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TELLUS

SHORT CONTRIBUTION

A condition on the average Richardson number for weak non-linearity of internal gravity waves

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ABSTRACT

A condition on the average Richardson number, Ri, for weak non-linearity of an internal gravity wavefield is derived using a quasi-normal assumption. For weak non-linearity to be satisfied it is required that $Ri^{-1} \ll 0.5$. This condition is very rarely satisfied in the ocean at vertical scales up to the order of 100 m, for which it is often found that $Ri^{-1} \sim 1$. The analysis suggests that non-linear effects are of no less importance than linear effects in the dynamics of the interior of the ocean at these scales.

1. Introduction

It is generally believed that the dynamics of the interior of the ocean at vertical scales of the order of 100 m and down to about 10 m is dominated by linear or weakly non-linear internal gravity wave (Garret and Munk, 1972, 1975, 1979). It is also generally recognized that non-linear effects become more and more important with decreasing vertical scale. However, there is no general consensus on the minimum vertical wavelength at which the assumption of weak non-linearity is valid. McComas and Müller (1981) compared interaction time scales of weak interaction theory (McComas and Bretherton, 1977) with typical wave periods and concluded that the internal wavefield is weakly non-linear down to typical vertical wavelengths of about 5 m. By comparing relaxation times of weak interaction theory with typical wave periods and also by comparing typical phase speeds with fluid particle speeds, Holloway (1980, 1982) argued that weak non-linearity is not generally satisfied for vertical wavelengths which are smaller than 60 m. The analysis of Holloway suggests that in most cases the transition wavelength is in fact larger than 60 m. Using a stability argument Munk (1981) suggested that the transition scale at which non-linear effects become of leading order is determined by the average Richardson number, which can be written as

$$Ri = \frac{N^2}{\left\langle \frac{\partial u}{\partial z} \frac{\partial u}{\partial z} \right\rangle},$$

where u is a horizontal velocity component, z is the vertical coordinate, N is the Brunt–Väisälä frequency and $\langle \cdots \rangle$ is a space–time average over all scales of interest. The smaller the vertical scales which are included in the procedure of averaging Ri, the smaller the value of Ri which is obtain. According to the argument by Munk, the vertical transition wavenumber, m_c, at which linear waves can no longer be dominant, can be determined by some typical lower limit of Ri. Using the terminology of D'Asaro and Lien (2000), m_c is determined by the condition

$$Fr(m_c) = \int_{m_1}^{m_c} \frac{\Phi(m)}{N^2}\, \mathrm{d}m = Fr_c, \qquad (1)$$

where m_1 is the smallest vertical wavenumber of the internal wavefield, Φ is the shear spectrum and Fr_c is some critical value of the Froude function, Fr. The lower integration limit, m_1, may be replaced by zero, since the contribution to the total shear from very large vertical scales can be assumed to be negligible. The Richardson number can be calculated as the inverse of the Froude function, formally letting $m_c \to \infty$. With a finite upper integration limit we can define a scale dependent Richardson number as $Ri(m_c) = [Fr(m_c)]^{-1}$ and (1) can be written as $Ri(m_c) = Fr_c^{-1}$. Different values of Fr_c have been proposed in the literature. Polzin et al. (1995) assume that $Fr_c = 0.7$, Sherman and Pinkel (1991) assume that $Fr_c = 0.3$, Duda and Cox (1989) assume that $Fr_c > 1$ while Gregg et al. (2003) use the value $Fr_c =$

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0.661. D'Asaro and Lien (2000) point out that the experimental evidence and theoretical support for the condition (1) is not very strong. In this paper, we shall give the condition (1) a more firm theoretical foundation. Instead of using a definition of weak non-linearity which is based on concepts from weak interaction theory, or by a comparison between phase speed and fluid particle speed, we will use an even more fundamental approach of directly comparing the magnitude of the non-linear terms with the magnitude of the linear terms in the equations of motion for an internal wavefield. On the basis of such a comparison and a quasi-normal approximation we shall derive a condition on the Richardson number, or the Froude function, for non-linear effects to be of second order in an internal gravity wavefield.

2. Derivation of the Richardson number condition

The elementary wave solution of the linearized Navier–Stokes equations subject to the Boussinesq approximation, with constant Brunt–Väisälä frequency, N, and without rotation can be written as

$$\hat{w} = w_o \exp[\mathrm{i}(kx + ly + mz + \omega t)] \tag{2}$$

$$\hat{u} = -\frac{km}{k^2 + l^2}\hat{w} \tag{3}$$

$$\hat{v} = -\frac{lm}{k^2 + l^2}\hat{w} \tag{4}$$

$$\omega^2 = \frac{k^2 + l^2}{k^2 + l^2 + m^2} N^2, \tag{5}$$

where $\hat{w}$ is the vertical velocity component and $\hat{u}$ and $\hat{v}$ are the two horizontal velocity components of the elementary wave. We consider the case when the velocity field, $\mathbf{u} = (u, v, w)$, can be written as a sum or an integral over an infinite number of elementary waves and can be regarded as a random function. We further assume that the statistical properties of the field are axisymmetric with respect to the vertical axis, which means that they are invariant under rotations and reflections in a vertical plane. We shall also assume that the statistical moments are homogeneous or weakly inhomogeneous with respect to the vertical axis.

