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A high-latitude quasi-geostrophic delta plane model derived from spherical geometry

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ABSTRACT

For quasi-geostrophic models, the beta plane approximation is well established and can be derived from spherical geographic coordinates. It has been argued that such a connection does not exist for a higher-order approximation, the so-called delta plane. Here it will be demonstrated that a quasi-geostrophic potential vorticity equation on the delta plane can formally be derived using rotated geographic instead of geographic coordinates. The rigorous derivation of such a model from the shallow-water equations leads to a correction of previous more intuitive-based formulations of the delta plane model. Some applications of the corrected delta plane model are given. It is shown that the delta plane model describes well the low-frequency basin modes of a polar plane shallow-water model. Moreover, it is found that the westward phase speed of the delta plane model shows a dependency on latitude comparable to a model on the sphere. The ratio of delta to beta plane zonal phase speed decreases monotonically with increasing latitude, in qualitative agreement with the phase speed ratio obtained by comparing a spherical to a beta plane model. Finally, it is demonstrated analytically that Rossby wave energy rays are curved on the delta plane, in contrast to the beta plane. Ray curvature is important for a realistic description of energy dispersion at high latitudes. The results suggest that the quasi-geostrophic delta plane model is a suitable tool for conceptual studies on Rossby wave dynamics at high latitudes.

1. Introduction

Rossby (1939) introduced the beta plane as a model describing quantitatively the effect of a background potential vorticity gradient on planetary motions. He did not relate the beta plane to the actual spherical geometry of the Earth. Fortunately, when the length-scale L is smaller than the planet's radius a and the Rossby number is small, the quasi-geostrophic model on the beta plane can be derived from equations on the sphere (Phillips, 1963; Pedlosky, 1987; Verkley, 1990). Thereby, the Coriolis parameter and the metric terms have to be expanded in Taylor series around a certain latitude. Then, keeping only terms up to order L/a, the quasi-geostrophic beta plane model results.

To study qualitatively high-latitude effects on Rossby wave propagation, Yang (1987) introduced the so-called delta plane[1] by considering the quasi-geostrophic model with $\beta - \delta y$, where β and δ are constants, instead of β alone (see also Yang, 1991). Others followed this approach (Nof, 1990; Nezlin and Snezhkin, 1993; Harlander et al., 2000) and discussed several dynamical effects of the delta plane approximation. These studies give important insight into Rossby wave dynamics on a 'perturbed beta plane'. Nevertheless, such a quasi-geostrophic delta plane model cannot be related to spherical geometry by using the approach which works for the beta plane model. The reason is that Taylor series of metric coefficients become divergent if the central latitude is shifted towards the pole.

The purpose of the present paper is to derive a high-latitude quasi-geostrophic delta plane model from spherical geometry. In contrast to the 'traditional' analysis, which fails for high latitudes, we use the rotated geographical coordinate system, introduced by Verkley (1984, 1990).

The paper is organized as follows. In the next section we discuss the shallow-water model on the sphere for geographical coordinates, and, following Verkley (1990), for rotated geographical coordinates. We do this for the reader's convenience. The only difference from Verkley (1990) is that we write down the equations in non-dimensional form so that the Rossby number shows up in the coefficients of the shallow-water equations. In Section 3 we derive the quasi-geostrophic model on the high-latitude delta plane by performing a perturbation analysis using the Rossby number as a small perturbation parameter (Pedlosky, 1987). First, we show that the 'traditional' analysis fails for the geographical coordinate system. Subsequently, the analysis is applied successfully by using Verkley's rotated coordinate

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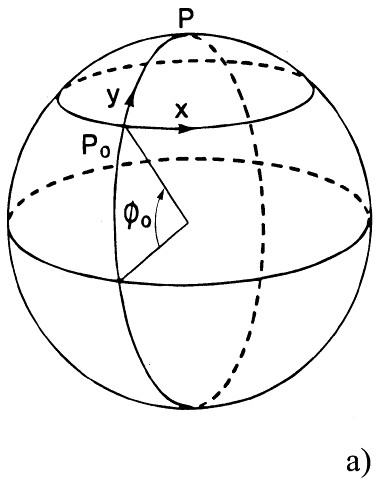
[1]Note that other authors prefer the name gamma plane (Nof, 1990).



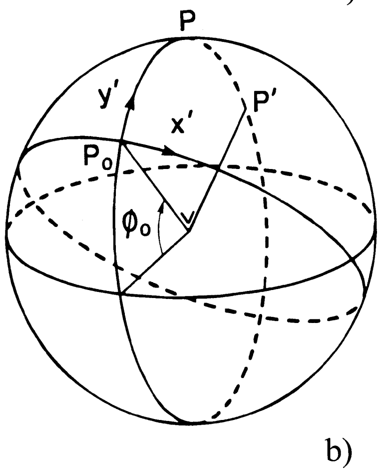


Fig 1. The xy coordinate system used to derive the mid-latitude beta plane model (a), and the $x'y'$ coordinate system we use to derive the delta plane model (b). P (P') denotes the geographical pole (the pole of the rotated system), and ϕ_0 is the latitude of the centre P_0 of both coordinate systems. It is obvious that P_0 is situated at the rotated system's equator. Adapted from Verkley (1990).

system. In Section 4 we discuss solutions for the linearized delta plane model. We compare eigensolutions found analytically for circular basins to the one found in polar plane models (LeBlond, 1964; Haurwitz, 1975; Bridger and Stevens, 1980). We find that polar plane Rossby wave solutions correspond to that of the delta plane when the central latitude of the delta plane is located at the geographical pole. Next we compute ray solutions for the delta plane model analytically. It can be shown that ray curvature, typical for spherical models (Longuet-Higgins, 1964a), is retained for the delta plane, in contrast to the beta plane model. Finally, we discuss phase speed differences at high latitudes between the classical beta plane and the delta plane model. The motivation for such a comparison comes from recent discussions of Rossby wave phase speeds estimated from satellite altimeter data (Lee and Cazenave, 2000). In Section 5 we summarize our findings and give conclusions.

2. Shallow-water equations on the sphere

2.1. Geographical coordinate system

In the study of the equations of motion on a rotating sphere the geographical coordinate system is usually used (see Fig. 1a). This system has longitude λ and latitude ϕ as coordinates, with unit vectors pointing in the direction of increasing λ and ϕ, respectively. The delta plane approximation to be discussed in this section is based on this coordinate system, and we begin by selecting a particular point $\lambda = 0$, $\phi = \phi_0$ as the centre of a new coordinate system, defined by

$$x = \left(\frac{a}{L}\right) \cos\phi_0 \lambda, \tag{1}$$

$$y = \left(\frac{a}{L}\right) (\phi - \phi_0), \tag{2}$$

where a is the planet's radius and L is the typical length-scale of synoptic-scale Rossby waves. For the atmosphere, the standard value is $L \sim \mathcal{O}$ (1000 km), for the ocean $L \sim \mathcal{O}$ (100 km). It is important to note that for the derivation of the beta or the delta plane model from spherical geometry, the ratio L/a must be smaller than one. In the literature (see, for example, Pedlosky, 1987), the size of L/a is connected to the perturbation parameter which is used to derive the quasi-geostrophic model from the shallow-water model. We will also follow this approach (see Section 3). Note that the coordinates x and y are non-dimensionalized. The metric coefficients h_x and h_y specify the nature of the coordinate system by determining distances between neighbouring points. For spherical polar coordinates, they read

$$h_\lambda = a\cos\phi, \qquad h_\phi = a. \tag{3}$$

Using the fact that the Jacobian transformation matrix $J = \partial(\lambda, \phi)/\partial(x, y)$ is diagonal, the metric coefficients, written in the coordinates x and y, read

$$h_x = \frac{L}{a}(\cos\phi_0)^{-1} h_\lambda = L\frac{\cos[\phi_0 + (L/a)y]}{\cos\phi_0}, \tag{4}$$

$$h_y = \frac{L}{a} h_\phi = L. \tag{5}$$

Because $L/a \ll 1$ is assumed, we can expand the metric coefficients in powers of L/a by using a Taylor series expansion about the latitude ϕ_0. We find to order $(L/a)^2$

$$h_x \approx L\left[1 - (L/a)\tan\phi_0 y - \frac{1}{2}(L/a)^2 y^2\right], \tag{6}$$

$$\frac{1}{h_x} \approx \frac{1}{L}\left[1 + (L/a)\tan\phi_0 y + \frac{1}{2}(L/a)^2(1 + 2\tan^2\phi_0)y^2\right], \tag{7}$$

$$\frac{\partial h_x}{\partial y} \approx -L\left[(L/a)\tan\phi_0 + (L/a)^2 y\right]. \tag{8}$$

