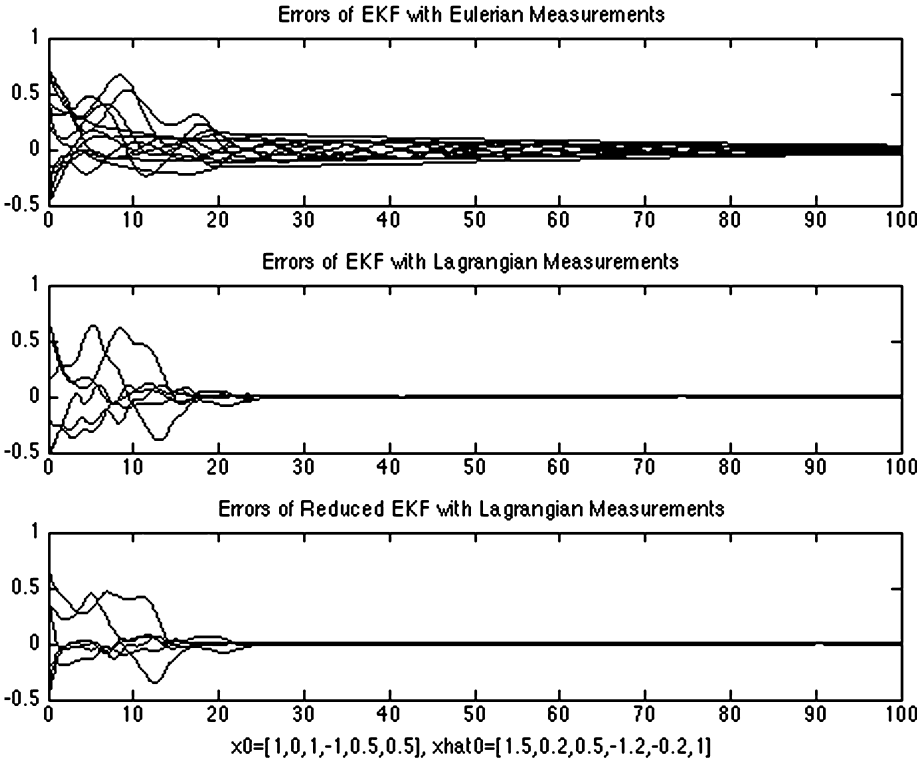

Fig. 9. Filter errors, unequal vortices not collinear with the origin.