It is readily verified that each elementary wave satisfies the relation

$$N^2 \hat{w}\hat{w} = \omega^2(\hat{u}\hat{u} + \hat{v}\hat{v} + \hat{w}\hat{w}). \tag{6}$$

By using Parseval's theorem we find

$$N^2\langle ww\rangle = \left\langle \frac{\partial u}{\partial t}\frac{\partial u}{\partial t}\right\rangle + \left\langle \frac{\partial v}{\partial t}\frac{\partial v}{\partial t}\right\rangle + \left\langle \frac{\partial w}{\partial t}\frac{\partial w}{\partial t}\right\rangle, \tag{7}$$

where $\langle\cdots\rangle$ is the space–time average. The condition for weak non-linearity in the equation for u can be written

$$\left|\frac{\partial u}{\partial t}\right| \gg \left|u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right|. \tag{8}$$

Combining (7) and (8) and using the assumption of axisymmetry we find

$$N^2\langle ww\rangle > 2\left\langle \frac{\partial u}{\partial t}\frac{\partial u}{\partial t}\right\rangle \gg 2\left\langle \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)^2\right\rangle. \tag{9}$$

We now make the quasi-normal approximation, that is, we assume that fourth-order statistical moments of the velocity components and their derivatives can be factorized into second-order moments as if they were normally distributed (Lesieur, 1987). This assumption is often used in the weakly non-linear theories, and is a key part of some theories for strongly non-linear flows as well (e.g. Godeferd et al., 2001). Using it, the right-hand side of (9) can be expanded as

$$\begin{aligned}
&2\left(\langle uu\rangle\left\langle\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right\rangle + \langle vv\rangle\left\langle\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right\rangle + \langle ww\rangle\left\langle\frac{\partial u}{\partial z}\frac{\partial u}{\partial z}\right\rangle\right)\\
&\quad + 4\left(\left\langle u\frac{\partial u}{\partial x}\right\rangle^2 + \left\langle v\frac{\partial u}{\partial y}\right\rangle^2 + \left\langle w\frac{\partial u}{\partial z}\right\rangle^2\right)\\
&\quad + 4\left(\langle uv\rangle\left\langle\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right\rangle + \left\langle u\frac{\partial u}{\partial x}\right\rangle\left\langle v\frac{\partial u}{\partial y}\right\rangle + \left\langle u\frac{\partial u}{\partial y}\right\rangle\left\langle v\frac{\partial u}{\partial x}\right\rangle\right)\\
&\quad + 4\left(\langle uw\rangle\left\langle\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right\rangle + \left\langle u\frac{\partial u}{\partial x}\right\rangle\left\langle w\frac{\partial u}{\partial z}\right\rangle + \left\langle u\frac{\partial u}{\partial z}\right\rangle\left\langle w\frac{\partial u}{\partial x}\right\rangle\right)\\
&\quad + 4\left(\langle vw\rangle\left\langle\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\right\rangle + \left\langle v\frac{\partial u}{\partial y}\right\rangle\left\langle w\frac{\partial u}{\partial z}\right\rangle + \left\langle v\frac{\partial u}{\partial z}\right\rangle\left\langle w\frac{\partial u}{\partial y}\right\rangle\right).
\end{aligned} \tag{10}$$

All terms in (10) except the ones on the top row cancel on the basis of axisymmetry and weak inhomogeneity with respect to the vertical direction. By incompressibility and axisymmetry the last term on the fourth row can be written as

$$\begin{aligned}
\left\langle u\frac{\partial u}{\partial z}\right\rangle\left\langle w\frac{\partial u}{\partial x}\right\rangle &= \left\langle\frac{1}{2}\frac{\partial}{\partial z}u^2\right\rangle\left\langle w\frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right\rangle\\
&= -\frac{1}{4}\frac{\partial}{\partial z}\langle uu\rangle\left\langle w\frac{\partial w}{\partial z}\right\rangle = -\frac{1}{8}\frac{\partial}{\partial z}\langle uu\rangle\frac{\partial}{\partial z}\langle ww\rangle.
\end{aligned} \tag{11}$$