The Coriolis parameter is

$$f = 2\Omega\sin\phi. \tag{9}$$

Keeping only terms to the order $(L/a)^2$ and using

$$f \approx 2\Omega \left[\sin\phi_0 + \left(\frac{L}{a}\right) \cos\phi_0 y - \frac{1}{2}\left(\frac{L}{a}\right)^2 \sin\phi_0 y^2 \right] \quad (10)$$

we find (from eq. (5)–(8) and (16)–(19), respectively, of Verkley, 1990) for the non-dimensional shallow-water equations

$$\begin{aligned} &\epsilon \left\{ \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) + \left(\frac{L}{a}\right) \tan\phi_0 \left(y\frac{\partial u}{\partial x} - v \right) u \right. \\ &\left. + \frac{1}{2}\left(\frac{L}{a}\right)^2 \frac{1}{\cos^2\phi_0} \left[\left(1+\sin^2\phi_0\right) y\frac{\partial u}{\partial x} - 2v \right] yu \right\} \\ &- \left[1 + \beta y - \frac{1}{2}\delta y^2 \right] v + \left[1 + \left(\frac{L}{a}\right) \tan\phi_0 y \right. \\ &\left. + \frac{1}{2}\left(\frac{L}{a}\right)^2 \frac{1}{\cos^2\phi_0} \left(1+\sin^2\phi_0\right) y^2 \right] \frac{\partial \eta}{\partial x} = 0 \end{aligned} \quad (11)$$

$$\begin{aligned} &\epsilon \left\{ \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) + \left(\frac{L}{a}\right) \tan\phi_0 \left(y\frac{\partial v}{\partial x} + u \right) u \right. \\ &\left. + \frac{1}{2}\left(\frac{L}{a}\right)^2 \frac{1}{\cos^2\phi_0} \left[\left(1+\sin^2\phi_0\right) y\frac{\partial v}{\partial x} + 2u \right] yu \right\} \\ &+ \left[1 + \beta y - \frac{1}{2}\delta y^2 \right] u + \frac{\partial \eta}{\partial y} = 0 \end{aligned} \quad (12)$$

$$\begin{aligned} &\epsilon \left\{ F \left(\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \right) - u\frac{\partial \eta_B}{\partial x} - v\frac{\partial \eta_B}{\partial y} \right. \\ &+ \left(\frac{L}{a}\right) \tan\phi_0 y \left(F u\frac{\partial \eta}{\partial x} - u\frac{\partial \eta_B}{\partial x} \right) \\ &\left. + \frac{1}{2}\left(\frac{L}{a}\right)^2 \frac{1}{\cos^2\phi_0} \left(1+\sin^2\phi_0\right) y^2 \left(F u\frac{\partial \eta}{\partial x} - u\frac{\partial \eta_B}{\partial x} \right) \right\} \\ &+ (1 + \epsilon F\eta - \epsilon\eta_B) \left\{ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \left(\frac{L}{a}\right) \tan\phi_0 \left(y\frac{\partial u}{\partial x} - v \right) \right. \\ &\left. + \frac{1}{2}\left(\frac{L}{a}\right)^2 \frac{1}{\cos^2\phi_0} \left[(1+\sin^2\phi_0) y\frac{\partial u}{\partial x} - 2v \right] y \right\} = 0. \end{aligned} \quad (13)$$

Here, $\epsilon = U/(f_0 L)$ is the Rossby number, where $f_0 = 2\Omega \sin\phi_0$, Ω is the planet's angular velocity, and U is the velocity scale. In the context of large-scale dynamics U can be assumed to be of the order of 10 m s^{-1} for the atmosphere and 1 m s^{-1} for the ocean. Beta and delta are given as

$$\beta = \frac{L}{a}\frac{1}{\tan\phi_0}, \qquad \delta = \left(\frac{L}{a}\right)^2 \qquad \text{and} \qquad F = \frac{f_0^2 L^2}{gD}.$$

D is the mean depth of the fluid, and g is the constant of gravity. The dimensional fluid depth is given as

$$H = D + \eta^* - h_B = D\left(1 + \frac{N_0}{D}\eta - \frac{h_B}{D}\right).$$

The surface variations η are scaled by $N_0 = (f_0 UL)/g$ and further it is assumed that $h_B/D = \epsilon\eta_B$. Therefore, the non-dimensional fluid depth is $h = 1 + \epsilon F\eta - \epsilon\eta_B$. Applying this scaling we follow Pedlosky (1987, chapter 3.12). Note that $1/\cos^2\phi_0 = 1 + \tan^2\phi_0$ and $(1+\sin^2\phi_0)/\cos^2\phi_0 = 1 + 2\tan^2\phi_0$. It should be mentioned that if we fix the Rossby number to the typical value of 0.1 for the atmosphere and 0.01 for the ocean, and we represent the size of the ratio L/a in terms of the Rossby number, the length-scale L as well as the velocity scale U are determined. To be consistent these scales should have the orders of magnitude given above, i.e. L of the order of 1000 km (100 km), and U of the order of 10 m s^{-1} (1 m s^{-1}) for the atmosphere (ocean).

2.2. *Rotated geographical coordinates*

Rotated geographical coordinates were used by Verkley (1984, 1990). Here we assume that the pole of this rotated system has the coordinates $\lambda = -\pi = \lambda'_p$ and $\phi = \pi/2 - \phi_0 = \phi'_p$. As shown in Fig. 1b, we define new (geodesic) coordinates with respect to the rotated geographical coordinates

$$x' = \left(\frac{a}{L}\right)(\lambda' - \pi), \quad (14)$$

$$y' = \left(\frac{a}{L}\right)\phi'. \quad (15)$$

Here λ' and ϕ' refer to longitude and latitude of the rotated coordinate system. Note that the origin of the x', y' system is the same as that of the x, y system. In the geographical coordinates it has the coordinates $\lambda = 0$, $\phi = \phi_0$; in the rotated system the origin is given by $\lambda' = \lambda'_0 = \pi$, $\phi' = \phi'_0 = 0$, i.e. the origin is located on the equator of the rotated geographical system.[2]

The Coriolis parameter can be obtained by substituting

$$\sin\phi = \sin\phi' \sin\phi'_p - \cos\phi' \cos\phi'_p \cos\lambda' \quad (16)$$

into eq. (9) (Verkley, 1990). To order $(L/a)^2$ we find

$$f \approx 2\Omega \left[\sin\phi_0 + (L/a)\cos\phi_0 y' - \frac{1}{2}(L/a)^2 \sin\phi_0 (x'^2 + y'^2) \right]. \quad (17)$$

The metric coefficients read

$$h_{x'} = L\cos[(L/a)y'], \quad (18)$$

$$h_{y'} = L. \quad (19)$$

Once again we can expand the metric coefficients in powers of L/a by using Taylor series expansion about $y' = 0$. For the primed system, the metric coefficients and the equations of motion can be obtained from eqs. (6–8) and (11)–(13) by (i) adding primes to u, v, x, y, (ii) replacing δy^2 by $\delta\,(x'^2 + y'^2)$, and (iii) by replacing

[2]In setting the origin of the primed longitude λ', we use the convention (Verkley, 1990) that $\lambda' = \pi$ for the unprimed pole.

