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SHORT CONTRIBUTION

A note on time averages in turbulence with reference to geophysical applications

By P. C. CHATWIN and C. M. ALLEN, *Department of Applied Mathematics and Theoretical Physics, University of Liverpool, P.O. Box 147, Liverpool L69 3BX, England*

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1. Introduction

The fundamental average in turbulence is the probability, or ensemble, average. Suppose that Γ denotes any randomly fluctuating variable, such as temperature or one component of velocity, and let $\Gamma^{(n)}(\boldsymbol{x}, t)$ be its value at position $\boldsymbol{x}$ and time t in the nth of N realizations. The ensemble average, or mean, of $\Gamma(\boldsymbol{x}, t)$, denoted by $C(\boldsymbol{x}, t)$ satisfies

$$C(\boldsymbol{x}, t) = \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=1}^{N} \Gamma^{(n)}(\boldsymbol{x}, t) \right\}, \tag{1}$$

and $C(\boldsymbol{x}, t)$ can be estimated from experiments by determining the quantity in curly brackets for a large, but finite, value of N. Note that each of the values of $\Gamma^{(n)}$ must be determined at the same position and time, and that (in general) C depends explicitly on all components of $\boldsymbol{x}$, and on t.

However, in certain circumstances, C is also equal to the limit of an integral. Thus suppose $\Gamma(\boldsymbol{x}, t)$ is statistically stationary so that C is independent of t. According to ergodic theory, C then satisfies

$$C(\boldsymbol{x}) = \lim_{T \to \infty} \left\{ \frac{1}{T} \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} \Gamma(\boldsymbol{x}, s)\, \mathrm{d}s \right\}. \tag{2}$$

Similarly, when Γ is statistically homogeneous in one spatial coordinate, C is the limit of the integral of Γ over this coordinate. Although the rest of this note refers to time averages only, it will be obvious that all the conclusions apply equally when such spatial averages are used instead.

Denote the fluctuation in $\Gamma(\boldsymbol{x}, t)$ by $c(\boldsymbol{x}, t)$, so that

$$c(\boldsymbol{x}, t) = \Gamma(\boldsymbol{x}, t) - C(\boldsymbol{x}, t). \tag{3}$$

An overbar will stand for the averaging operation on the right-hand side of (1) (and of (2) when this is equivalent); thus $C(\boldsymbol{x}, t) = \bar{\Gamma}(\boldsymbol{x}, t)$. It is an exact consequence of (1) and (3) that

$$\bar{c}(\boldsymbol{x}, t) = 0 \tag{4}$$

because

$$\bar{c}(\boldsymbol{x}, t) = \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=1}^{\infty} [\Gamma^{(n)}(\boldsymbol{x}, t) - C(\boldsymbol{x}, t)] \right\}$$

$$= \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=1}^{\infty} \Gamma^{(n)}(\boldsymbol{x}, t) \right\} - C(\boldsymbol{x}, t),$$

which is identically zero.

Unfortunately, the random variables of interest in geophysical fluid dynamics are but rarely statistically steady or statistically homogeneous, and the cost of performing sufficient repetitions for stable estimates of C to be obtained using (1) is invariably prohibitive. It is therefore standard practice (see e.g. Dyer, 1973; Pasquill, 1974; Officer, 1976) to consider $C_T(\boldsymbol{x}, t)$ defined by

$$C_T(\boldsymbol{x}, t) = \frac{1}{T} \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} \Gamma(\boldsymbol{x}, s)\, \mathrm{d}s, \tag{5}$$

for some preselected finite value of T, and to regard $C_T(\boldsymbol{x}, t)$ as equivalent to, or an estimate of, $C(\boldsymbol{x}, t)$.

Associated with C_T is the corresponding fluctuation $c_T(\boldsymbol{x}, t)$ satisfying

$$c_T(\boldsymbol{x}, t) = \Gamma(\boldsymbol{x}, t) - C_T(\boldsymbol{x}, t). \tag{6}$$

During the preparation of a review article (Chatwin and Allen, 1985), it was noticed that it is assumed in many books and papers that $\langle c_T \rangle_T$ is identically zero, where the suffix $\langle \ \rangle_T$ denotes the averaging operation on the right-hand side of (5). That this assumption is wrong was noted in that article, and very briefly discussed, but there was insufficient space for an adequate treatment. The error is potentially important, so a fuller treatment is given here. The discussion here is in one sense an extension of arguments given by Lester (1972); he stated that the Reynolds averaging techniques hold for equations like (5) only if $\Gamma(\boldsymbol{x}, t)$ is a constant or a linear function of time; this is, of course, not usually the case in the regions of clear air turbulence that he was considering. Wyngaard (1973) also estimated averaging times of different parameters in the atmospheric surface layer; we believe, however, that the present account is justified by the potential practical importance of many non-stationary flows, an example of which is the recent increased interest in the dispersion of finite clouds of pollutant.