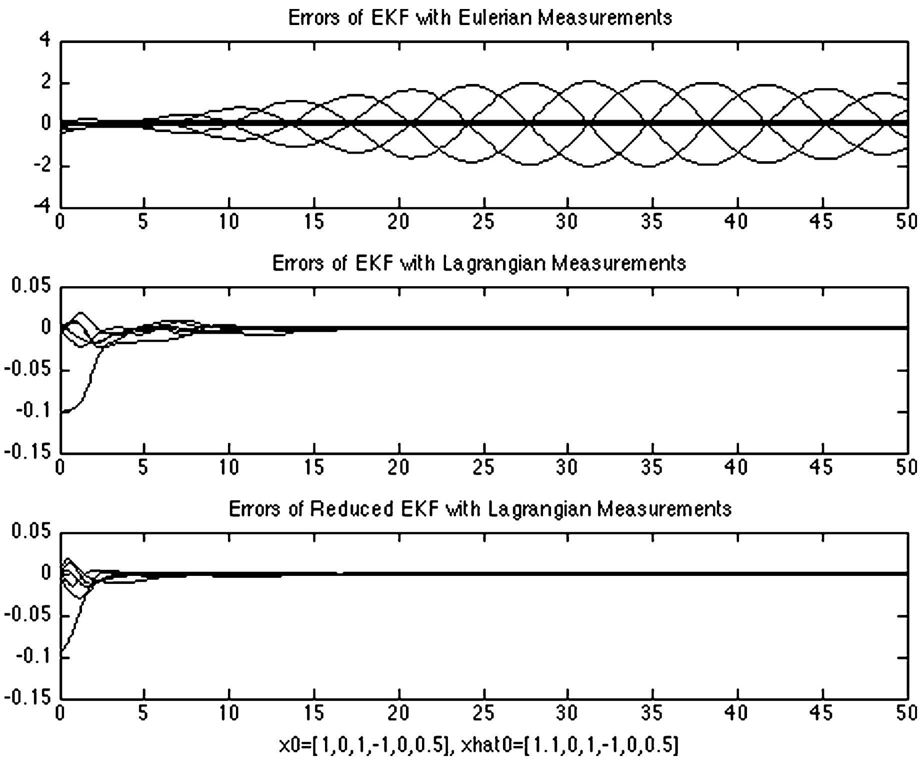

Fig. 10. Filter errors, unequal vortices collinear with the observation.

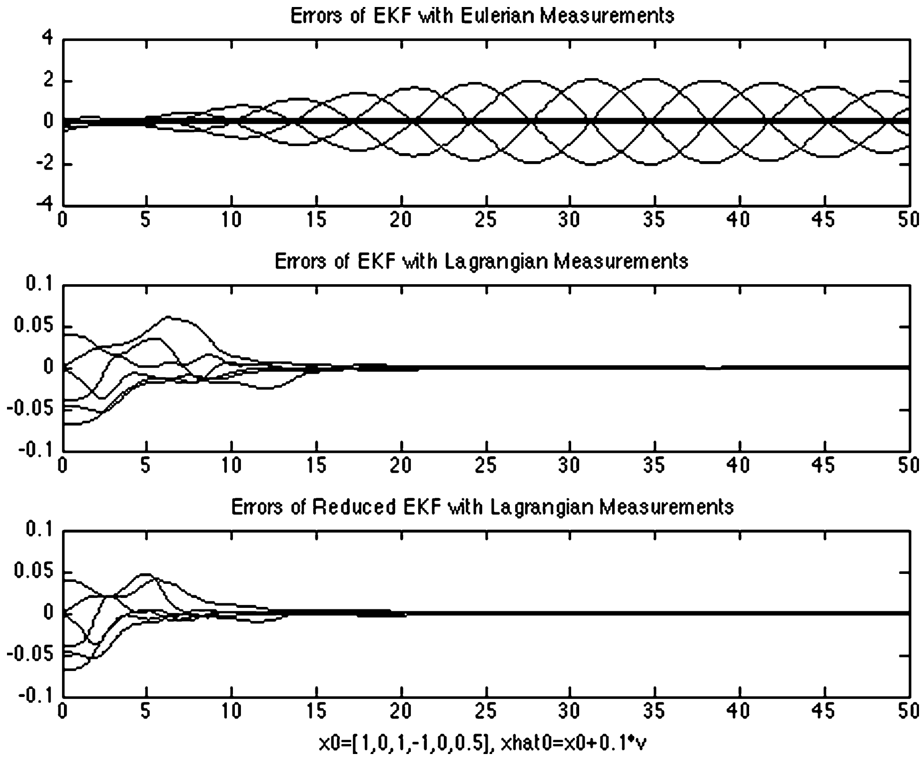

Fig. 11. Filter errors, unequal vortices collinear with the observation. Initial estimation error in the null space of the SORC one-forms.