ϕ_0 by ϕ_0'. They read

$$h_{x'} \approx L\left[1 - \frac{1}{2}(L/a)^2 y^2\right], \tag{20}$$

$$\frac{1}{h_{x'}} \approx \frac{1}{L}\left[1 + \frac{1}{2}(L/a)^2 y^2\right], \tag{21}$$

$$\frac{\partial h_{x'}}{\partial y'} \approx -L(L/a)^2 y, \tag{22}$$

and

$$\begin{aligned}\epsilon &\left\{\left(\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) + \frac{1}{2}\left(\frac{L}{a}\right)^2\left[y'\frac{\partial u'}{\partial x'} - 2v'\right]y'u'\right\}\\ &-\left[1 + \beta y' - \frac{1}{2}\delta(x'^2 + y'^2)\right]v'\\ &+\left[1 + \frac{1}{2}\left(\frac{L}{a}\right)^2 y'^2\right]\frac{\partial \eta'}{\partial x'} = 0\end{aligned} \tag{23}$$

$$\begin{aligned}\epsilon &\left\{\left(\frac{\partial v'}{\partial t} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) + \frac{1}{2}\left(\frac{L}{a}\right)^2\left[y'\frac{\partial v'}{\partial x'} + 2u'\right]y'u'\right\}\\ &+\left[1 + \beta y' - \frac{1}{2}\delta(x'^2 + y'^2)\right]u' + \frac{\partial \eta'}{\partial y'} = 0\end{aligned} \tag{24}$$

$$\begin{aligned}\epsilon &\left\{F\left(\frac{\partial \eta'}{\partial t} + u'\frac{\partial \eta'}{\partial x'} + v'\frac{\partial \eta'}{\partial y'}\right) - u'\frac{\partial \eta_B}{\partial x'} - v'\frac{\partial \eta_B}{\partial y'}\right.\\ &\left.+\frac{1}{2}\left(\frac{L}{a}\right)^2 y'^2\left[Fu'\frac{\partial \eta'}{\partial x'} - u'\frac{\partial \eta_B}{\partial x'}\right]\right\}\\ &+(1 + \epsilon F\eta' - \epsilon\eta_B)\left\{\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right.\\ &\left.+\frac{1}{2}\left(\frac{L}{a}\right)^2\left[y'\frac{\partial u'}{\partial x'} - 2v'\right]y'\right\} = 0.\end{aligned} \tag{25}$$

Of course, ϕ_0 must not be replaced by ϕ'_0 in eq. (17) because the axis of rotation intersects the geographical pole and not the pole of the rotated system.

3. Quasi-geostrophic potential vorticity equation

In the following we derive the barotropic quasi-geostrophic potential vorticity equation defining special order of magnitudes for the centre latitude ϕ_0, L/a, and the Rossby number. Therefore, we assume that the Rossby number $\epsilon \ll 1$ so that we can expand u, v, η as

$$\begin{aligned}u &= \epsilon^0 u_0 + \epsilon^1 u_1 + \epsilon^2 u_2 + \cdots\\ v &= \epsilon^0 v_0 + \epsilon^1 v_1 + \epsilon^2 v_2 + \cdots\\ \eta &= \epsilon^0 \eta_0 + \epsilon^1 \eta_1 + \epsilon^2 \eta_2 + \cdots.\end{aligned} \tag{26}$$

The goal is to derive from eqs. (11)–(13) and (23)–(25), respectively, a single equation for the stream function ψ related to η_0, u_0, v_0 via

$$\begin{aligned}\eta_0 &= \psi\\ u_0 &= -\frac{\partial \psi}{\partial y}\\ v_0 &= \frac{\partial \psi}{\partial x},\end{aligned}$$

by truncating eq. (26) at a certain order.

Let us first briefly explain why a high-latitude delta plane equation cannot be derived from the shallow-water model on the sphere when geographical coordinates are used, in contrast to the classical mid-latitude beta plane model. In the standard perturbation analysis used to derive the quasi-geostrophic equation, only terms of order ϵ are considered. Keeping this in mind we focus on a central latitude ϕ_0 and a ratio L/a such that $\beta \sim \delta \sim \mathcal{O}(\epsilon)$. Therefore, we have to assume $(L/a) \sim \mathcal{O}(\epsilon^{1/2})$ and $\tan\phi_0 \sim \mathcal{O}(\epsilon^{-1/2})$, leading to $\mathcal{O}(\beta) = \mathcal{O}(\delta) = \mathcal{O}(\epsilon)$. Unfortunately, for such latitudes the Taylor expansion of $1/h_x$ becomes divergent for $y \geq 1$. As a consequence, we are not allowed to truncate the series to obtain eq. (7). This means that we cannot proceed with the perturbation analysis to derive a high-latitude quasi-geostrophic model as long as we use the standard geographical coordinate system displayed in Fig. 1a.

For the rotated geographical coordinate system (Fig. 1b), however, it is straightforward to derive a delta plane equation for high latitudes. The reason is that, for the rotated system, the Taylor expansions do not contain the central latitude ϕ_0 and therefore there is no danger in working with divergent Taylor series as long as $L/a \ll 1$ holds. As said before, with the assumptions $(L/a) \sim \mathcal{O}(\epsilon^{1/2})$, $\tan\phi_0 \sim \mathcal{O}(\epsilon^{-1/2})$ we create a situation where the beta term and the delta term have the same order of magnitude, namely $\mathcal{O}(\epsilon)$. Forming the vorticity equation by combining the $\mathcal{O}(\epsilon^0)$ equations and the $\mathcal{O}(\epsilon^1)$ equations resulting from eqs. (23)–(25) we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\nabla'^2\psi - F\psi\right) + J'(\psi, \nabla'^2\psi) + J'(\psi, \eta_B)\\ &+\left(\beta_0 - \delta_0 y'\right)\frac{\partial \psi}{\partial x'} + \delta_0 x'\frac{\partial \psi}{\partial y'} = 0,\end{aligned} \tag{27}$$

where $\beta_0 = \beta/\epsilon$ and $\delta_0 = \delta/\epsilon$. This is the delta plane potential vorticity equation derived from the rotated geographical coordinate system. These coordinates were previously used by Verkley (1984, 1990). Equation (27) differs from the delta plane equation used by Yang (1987), Nof (1990), and Harlander et al. (2000) by an additional term, namely $\delta_0 x'\partial\psi/\partial y'$. Notice that it is not problematic to move P_0 (see Fig. 1) further to the pole. The polar delta plane results by assuming $(L/a) \sim \mathcal{O}(\epsilon^{1/2})$, $\phi_0 = \pi/2$. For the primed system, $\tan\phi_0$ shows up in β alone and not in eqs. (20)–(22) and (23)–(25). Therefore, we can apply this scaling without blowing up metric coefficients. We simply create a situation where $\beta = 0$ and $\delta \sim \mathcal{O}(\epsilon)$. Considering the shallow-water equations to $\mathcal{O}(\epsilon)$ we obtain eq. (27) but without the beta term.

4. Applications

In the following we focus on applications for the delta plane equation (27). First of all, we should point out that eq. (27) holds when δ has a significant impact, i.e. if β_0 and δ_0 are of order one. Of course, the equation will also be valid for cases where δ_0 (or β_0) is smaller than this. Let us now discuss briefly for which geographical region δ has the same order of magnitude than β, namely $\mathcal{O}(\epsilon)$. We study two typical cases: (i) the Earth's atmosphere with a typical Rossby number of $\epsilon = 0.1$; (ii) the Earth's oceans where $\epsilon = 0.01$ typically holds. From these two choices for ϵ we obtain typical values for U, L, and ϕ_0, summarized in Table 1. It is important to note that, indeed, the values of U and L have the order of magnitude mentioned in Section 2.1. Figure 2a shows the magnitude of ϵ, β, and δ as a function of latitude for atmospheric conditions in the range $\phi = 40°$–$89°$ (note that a logarithmic ordinate has been used, i.e. -1 stands for $\epsilon, \beta, \delta = 10^{-1}$). The value of the Rossby number ϵ hardly changes in that range, and δ is constant. On the other hand, β varies from $10^{-0.4}$ to $10^{-2.2}$. Nevertheless, from about 60°–80°, β, δ, and ϵ stay in the same order of magnitude. This means that in this latitude band (or further poleward) the delta plane can be located. Note that only for polar regions the beta effect can be neglected.