An alternative method of treatment commonly used in atmospheric and oceanic turbulent flows and associated dispersion problems is the use of a spectral approach (Dyer, 1973; Pasquill, 1974). However one main advantage of spectral analysis is usually thought to be the simplification in the governing equations (replacing derivatives by simple algebraic expressions etc.), and this simplification is lost when all Fourier components depend on $\boldsymbol{x}$ and t as occurs when conditions are not statistically stationary or homogeneous. Accordingly, various data processing techniques must be applied to try and produce quasi-stationary series for different parameters (Soulsby, 1980).

2. Some simple examples

We begin by discussing some simple theoretical examples which illustrate the points made above.

Suppose first that

$$\Gamma(\boldsymbol{x}, t) = \Gamma_0\{1 + \alpha t^2\}, \tag{7}$$

where Γ_0 is a constant and α is independent of t (but can depend on $\boldsymbol{x}$ and can be random). A straightforward application of (5) gives

$$C_T(\boldsymbol{x}, t) = \Gamma_0\{1 + \alpha t^2 + \tfrac{1}{12}\alpha T^2\} \tag{8}$$

so that, from (6),

$$c_T(\boldsymbol{x}, t) = -\tfrac{1}{12}\Gamma_0\, \alpha T^2 \tag{9}$$

Thus $\langle c_T \rangle_T$, the time average of c_T, also satisfies

$$\langle c_T \rangle_T(\boldsymbol{x}, t) = -\tfrac{1}{12}\Gamma_0\, \alpha T^2, \tag{10}$$

since c_T is itself independent of t. From the point of view of turbulence, a more realistic example is perhaps given by

$$\Gamma(\boldsymbol{x}, t) = \Gamma_0\{1 + \alpha \cos(\omega t - \phi)\}, \tag{11}$$

where ω and ϕ are further constants. Application of (5) now gives

$$C_T(\boldsymbol{x}, t) = \Gamma_0\left\{1 + \alpha \cos(\omega t - \phi)\left(\frac{\sin \frac{1}{2}\omega T}{\frac{1}{2}\omega T}\right)\right\}. \tag{12}$$

Hence, from (6),

$$c_T(\boldsymbol{x}, t) = \Gamma_0\, \alpha \cos(\omega t - \phi)\left\{1 - \left(\frac{\sin \frac{1}{2}\omega T}{\frac{1}{2}\omega T}\right)\right\}. \tag{13}$$

and so, after the further averaging operation,

$$\langle c_T \rangle_T(\boldsymbol{x}, t) = \Gamma_0\, \alpha \cos(\omega t - \phi)\left\{\left(\frac{\sin \frac{1}{2}\omega T}{\frac{1}{2}\omega T}\right) - \left(\frac{\sin \frac{1}{2}\omega T}{\frac{1}{2}\omega T}\right)^2\right\}. \tag{14}$$

Thus $\langle c_T \rangle_T$ is non-zero in both of these examples, and it can be readily confirmed that this is true for other simple cases. The reason why $\langle c_T \rangle_T$ is non-zero in general can be seen by applying the averaging operation directly to (6), obtaining

$$\langle c_T \rangle_T(\boldsymbol{x}, t) = \frac{1}{T}\int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} \{\Gamma(\boldsymbol{x}, s) - C_T(\boldsymbol{x}, s)\}\, ds$$

$$= C_T(\boldsymbol{x}, t) - \frac{1}{T}\int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} C_T(\boldsymbol{x}, s)\, ds. \tag{15}$$

From (5), $C_T(\boldsymbol{x}, s)$ depends on the values of $\Gamma(\boldsymbol{x}, s')$ only for times s' such that $s - \frac{1}{2}T < s' < s + \frac{1}{2}T$. Thus, on the right-hand side of (15), the value of $C_T(\boldsymbol{x}, t)$ depends on the values of $\Gamma(\boldsymbol{x}, s')$ only for times s' such that $t - \frac{1}{2}T <$

$s' < t + \frac{1}{2}T$, whereas the integral involves values of $C_T(\boldsymbol{x}, s)$ for all times s such that $t - \frac{1}{2}T < s < t + \frac{1}{2}T$ and hence on values of $\Gamma(\boldsymbol{x}, s')$ for all times s' such that $t - T < s' < t + T$. Equality of the two terms on the right-hand side of (15) therefore requires exceptional behaviour of $\Gamma(\boldsymbol{x}, s')$ for times s' such that $t - T < s' < t - \frac{1}{2}T$ and $t + \frac{1}{2}T < s' < t + T$. Granted that Γ is a random process, such equality will occur almost never (but note that the earlier argument does not depend on randomness).

Finally it is worth emphasizing here that the averaging operation in (2) is fundamentally different from that in (5) because:

(i) the operation in (2) can be applied only when Γ is statistically stationary;

(ii) the value of T in (2) can be made arbitrarily large.

These simple theoretical examples were chosen because we feel that they best illustrate the basic point we are making. However, we now present some geophysical applications which reinforce our views.