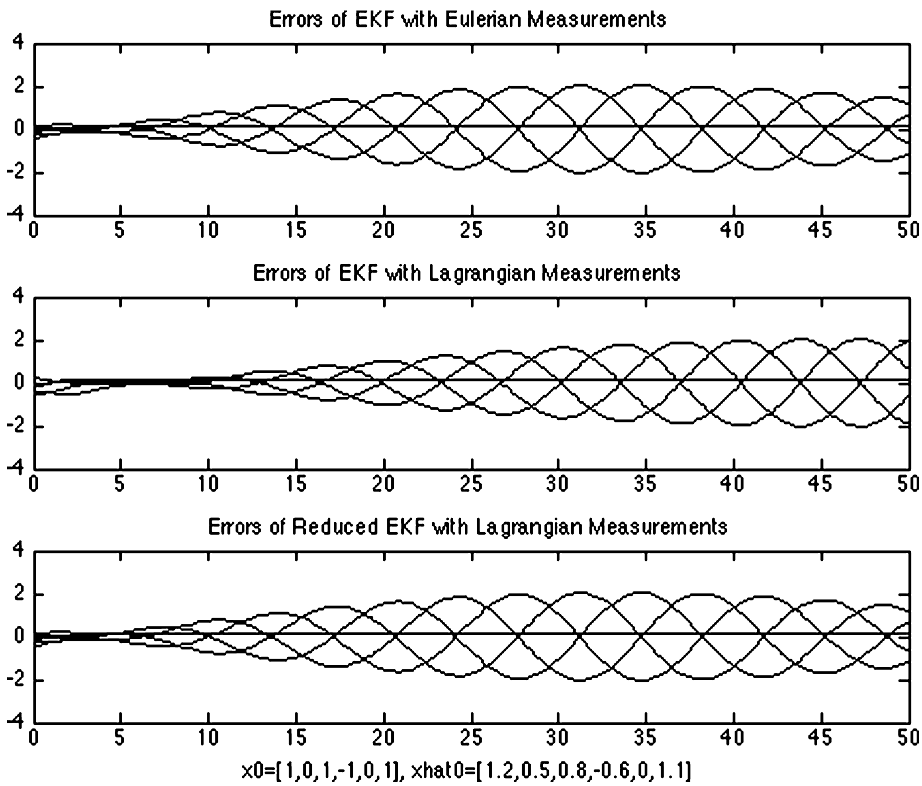

Fig. 12. Filter errors, equal vortices symmetric with respect to the observation.

and apparently this leads to the rapid convergence of the corresponding full and reduced EKFs with such observations . The Eulerian observation does not move and divergence results. Note that $\|x(0) - \hat{x}(0)\|_2 = 0.1$ is relatively small, still the Eulerian EKF diverges.

In Fig. 11, we take the vortices in the same initial location as Fig. 10 but the initial estimation error v is in the null space of the SORC one-forms (39). The size of the initial errors in Figs. 10 and 11 are equal, $\|v\|_2 = 0.1$. The direction v amounts to moving the vortices further apart along the horizontal axis and increasing their strengths. Taking the initial error in this direction slows down the convergence of the Lagrangian filters. Note the differences in the vertical scales between Figs. 10 and 11.

The final example, Fig. 12, is the symmetric case where the ORC condition fails. None of the filters converge.

In the simulations that we have run, we generally found that the REKF with Lagrangian observations converges the fastest, followed by EKF with Lagrangian observations, followed by the EKF with Eulerian observations. However, there were simulations where any or all of the filters failed to converge. This will almost certainly happen if the initial state estimation error, $x(0) - \hat{x}(0)$, is large enough.

10. Conclusion

We have studied the observability of one and two point vortex flow under one or two Eulerian or Lagrangian observations. Admittedly, this is an idealized situation but one hopes that the answers obtained will illuminate the observability of more realistic flows and observations.

Although we are not able to prove it in all cases, apparently the extra term (20) in the ORC for Lagrangian observations has a positive impact on the observability of the flow.

We also studied the perfomance of extended Kalman filters applied to the flow of two vortices with either Eulerian or Lagrangian observation and a reduced order extended Kalman filter with a Lagrangian observation. Our impression is that the reduced order extended Kalman filter with a Lagrangian observation generally performs the best whereas the extended Kalman filter with an Eulerian observation generally performs the worst.

Of course in realistic problems of data assimilation, the measurements are not continuous, and therefore, a reduced order extended Kalman filter may not be possible.

The next step is to extend these results to multi vortex and other higher dimensional flows. There are similar symmetric configurations in multi vortex flow where the vortices rotate in a symmetric fashion around the origin, which is a stagnation point of the flow (Acheson, 1990, p. 184). One expects that such configurations will be unobservable under one Eulerian or Lagrangian observation at the origin. Furthermore, it is known that the dynamics of four point vortices can be chaotic (Aref, 1983). It would be interesting to study the observability of such a system. Other interesting cases are spatially discretized Euler or Navier Stokes equations. These latter studies are probably not possible analytically but could be done numerically.

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