For the ocean scaling (see Table 1 and Fig. 2b) β covers three orders of magnitude from 40° to 89°, whereas ϵ remains in the order $\mathcal{O}(10^{-2})$, and δ is constant. It can be seen that for small Rossby numbers the delta plane is located more poleward than for typical atmospheric conditions. The centre of the delta plane should be located north (south) of 70°N (S).

In the following we study some solutions of the delta plane equation mainly to stimulate interest for further investigations of this extended quasi-geostrophic model.

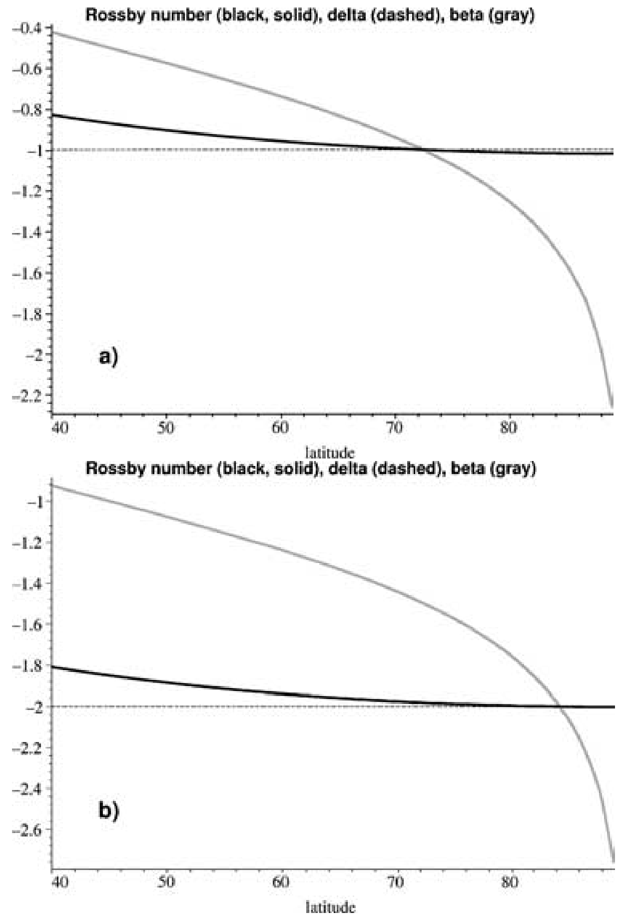


Fig 2. (a) Magnitude of ϵ, β, and δ as a function of latitude (a) for the Earth's atmosphere and (b) for the Earth's oceans. Note that $(L/a) \sim 1/\tan\phi_0 \sim \mathcal{O}(\epsilon^{1/2})$. The grey line corresponds to β, the thick solid line to ϵ, and the thin dashed line to δ. Note further that a logarithmic ordinate has been used, i.e. -1 represents $\epsilon, \beta, \delta = 10^{-1}$.

4.1. *Comparison of ocean basin eigenmodes obtained from different polar plane models*

4.1.1. Eigenmodes found from the delta plane model. Here we study eigensolutions of a high-latitude ocean basin. For enclosed basins in spherical geometry other than those whose boundaries are exclusively circles of latitude, only very few solutions have been evaluated. On the other hand, for beta plane geometry, analytical solutions for basins with rectangular, circular, and triangular shape are known. The delta plane discussed here offers the possibility to discuss analytical solutions for enclosed basins in regions where the classical beta plane approximation is not valid.

Using x', $y' = \hat{x}$, $\hat{y} + \beta_0/\delta_0$, the linearized version of eq. (27) without topography reads (by dropping the hats)

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) - \delta_0 y\frac{\partial\psi}{\partial x} + \delta_0 x\frac{\partial\psi}{\partial y} = 0. \tag{28}$$

Using the fact that in circular cylindrical coordinates the operator $x\partial/\partial y - y\partial/\partial x = \partial/\partial\varphi$, eq. (27) becomes separable for circular ocean basins. If we substitute

$$\psi = \tilde{\psi}(r)\exp[-\mathrm{i}(m\varphi + \omega t)] \tag{29}$$

into eq. (27) written in circular cylindrical coordinates we obtain

$$r^2\tilde{\psi}'' + r\tilde{\psi}' + (K^2r^2 - m^2)\tilde{\psi} = 0, \quad m = 0, 1, 2, \cdots, \tag{30}$$

with

$$K^2 = \frac{\delta_0 m}{\omega} - F. \tag{31}$$

Table 1. Scales for the situations discussed in the text. U is in m s^{-1}, L in 10^3 km, and ϕ_0 in degrees. The Rossby number is denoted by ϵ

	$U \sim \mathcal{O}(2\Omega\sin\phi_0 L\epsilon)$	$L \sim \mathcal{O}(\epsilon^{1/2}a)$	$\phi_0 \sim \mathcal{O}(\arctan\epsilon^{-1/2})$	ϵ
Atmosphere	28	2.02	72	0.1
Ocean	0.9	0.64	84	0.01

Note that eq. (31) is the beta plane dispersion relation for Rossby waves, but with β_0 replaced by $-\delta_0$. The total wavenumber K corresponds to the eigenvalues of the eigenproblem (30). The general solution of eq. (30) reads for $m \neq 0$

$$\tilde{\psi} = A_m J_m(Kr) + B_m Y_m(Kr), \tag{32}$$

where $J_m(x)$ and $Y_m(x)$ are the Bessel functions of first and second kinds, respectively (Temme, 1996). The eigenvalues can be found from the boundary conditions $\psi(r_1, \varphi, t) = \psi(r_2, \varphi, t) = 0$, where r_1 and r_2 define the boundaries of the circular shaped ocean. If $r = 0$ is part of the ocean (i.e. the ocean is simply connected), $B_m = 0$ for regularity of ψ. K is then given by the zeros of $J_m(Kr_2)$. Otherwise (e.g. if we consider a polar ocean with a continent in the centre) we need to solve

$$J_m(Kr_1)Y_m(Kr_2) - J_m(Kr_2)Y_m(Kr_1) = 0, \tag{33}$$

$$B_m = -A_m J_m(Kr_1)/Y_m(Kr_1). \tag{34}$$

For $m = 0$ the general solution of eq. (30) reads

$$\tilde{\psi} = A_0 I_0(Kr) + B_0 K_0(Kr), \tag{35}$$

where $I_0(x)$ and $K_0(x)$ are the modified Bessel functions (Temme, 1996). These functions have the properties $I_0(x) \to \infty$ for $x \to \infty$, $K_0(x) \to 0$ for $x \to \infty$ and $I_0(x)$ finite for $x \to 0$, $K_0(x) \to \infty$ for $x \to 0$.

If we use the original primed coordinates as in eq. (27), we see the effect of β_0 on the solutions: the centre of the ocean is given by x_0, $y_0 = 0$, $-\beta_0/\delta_0$, which means that the location of the ocean basin is determined by β_0/δ_0. The structure of the eigensolutions as well as the eigenvalues are not affected by β_0. This, of course, might be different for other ocean basin geometries.

4.1.2. Eigenmodes from polar plane geometries. When the centre of the δ plane is situated at the geographical pole (i.e. $\beta = 0$), eq. (27) describes quasi-geostrophic motions on polar plane geometry.

Other authors have derived polar plane approximations by completely different approaches than the one discussed here (LeBlond, 1964; Haurwitz, 1975; Bridger and Stevens, 1980). LeBlond projected the surface of the sphere in a simple orthographic manner to a plane tangent to the sphere at the pole. He retained only first-order terms in r/a to keep the influence of the terrestrial curvature. Haurwitz (1975) and Bridger and Stevens (1980) used a cylindrical coordinate system with the plane $z = 0$ tangential to the Earth's surface at the pole. Like we have done, they expanded f in a Taylor series and truncated it after the quadratic term and restricted the analysis to small colatitude angles. It is satisfactory that for Rossby waves (i.e. $\omega \ll f$) their solutions correspond to the solutions of the polar delta plane (31) and (32). This can be seen by comparing eq. (29) of LeBlond and the solution of eq. (3.9) by Bridger and Stevens with eq. (32). To compare the eigenvalues, eq. (31) of LeBlond has to be solved for $\beta^2_{k,n}$, which corresponds to K^2 in eq. (31). If the second and the last term on the right-hand side of eq. (3.10) of Bridger and Stevens are neglected (using $\omega \ll f$), m^2 in Bridger and Stevens corresponds to K^2 in eq. (31). We think that the correspondence of polar delta plane solutions to solutions of other polar plane models gives the delta plane model an additional validity. Notice that LeBlond (1964), Haurwitz (1975), and Bridger and Stevens (1980) compared the polar plane solutions to solutions for spherical geometry given by Longuet-Higgins (1964a). LeBlond concluded that 'the polar plane approximation used here is a very good one indeed and can be employed in the study of polar oceanographic phenomena with the same confidence that is bestowed upon the beta plane approximation in the study of mid-latitude phenomena.'