The ratio between this term and the last term in the first row can be estimated as

$$\frac{\langle uu\rangle}{h^2\left\langle\frac{\partial u}{\partial z}\frac{\partial u}{\partial z}\right\rangle}, \tag{12}$$

where h is the vertical length scale over which the statistical properties of the flow field vary, which can be taken as the depth of the thermocline. By the assumption of weak inhomogeneity the ratio (12) is much smaller than unity which means that the term (11) can be neglected. The rest of the terms on row two to five cancel by axisymmetry, which can be seen from the fact that each mean value must be invariant with respect to each of the transformations: $(x, u) \to (-\mathrm{x}, -\mathrm{u})$ and $(y, v) \to (-\mathrm{y}, -\mathrm{v})$. Thus

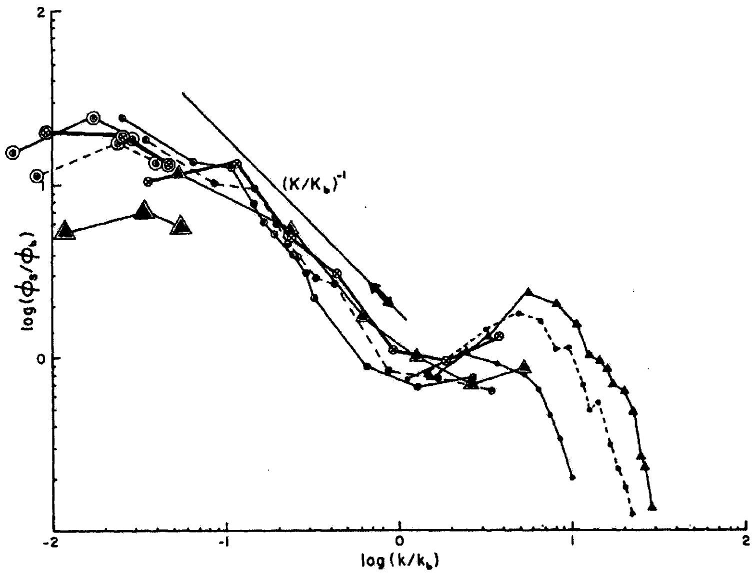

Fig. 1. Vertical wavenumber spectra of vertical shear in the ocean. The spectra are normalized by $\phi_b = (\epsilon N)^{1/2}$ and the wave number is normalized by $k_b = N^{3/2}/\epsilon^{1/2}$. Reproduced from Gargett et al. (1981).

we find

$$N^2\langle ww\rangle \gg 2\left(\langle uu\rangle\left\langle\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right\rangle + \langle vv\rangle\left\langle\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right\rangle + \langle ww\rangle\left\langle\frac{\partial u}{\partial z}\frac{\partial u}{\partial z}\right\rangle\right) > 2\langle ww\rangle\left\langle\frac{\partial u}{\partial z}\frac{\partial u}{\partial z}\right\rangle. \tag{13}$$

Hence, a necessary condition for weak non-linearity is

$$Ri^{-1} = \frac{\left\langle\frac{\partial u}{\partial z}\frac{\partial u}{\partial z}\right\rangle}{N^2} \ll \frac{1}{2}. \tag{14}$$

We have derived the condition (14) for a field of internal gravity waves without introducing any effects of rotation. We shall now argue that the result also is valid for inertia-gravity waves if it is assumed that the frequency spectra of the horizontal and vertical velocities are reasonably close to empirical spectra. If the relation (7) holds also for a field of inertia-gravity waves, the condition (14) must also be valid for such a field. The relation (7) was derived by performing a space–time averaging of (6). By Parseval's theorem this means that we have integrated (7) over all wavenumbers and frequencies. For inertia-gravity waves the relation (6) is replaced by (Fofonoff, 1969)

$$(N^2 - \omega^2)(\omega^2 + f^2)\hat{w}\hat{w} = \omega^2(\omega^2 - f^2)(\hat{u}\hat{u} + \hat{v}\hat{v}), \tag{15}$$

where f is the Coriolis parameter. Dividing by ω^2 and rearranging the terms we find

$$N^2\left(1 + \frac{f^2}{\omega^2} - \frac{f^2}{N^2}\right)\hat{w}\hat{w} = \omega^2(\hat{u}\hat{u} + \hat{v}\hat{v} + \hat{w}\hat{w}) - f^2(\hat{u}\hat{u} + \hat{v}\hat{v}). \tag{16}$$

If the frequency spectra, $E_u(\omega)$ and $E_w(\omega)$, of horizontal and vertical velocity are reasonably close to empirical spectra, all terms including f in (16) will be negligible after integration over