If the geophysical flow is stationary, then there will be a spectral gap and an average over a time T can be taken. A typical example in the marine environment would be if there were a relatively long steady phase during the tidal cycle where the flow is quasi-stationary (Soulsby, 1980) but not during the accelerating or decelerating phases when the flow is not steady. In the atmospheric environment, there are some instances where the flow is stationary, e.g., Leyi and Panofsky (1983) analysed experimental data from Boulder Tower and found a "gap" in the spectrum for the particular case where there was very stable air and light winds. Similarly, Sreenivasan et al. (1978) derived a detailed accuracy analysis for moments of velocity and scalar fluctuations of data measured in the marine surface layer in Bass Strait, Australia. They stress that it is the observed stationarity in their data set that enables a meaningful definition of integral scales and high-order moments to be made.

However, many geophysical flows are not stationary; in estuaries there is often no long period of steady flow during the tidal cycle. Similarly, Caughey et al. (1979) reported that the stable atmospheric boundary layer observed in Minnesota shortly after the evening transition was "far from stationary". In both these cases, the types of error which are discussed in this note will be introduced.

3. Discussion

The demonstration that a basic mathematical error exists in many authoritative accounts of the fundamental equations of turbulence and turbulent diffusion is not the main reason for writing this note. Rather the justification is that the error has important practical implications which can be illustrated by considering the case when Γ is the concentration of a scalar satisfying the equation

$$\frac{\partial \Gamma}{\partial t} + \nabla \cdot (\mathbf{Y}\Gamma) = \kappa \nabla^2 \Gamma, \tag{16}$$

where $\mathbf{Y} = \mathbf{Y}(\boldsymbol{x}, t)$ is the random velocity field and κ is the molecular diffusivity. Let $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{x}, t)$ and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ be the ensemble average and fluctuation of $\mathbf{Y}$, defined by equations analogous to (1) and (3). The standard Reynolds procedure then yields the exact equation

$$\frac{\partial C}{\partial t} + \nabla \cdot (\boldsymbol{U}C) + \nabla \cdot (\overline{\boldsymbol{u}c}) = \kappa \nabla^2 C. \tag{17}$$

In deriving (17) of course, no assumption about the statistical steadiness or homogeneity of $\mathbf{Y}$ or Γ is necessary.

Now apply the time averaging operation in (5) to eq. (16), obtaining (since $\langle\ \rangle_T$ and spatial differentiation obviously commute)

$$\left\langle \frac{\partial \Gamma}{\partial t} \right\rangle_T + \nabla \cdot \{\langle \mathbf{Y}\Gamma \rangle_T\} = \kappa \nabla^2 C_T. \tag{18}$$

Although it is *not* true that $\langle \partial C_T/\partial t \rangle_T = (\partial C_T/\partial t)$, it can be shown that $\langle \partial \Gamma/\partial t \rangle_T = (\partial C_T/\partial t)$ since

$$\left\langle \frac{\partial \Gamma}{\partial t} \right\rangle_T = \frac{1}{T} \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} \frac{\partial \Gamma}{\partial s} \, ds = \frac{1}{T} \{\Gamma(\boldsymbol{x}, t + \tfrac{1}{2}T) - \Gamma(\boldsymbol{x}, t - \tfrac{1}{2}T)\} = \frac{\partial C_T}{\partial t},$$

using (5). In the term $\langle \mathbf{Y}\Gamma \rangle_T$ it is normal to write $\Gamma = C_T + c_T$ and $\mathbf{Y} = \boldsymbol{U}_T + \boldsymbol{u}_T$ (where $\boldsymbol{U}_T$ and $\boldsymbol{u}_T$ are defined by the obvious analogues of (5) and (6)), obtaining

$$\langle \mathbf{Y}\Gamma \rangle_T = \langle \boldsymbol{U}_T C_T \rangle_T + \langle \boldsymbol{U}_T c_T \rangle_T + \langle \boldsymbol{u}_T C_T \rangle_T + \langle \boldsymbol{u}_T c_T \rangle_T. \tag{19}$$

The work in the previous section makes it obvious that neither $\langle \boldsymbol{U}_T c_T \rangle_T$ nor $\langle \boldsymbol{u}_T C_T \rangle_T$ is identically zero, and nor is $\langle U_T C_T \rangle_T$ equal to $(U_T C_T)$. The "standard" equation, *viz.*

$$\frac{\partial C_T}{\partial t} + \nabla\cdot(U_T C_T) + \nabla\cdot\{\langle \boldsymbol{u}_T c_T \rangle_T\} = \kappa\nabla^2 C_T, \quad (20)$$

which is the obvious analogue of (17), is therefore *not* a correct equation, nor is it clear that the errors involved in using it are small. For it is difficult to assert, or even believe, that the magnitudes of $\langle U_T c_T \rangle_T$ and $\langle \boldsymbol{u}_T C_T \rangle_T$ are much less than that of $\langle \boldsymbol{u}_T c_T \rangle_T$.

Therefore, the only justifications that can be provided for (20) are its superficial analogue to (17), and its consequent status as a manageable model equation. Two steps seem desirable in subsequent investigations. These are:

(i) careful measurements designed to quantify the various terms in (19);

(ii) a search for other model equations that may be as useful as (20).

In investigating (i) it will be necessary to take account also of the dependence of the statistical properties of C_T and c_T on T; such dependence is inevitable as shown in (9) and (13).

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