In Figs. 3 and 4 we show some of the gravest eigenmodes for $m = 1, 2$ for a simply connected polar ocean (in Fig. 3a), and the case with a central 'continent' (in Fig. 3b), a configuration not investigated by LeBlond (1964). We will not study these eigenmodes in detail here, however, it should be mentioned that such modes might play an important role in recently discussed Antarctic circumpolar waves (White and Peterson, 1996; Haarsma et al., 2000).

4.2. *Rays of wave packet propagation*

The study of energy dispersion helps us to understand the propagation of local disturbances. Therefore, the ray patterns, in addition to eigenmodes, complete the picture one can gain from linear wave theory. On a beta plane without a shear flow, the energy rays are straight lines. Nevertheless, it is known that on a sphere, Rossby wave packets follow great circles whose plane rotates round the axis of rotation with angular velocity ω/k, where ω (k) is the wave's frequency (zonal wavenumber) (Longuet-Higgins, 1964b). This means that on a beta plane the curvature of the ray paths is lost, which is certainly unsatisfactory for higher latitudes. Therefore, it is logical to study to what extent ray curvature is retained by the delta plane model (27). The ray tracing equations can be obtained from a higher-dimensional WKB method, called, for short, the ray method (for details see Yang, 1991, pp. 29 and 30). The equations corresponding to the linear version of eq. (27) with $(\hat{x}, \hat{y}) = (x', y' - \beta_0/\delta_0)$ read (hats are dropped again)

$$-\frac{\partial \omega}{\partial x} = \frac{\mathrm{d}k}{\mathrm{d}t} = +\frac{l}{K^2}\delta_0 \tag{36}$$

$$-\frac{\partial \omega}{\partial y} = \frac{\mathrm{d}l}{\mathrm{d}t} = -\frac{k}{K^2}\delta_0 \tag{37}$$

$$\frac{\partial \omega}{\partial k} = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\delta_0}{K^4}[(l^2 - k^2 + F)y + 2klx] \tag{38}$$

$$\frac{\partial \omega}{\partial l} = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\delta_0}{K^4}[(l^2 - k^2 - F)x - 2kly], \tag{39}$$

with the dispersion relation

$$\omega = \frac{\delta_0}{K^2}(ky - lx). \tag{40}$$

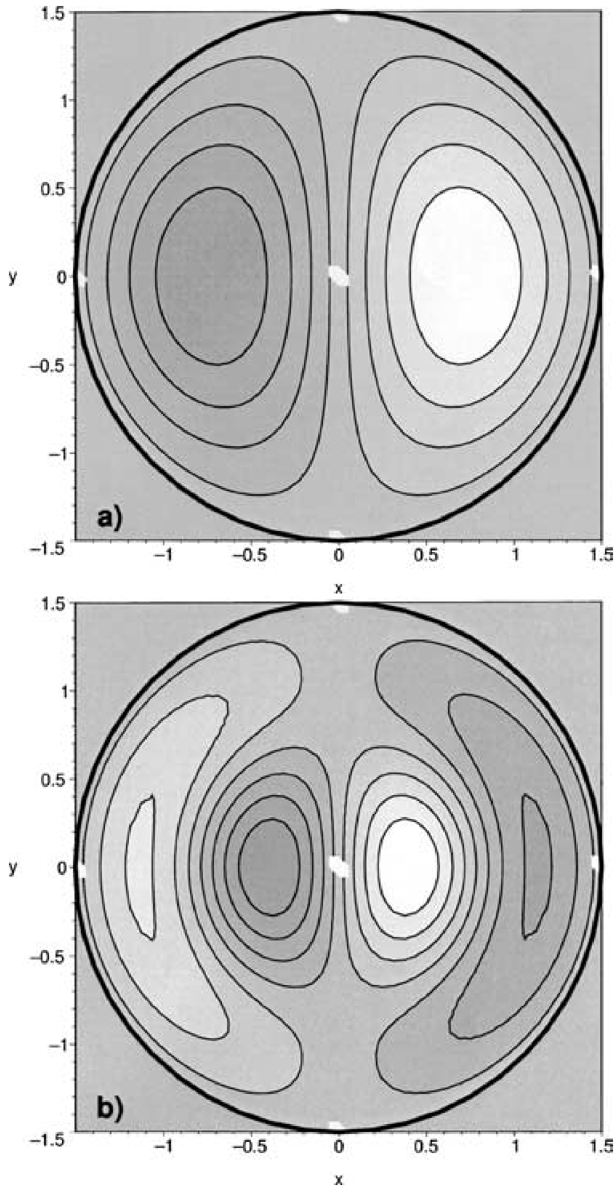


Fig 3. Eigenfunctions of a circular polar ocean basin with $r_2 = 1.5$. (a) $m = 1$, $K = 2.5544$ (first mode with zonal wavenumber 1); (b) $m = 1$, $K = 4.6770$ (second mode with zonal wavenumber 1); (c) $m = 2$, $K = 3.4237$ (first mode with zonal wavenumber 2).

The wave packet propagates with the group velocity $\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t$; k and l are the local wavenumbers in zonal and meridional directions (with respect to the primed system), $K^2 = k^2 + l^2 + F$ is total wavenumber, and $x(t)$ and $y(t)$ give the position of the wave packet for each t. Equations (36)–(39) form a complicated nonlinear system. Nevertheless, analytical solutions can be found. First, we observe that eqs. (36) and (37) are decoupled from eqs. (38) and (39), i.e. the wavenumber evolution can be computed independently from the path. Secondly, multiplying

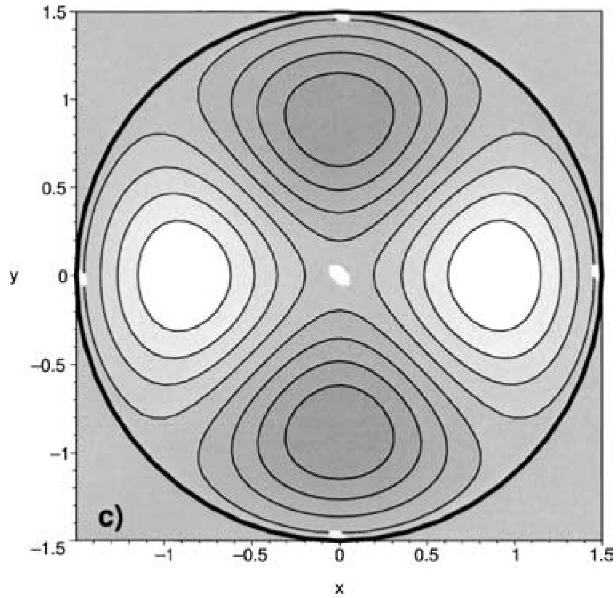


Fig 3. (cont'd).