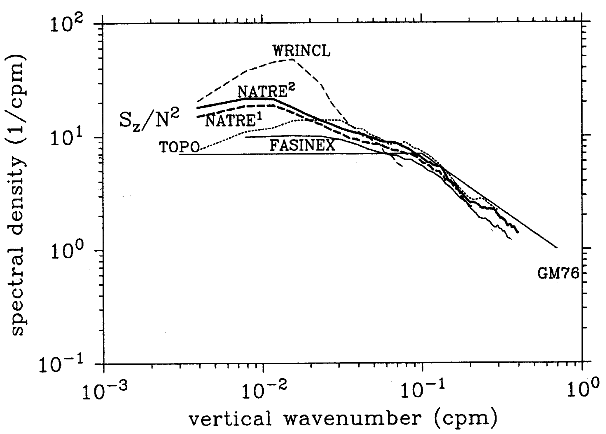

Fig. 2. Vertical wavenumber spectra of vertical shear from five different oceanic data sets. Reproduced from Polzin et al. (2003).

all wavenumbers and frequencies. To see this, we assume that the spectra have power law form for frequencies between f and N, that is,

$$E_u(\omega) = K_1\omega^{-p} \qquad E_w(\omega) = K_2\omega^{-n}, \tag{17}$$

where $1 < p < 3$ and $-1 < n < 1$. After integration over all frequencies and wavenumbers, the last term on the right-hand side of (16) will be of the order of $(f/N)^{3-p}$ as compared to the first term on the right-hand side. The second term on the left-hand side will be of the order of $(f/N)^{1-n}$ as compared to the first term and the last term will be of the order of $(f/N)^2$ as compared to the first term. According to the Garret & Munk model $p = 2$ and $n = 0$. These values are also sufficiently close to empirical data (see D'Asaro and Lien, 2000) to motivate why all the terms including f in (16) can be neglected after integration. The relation (7) is therefore approximately valid for inertia-gravity waves with frequency spectra which are reasonably close to empirical oceanic spectra. Consequently, the condition (14) should also be satisfied.

3. Conclusions

Our derivation confirms the conjecture that the inverse of the Richardson number provides a lower limit of the ratio between non-linear and linear terms and that the Froude function in eq. (1) therefore cannot become larger than unity for weak non-linearity to be satisfied. This may not come as a surprise. The merit of the derivation is that it also gives us, if not a precise number, a clear hint of what can be required for non-linear terms to be negligible small. It is, of course, still difficult to say what exact limit, F_c, of the Froude function we shall choose if we would like to determine if a certain range of scales is dominated by weakly non-linear internal waves. Our derivation shows, however, that we should choose a value which is considerably smaller than one half for the assumption of weak non-linearity to be valid. Note that in going from (8) to (14) we have squared the condition (8) which means that (14) should be interpreted in a strong sense. If we would interpret the condition (8) as saying that the magnitude of the non-linear terms in the momentum equation should be not larger than a tenth of the magnitude of the linear term, then we would obtain $F_c = 0.005$, which is two orders of magnitude smaller than values which are normally used. This may, of course, be a too low value. On the other hand, any value of F_c close to one half would lead us into the range of scales where non-linear effects are at least as large as linear effects.

In Fig. 1, we have reproduced a typical vertical wavenumber spectrum, $\Phi(k)$, of vertical shear in the ocean, from Gargett et al. (1981). The spectrum takes a local minimum around the wavenumber $k_b = 1/l_O$, where $l_O = \epsilon^{1/2}/N^{3/2}$ is the Ozmidov lengthscale. Here, ϵ is the mean dissipation of energy per unit mass. The minimum marks a transition between those scales of motion which are strongly influenced by stratification and those that are not. The observations reported in Fig. 1 are made at different depths at two different locations, one outside the island of Bermuda and one at the northern edge of the Gulf Stream. In all records $k_b \approx 1$ cycle per metre (cpm), which means that the spectrum range to the left of the minimum corresponds to spatial scales between 1 and 100 m. In Fig. 2, we have reproduced the vertical shear spectra from Polzin et al. (2003). The spectra are calculated from five different data sets, representing a large variation in depths and locations, and are divided by N^2; they are consistent with the data of Gargett et al. The spectra in Fig. 2 are given for vertical wavenumbers between 0.004 and 0.3 cpm. Integrating these spectra between wavenumbers 0.004 and 0.01 cpm, we obtain a value of the order of 0.1 of the Froude function, which may be sufficiently low for linear dynamics to dominate. Integrating either the spectra in Fig. 1 or in Fig. 2 between wavenumbers 0.01 and 0.1 cpm, we obtain a value of the order of unity or larger of the Froude function, suggesting that the spectra do not originate from weakly non-linear waves in this wavenumber range. Our analysis thus supports the conclusion by Holloway (1980) that oceanic motions at vertical scales of the order of 100 m and smaller are generally strongly influenced by non-linear effects.

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