eq. (36) by k, eq. (37) by l and adding both equations we obtain

$$\frac{\mathrm{d}K^2}{\mathrm{d}t} = 0, \tag{41}$$

i.e. the total wavenumber is an invariant of motion. This means that a propagating wave packet cannot gain or lose energy by a total wavenumber change, but only due to amplitude changes, in contrast to the situation in shear flows (Yang, 1991). Note that for spherical geometry the zonal wavenumber is constant along the path, but the meridional wavenumber changes. By applying eq. (41) we obtain from eqs. (36) and (37)

$$\frac{\mathrm{d}^2k}{\mathrm{d}t^2} + \hat{\delta}^2 k = 0, \tag{42}$$

where $\hat{\delta} = \delta_0/K^2$. This equation can be integrated immediately

$$k(\tau) = \kappa \sin\tau \tag{43}$$

$$l(\tau) = \kappa \cos\tau, \tag{44}$$

with $\kappa^2 = K^2 - F$, and $\tau = \hat{\delta}t + \nu$. The values for κ and ν are determined by the initial conditions. Another constraint of motion is ω. This can simply be shown by evaluating $\mathrm{d}K^2\omega/\mathrm{d}t = K^2\mathrm{d}\omega/\mathrm{d}t = 0$ using eqs. (36)–(41). Then, by writing eq. (40) in polar coordinates $x(\tau) = r(\tau)\cos\varphi(\tau)$, $y(\tau) = r(\tau)\sin\varphi(\tau)$, we find

$$r(\tau) = -\frac{\omega}{\hat{\delta}\kappa}(\cos[\tau + \varphi(\tau)])^{-1}. \tag{45}$$

Writing eqs. (38) and (39) in polar coordinates and substituting eqs. (43) and (44), an equation for $\varphi(\tau)$ can be found. First, we multiply eq. (38) by $\sin\varphi$, eq. (39) by $\cos\varphi$, then we subtract the

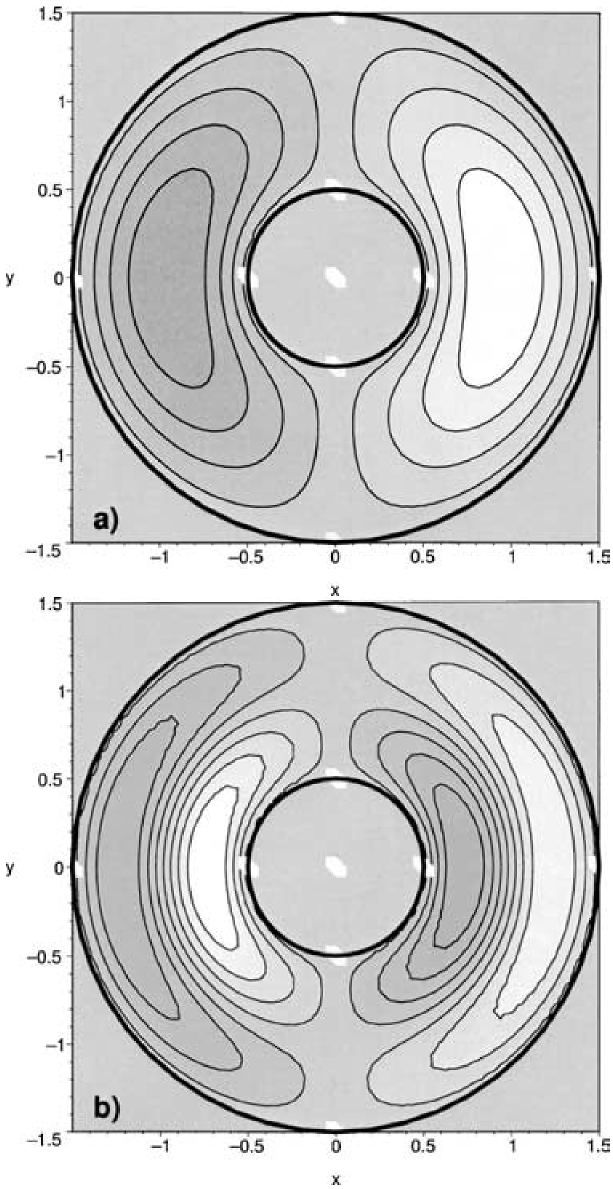


Fig 4. Eigenfunctions of a circular polar ocean annulus with $r_1 = 0.5$, $r_2 = 1.5$. (a) $m = 1$, $K = 3.2712$, $B/A = 1.7636$ (first mode with zonal wavenumber 1); (b) $m = 1$, $K = 6.3576$, $B/A = -0.7360$ (second mode with zonal wavenumber 1); (c) $m = 2$, $K = 3.7360$, $B/A = 0.4690$ (first mode with zonal wavenumber 2).

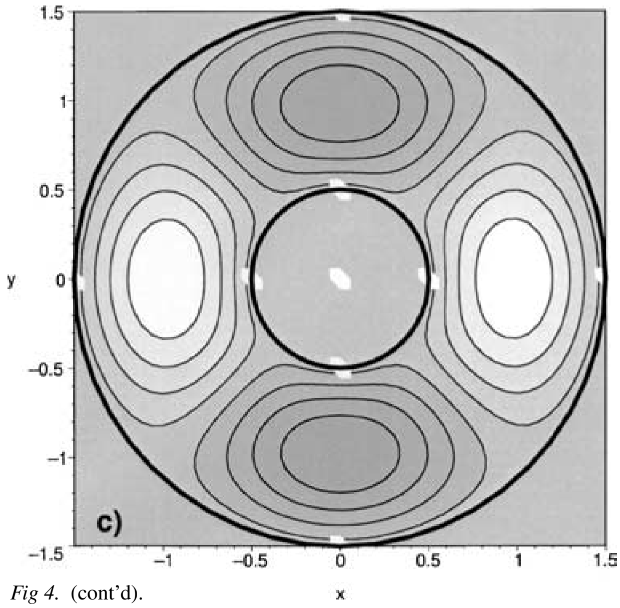


Fig 4. (cont'd).

equations. Using identities from trigonometry we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\kappa^2}{K^2}\cos[2(\varphi + \tau)] - \frac{F}{K^2}. \tag{46}$$

Now we introduce $\tilde{\varphi} = \varphi + \tau$ and, by noting that $1 - F/K^2 = \kappa^2/K^2$, $\cos(2\tilde{\varphi}) + 1 = 2\cos^2\tilde{\varphi}$ we find

$$\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau} = \frac{2\kappa^2}{K^2}\cos^2\tilde{\varphi}. \tag{47}$$

Finally, by applying $\mathrm{d}(\tan\tilde{\varphi} + 2\kappa^2 C/K^2)/\mathrm{d}\tilde{\varphi} = 1/\cos^2\tilde{\varphi}$, where C is an arbitrary constant, we obtain

$$\varphi = \arctan\left(\frac{2\kappa^2}{K^2}(\tau - C)\right) - \tau. \tag{48}$$

Inspecting eqs. (45) and (48) we can give a special solution immediately. Assuming a constant radius $r(\tau)$, we obtain from eq. (45), $\tau + \varphi(\tau) = \varphi_0$, where φ_0 is a constant. This is a solution of eq. (48) if $C \to \infty$ and $\varphi_0 = -\pi/2$, which implies $\omega = 0$. This solution describes 'inertial oscillations' of a wave packet around the delta plane's centre.

Figure 5a shows solutions computed from three different initial conditions, namely, $k(0) = l(0) = 3$, $x(0) = y(0) = 0.55$, $k(0) = l(0) = 6$, $x(0) = y(0) = -0.3$, and $k(0) = l(0) = 1$, $x(0) = 0.45$, $y(0) = 0.5$. The first two situations correspond to the special solution mentioned above, i.e. with $\omega = 0$; the last one (grey curve in the figure) is computed from eqs. (45) and (48) with $F = 0$ and shows more complex dynamics. It is clear from the figure that in contrast to the beta plane, ray curvature is retained; however, the rays look different than a projection of great circles on a polar plane. Finally, in Fig. 5b, we show two solutions corresponding to initial conditions $k(0) = 2.4$, $l(0) = 3$, $x(0) = y(0) = -0.5$, one with $F = 0$ and the other with $F = 1$. Again, the solution shows oscillations but with increasing r, like an orbit around a slightly unstable centre. Because the delta plane is useful for $r \leq \mathcal{O}(1)$ only, such rays show some similarity to great circles projected on the delta plane.

It is not the purpose of the study to discuss the complicated ray dynamics on the delta plane in detail. This will be done elsewhere. Nevertheless, it is clear that the ray dynamics are much richer on the delta than on the beta plane. Interestingly, some of the paths shown remain on spherical spirals: the route taken by

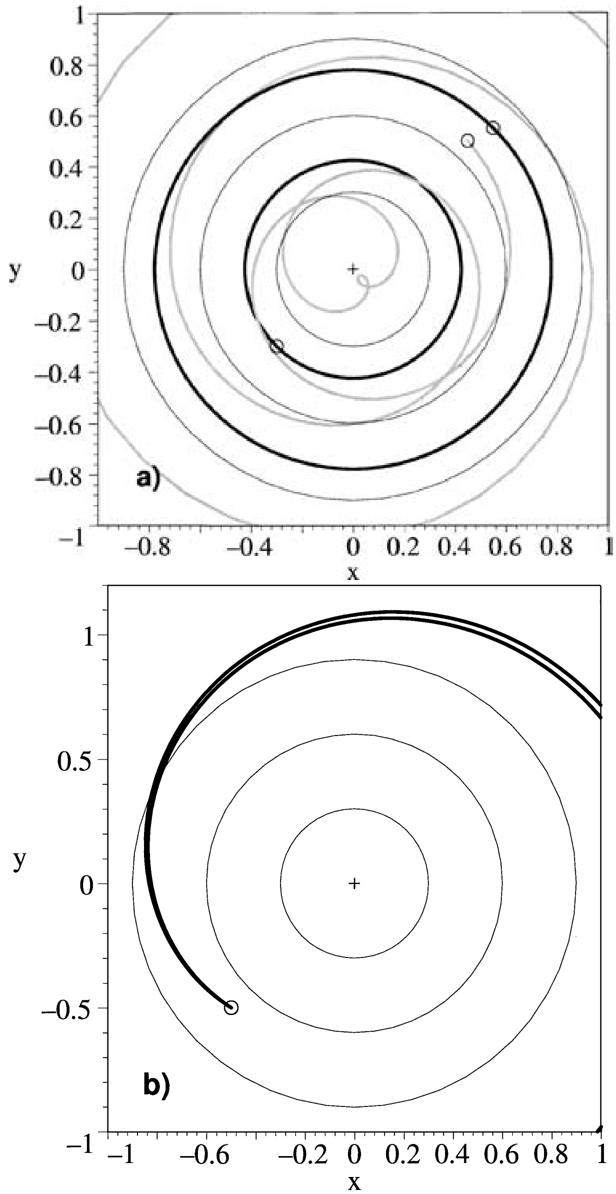


Fig 5. Ray paths on the delta plane. Initial conditions are $k(0) = l(0) = 3$, $x(0) = y(0) = 0.55$, $k(0) = l(0) = 6$, $x(0) = y(0) = -0.3$ (circles drawn by thick lines), and $k(0) = l(0) = 1$, $x(0) = 0.5$, $y(0) = 0.45$ (grey curve) with $F = 0$ in (a). In (b) the initial condition is $k(0) = 2.4$, $l(0) = 3$, $x(0) = y(0) = -0.5$, one path corresponds to $F = 0$, the other to $F = 1$. The paths are drawn by thick curves. The open dots show $x(0)$, $y(0)$, and the circles drawn by thin solid lines show $r = 0.3$; 0.6; 0.9.

a ship which travels from one pole to the other while keeping a fixed (but not right) angle with respect to the meridians.

4.3. *Phase speed differences on beta and delta plane*

Before we compare phase speeds on the beta and delta plane, we first consider the dispersion relation of a purely horizontal flow on the sphere. For such a flow, absolute vorticity is conserved and the vorticity equation reads in spherical coordinates (Hoskins and Karoly, 1981)

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{1}{a}\frac{\partial f}{\partial \phi}\frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda} = 0. \tag{49}$$

To derive the dispersion relation we follow Hoskins and Karoly (1981) and use the Mercator projection of the sphere (Haltiner and Williams, 1980):

$$\tilde{x} = a\cos(\phi_0)\lambda \tag{50}$$

$$\tilde{y} = a\cos(\phi_0)\ln[(1+\sin\phi)/\cos\phi] \tag{51}$$

$$\cos\phi = \left[\cosh\left(\frac{\tilde{y}}{a\cos\phi_0}\right)\right]^{-1}, \quad \sin\phi = \tanh\left(\frac{\tilde{y}}{a\cos\phi_0}\right). \tag{52}$$

Using the scaling of Section 2, the non-dimensional version of eq. (49) reads

$$\epsilon\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial\hat{x}^2} + \frac{\partial^2\psi}{\partial\hat{y}^2}\right) + \beta_s\frac{\partial\psi}{\partial\hat{x}} = 0, \tag{53}$$

where $L(\hat{x}, \hat{y}) = (\tilde{x}, \tilde{y})$ and

$$\beta_s = \frac{L\cos^2\phi}{a\cos\phi_0\sin\phi_0}. \tag{54}$$

From a WKB analysis we can find the local zonal (x-direction) phase speed as

$$\frac{\omega}{k} = c_s = -\frac{\beta_s/\epsilon}{k^2+l^2}. \tag{55}$$

Inspecting this equation it is clear that with respect to a central latitude ϕ_0, phase velocity decreases (increases) north (south) of ϕ_0. Obviously, this effect cannot be described by the beta plane model.

Considering now eq. (40) (and recalling $y = y' - \beta_0/\delta_0$) it is clear that for high latitudes the phase velocity of linear Rossby waves on the delta plane can be significantly different from the beta plane values. When ϕ_0 is fixed (say at high northern latitudes), the phase of waves north (south) of the delta plane's centre propagate slower (faster) to the west than predicted by linear theory on the beta plane. Therefore, the delta plane model captures the fact that the phase velocity is latitude dependent on the sphere. The local 'zonal' phase velocities for the beta and the delta plane read

$$c_\beta = -\frac{\beta_0}{K^2}, \qquad c_\delta = -\frac{\beta_0}{K'^2} + \frac{\delta_0}{K'^2}[y' - x'(l'/k')], \tag{56}$$

where $K^2 = k^2 + l^2 + F$ is the total wavenumber squared.

To obtain a quantitative picture, we have plotted the phase speed ratios c_s/c_β and c_δ/c_β as a function of latitude in Fig. 6, assuming an offset of $x', y' = 0, \pm 1/8$ of the wave packet from the centre of the plane (corresponding to 250 km (or $\phi' = 7.115°/\pi$) for $\epsilon = 0.1$ and to 80 km (or $\phi' = 9°/4\,\pi$) for $\epsilon = 0.01$). For the local wavenumbers we used $k = k' = 1$, $l = l' = 3$. For

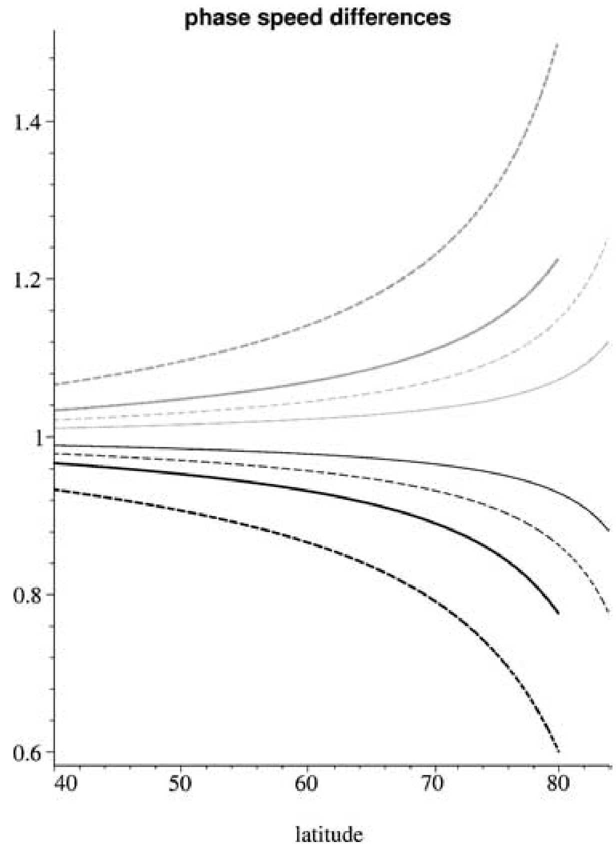


Fig 6. c_s/c_β (dashed lines), c_δ/c_β (solid lines) as a function of latitude. The thick lines correspond to the Earth's atmosphere, the thin lines to the Earth's oceans (see Table 1). The grey (black) lines correspond to negative (positive) offset. As offset we used $x' = 0$, $y' = \pm 1/8$. Moreover, $k = k' = 1$, $l = l' = 3$, $F = 0$ was used.

comparison with eq. (53) we set $F = 0$. The dashed (solid) lines in Fig. 6 correspond to c_s/c_β (c_δ/c_β), the thick lines correspond to the atmospheric scaling with $\epsilon = 0.1$, and the thin lines to the oceanic scaling with $\epsilon = 0.01$. For positive (negative) offset, lines are drawn in black (grey). The figure shows that the delta plane model is obviously superior to the beta plane model, but still it underestimates the effects related to Earth's curvature. However, it becomes clear from the figure that, compared to the beta plane, the delta plane model appears to be a useful extension for conceptual studies on high-latitude dynamics, where the delta effect becomes most important. It should be noted that, as long as the offset is small, we can compare solutions on the beta and delta plane directly because the unit vectors corresponding to the geographical and the rotated geographical system are essentially the same (see Fig. 1).

It is well known that Rossby wave phase speeds observed in the oceans by satellite altimetry do not correspond well to linear theory on the beta plane, in particular at high latitudes (for a review see Lee and Cazenave, 2000). Several aspects of the simple beta plane model are responsible for this mismatch, e.g. the neglect of a basic flow (or at least its shear), the neglect of bottom topography, the neglect of non-linear processes, the neglect of any coupling between oceanic Rossby waves and the wind-stress curl, etc. However, the delta effect also modifies the phase speeds (see Fig. 6). This fact might gain importance in the near future, when more accurate observational techniques allow for an investigation of Rossby waves poleward of 50°.

5. Discussion and conclusion

Before we discuss the findings, let us summarize the main results of the study. First, a formal derivation of the potential vorticity equation on the delta plane has been presented for the first time. We have seen that for the quasi-geostrophic model at high latitudes the delta plane approximation can be derived from spherical geometry. However, we need to do the analysis with a rotated geographical coordinate system. This system was used earlier by Verkley (1990) to derive a beta plane shallow-water model.[3] Secondly, eigenmodes of circular polar ocean basins have been computed, showing that the delta plane model describes well the low-frequency basin modes of a polar plane shallow-water model, discussed by other authors. Thirdly, rays of energy dispersion have been computed analytically for the delta plane model. We have found that ray curvature, typical for energy rays at high latitudes, is retained using the delta plane model, in contrast to the beta plane, where energy rays are straight lines. Fourthly, we estimated the westward phase speed of freely propagating Rossby waves on the delta plane and compared it to the phase speed on the sphere and on the beta plane. We have found that the westward phase speed of the delta plane model shows a dependency on latitude comparable to a model with spherical geometry. The ratio of the delta to the beta plane zonal phase speed changes monotonically with increasing latitude, in qualitative agreement with the phase speed ratio from models with spherical and beta plane geometry. These findings suggest that the quasi-geostrophic delta plane model is a suitable tool for conceptual studies on large-scale dynamics at high latitudes.

In the following we will make, first, some comments on the findings summarized above and, secondly, more general comments on the delta effect in other models of geophysical fluid dynamics. We have considered analytical solutions of ocean modes in circular closed basins. We have pointed out that for $\beta = 0$ (i.e. P_0 is located at the geographical pole) the solutions correspond to that found by LeBlond (1964), Haurwitz (1975), and Bridger and Stevens (1980). Their derivation of polar plane models, however, differs from our approach. Therefore, the correspondence of Rossby wave solutions for different polar plane models gives the delta plane model additional validity.

[3]Without giving details here we should mention that, in general, effects of curvature of the coordinate curves are as small as possible near the origin of the coordinate system when a geodesic system is applied (van der Toorn, 1997).

Notice that the high-latitude delta plane is not restricted to the pole. Our analysis shows that the centre can be located anywhere poleward to about 60° to be consistent with the scaling used (see Fig. 2a).

Listed as the third point in the summary above, we studied rays of energy propagation on the delta plane and showed how to obtain solutions analytically. Ray solutions have not been discussed for polar plane models so far, in contrast to solutions for beta plane and spherical models. This is surprising because it is an important question, to what degree polar plane models are able to map ray curvature, typical for the sphere. In contrast to the beta plane solutions, ray curvature is indeed retained on the delta plane, although a direct connection to the solutions on the sphere could not be made. For example, stationary ($\omega = 0$) wave packets propagate (with the group velocity) in a circle around the delta plane's centre whereas on the sphere such waves propagate along great circles.

Finally, in addition to the neglect of ray curvature, beta plane models do not contain the dependency of zonal phase speed with respect to latitude, which is typical for the situation on the sphere. We have compared zonal phase speeds when the standard beta plane approximation and the delta plane approximation are used. By comparison with phase speeds obtained from a model in spherical coordinates we conclude that, at high latitudes, observed Rossby wave speeds should better be compared with solutions from models with spherical or delta plane geometry because delta plane effects (i.e. Earth's sphericity) become important.

It is important to note that the delta plane applied by Yang (1987), Nof (1990), and Harlander et al. (2000) differs from the delta plane derived here. Replacing β simply by $\beta - \delta y$ (where y corresponds to the meridional direction) gives a quasi-geostrophic delta plane model which cannot be derived from spherical geometry. However, the purpose of our analysis is not to criticize these papers. Perturbing the standard quasi-geostrophic beta plane gives rise to new phenomena, e.g. Rossby wave vacillation (Yang, 1991). Such interesting findings should not be disregarded arguing that the used model differs from eq. (27). The reason is that beta plane perturbations can result from sphericity and topography effects; the latter effects were not considered here.

The delta effect shows up in laboratory experiments with quite different settings, e.g. rotating annulus experiments (Busse and Or, 1986; Or and Busse, 1987). From experiments it is well known that the shape and orientation of boundaries can have a strong influence on the dynamics of rotating fluids (Wimmer, 2000; Maas, 2001). If the lower boundary of an annulus has a conical shape with a small slope, the topographic effect can (in the barotropic case) simply be interpreted as the planetary beta effect. Our analysis makes it clear that, in the case of a rotating annulus with a paraboloidal lower boundary, the Rossby wave dynamics can be seen as a result of the planetary delta effect, resulting from the spherical geometry at high latitudes.

Finally, it should be mentioned that in other experiments, parabolically-shaped fast rotating bowls have been used. In a non-perturbed situation, the thickness of the fluid layer is (almost) constant due to the fast rotation (and not due to the small slope of the lower boundary). Experiments on Rossby vortices have been performed with this configuration (Nezlin and Snezhkin, 1993). The observed vortices have large amplitudes, i.e. the perturbation of the fluid depth is comparable to the equilibrium depth and the standard quasi-geostrophic theory is not valid (Nycander and Sutyrin, 1992). In the extended quasi-geostrophic equation of Nezlin and Snezhkin, additional non-linear terms appear and, interestingly, also a delta term [as $J(\psi, \beta - \delta y^2/2)$], considered as an 'enhancement' of the beta plane approximation (see eq. (5.4) on p. 46 in Nezlin and Snezhkin, 1993). According to Nezlin and Snezhkin, the delta term is of major importance in the context of cyclone/anticyclone dynamics because it determines the types of non-linear structures allowed by the extended quasi-geostrophic equation.

We should point out that the mid-latitude beta and high-latitude delta plane assumption are valid only around a particular point P_0, i.e. the approximations are local with respect to x, x' and y, y'. Thinking in terms of 'latitude band subdivision' we suggest the following simpler-than-spherical geometry for Rossby wave dynamics: the shallow-water equatorial beta plane near the equator (Matsuno, 1966), the quasi-geostrophic model on the beta plane for latitudes between 10° to 50° (Lindzen, 1967), the same model but on the delta plane for latitudes larger than 50°.

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