Dynamic Analysis and Synchronization of Two Uncoupled Chaotic Hindmarsh-Rose Neurons

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Abtract

In this paper, the problem of synchronization of two uncoupled chaotic Hindmarsh-Rose (HR) neurons is addressed. First, the dynamic behaviors of a single HR neuron stimulated by an external applied current are studied. By using the concept of fast/slow dynamic analysis, the bursting mechanism of the HR neuron is investigated. Considering the applied current as a bifurcation parameter, the chaotic behavior as well as other dynamic behaviors is reported. Second, the author formulated a method for synchronization of two uncoupled chaotic HR neurons. By using a Lyapunov function, a nonlinear feedback control law is designed that guarantees that the two uncoupled neurons are globally asymptotically synchronized. Finally, in order to verify the effectiveness of the proposed method, numerical simulations are carried out, the results of which are provided herein.

Keywords: Chaos, Bursting mechanism, Hindmarsh-Rose neuron, Lyapunov function, Synchronization

I. Introduction

Neurons are the basic building blocks of the nervous system. In order to understand the dynamic behaviors of individual neurons and further comprehend the biological information processing of neural networks, a variety of mathematical neuronal models have been proposed [1-4]. Among them, the Hodgkin-Huxley model [1] is the most important one. This model gives an explanation on the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon. A main conceptual drawback of the Hodgkin-Huxley model is that its numerical complexity (e.g., solve a large number of nonlinear differential equations). In addition, some important dynamic behaviors that are observed in real biological neurons such as bursting, chaos, etc. cannot be described by using the original Hodgkin-Huxley equations. The HR model [4], a simplification of the Hodgkin-Huxley model, can provide very realistic descriptions on a number of biological features such as rapid firing, bursting, and chaos. Therefore, the HR model is getting more attention in the study of many features of brain activity. Individual neurons can exhibit chaotic behavior, whereas ensembles of different neurons might synchronize in order to process biological information or to produce regular, rhythmical activity [5]. Therefore, the study of synchronization processes for populations of interacting neurons is basic to the understanding of some key issues in neuroscience. Recently, many researchers have focused on the synchronization of two chaotic neurons, which is one of the fundamental issues in understanding the neuronal behaviors in networks. The two neurons synchronization can be classified into two groups, namely, self-synchronization and controlled synchronization. The self-synchronization can be achieved when the intensity of an external noise exceeds a critical value [6, 7]. Other results have also shown that self-synchronization occurs when the coupling coefficient is strong enough [8, 9]. Alternatively, in the case that the conditions for self-synchronization do not satisfy, various modern control methods have been proposed to synchronize two coupled chaotic neurons [9-15]. In [10], two different adaptive control laws were proposed to synchronize two coupled chaotic HR neurons under the assumption that the structure of two neuron with unknown parameters is identical. A sufficient condition for self-synchronization of two coupled chaotic HR neurons that related to the coupling coefficient was clearly shown in [11]. In addition, another nonlinear control law was proposed to achieve the synchronization of two coupled chaotic neurons [11].

In the present study, we first investigate the dynamic behaviors of a single HR neuron under external electrical stimulation. Then the synchronization of two uncoupled HR neurons is studied. By using a Lyapunov function, a nonlinear feedback control law is proposed to guarantee that

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two neurons are globally asymptotically synchronized.

The layout of this paper is organized as follows: Section 2 describes the dynamics of a single HR neuron. In Section 3, the synchronization problem is studied. The details of the design of the nonlinear feedback control law for synchronization of two uncoupled chaotic HR neurons are provided. Finally, conclusions are drawn in Section 4.

2. Dynamics of a single HR neuron

2.1 Model description

The HR neuron model [4], a modification of the Hodgkin-Huxley model, is a genetic model of the membrane potential which enables to simulate spiking, bursting and chaos phenomena in real biological neurons. This model is described as follows:

$$\dot{x} = y - ax^3 + bx^2 - z + I\,, \tag{1}$$

$$\dot{y} = c - dx^2 - y\,, \tag{2}$$

$$\dot{z} = \varepsilon\left\{s\left(x - x_e\right) - z\right\}, \tag{3}$$

where x, y, and z represent the membrane potential, the recovery variable associated with the fast current of Na^+ or K^+ ions, and the adaptation current associated with the slow current of, for instance Ca^+ ions, respectively. I is the applied current that mimics the membrane input current in biological neurons, and a, b, c, d, ε, s, and x_e are the constant parameters. These parameters are set as: $a = 1.0$, $b = 3.0$, $c = 1.0$, $d = 5.0$, $\varepsilon = 0.006$, $s = 4.0$, and $x_e = -1.56$. By varying the amplitude of the applied current I, various firing patterns such as tonic spiking, regular bursting, chaotic bursting, etc. can be observed.

2.2 Bursting mechanism

To understand the bursting mechanism of the HR neuron, the fast/slow dynamic analysis method [16] is used. The idea of this method is to divide the neuronal system into two subsystems according to the different of time scales, in which the fast subsystem is responsible for the generation of spikes while the slow subsystem contributes to the variation of burst duration. For the HR neuron model, the membrane potential x and the recovery variable y are considered as fast variables, while the adaptation slow current z is considered as a slow variable.

$$\left.\begin{aligned} \dot{x} &= y - ax^3 + bx^2 - z + I \\ \dot{y} &= c - dx^2 - y \end{aligned}\right\} \text{Fast subsystem} \tag{4}$$

$$\dot{z} = \varepsilon\left\{s\left(x - x_e\right) - z\right\} \qquad \text{Slow subsystem} \tag{5}$$

Consider the fast subsystem (Eq. (4)) by setting $\varepsilon = 0$ and considering z as a bifurcation parameter. The bifurcation diagram of the membrane potential x versus the slow variable z is depicted in Fig.1. Here, the thick solid curve presents stable equilibria while the dotted curve presents the unstable ones. The maxima and minima of the stable limit cycle is indicated by the thin curve.

When z is small, the fast subsystem has a unique stable equilibrium corresponding to the resting state. As z is increased, the equilibrium loses its stability via a supercritical Hopf bifurcation and the fast subsystem oscillates periodically. In order to generate bursts, the fast subsystem must exhibit bistability for a certain range of the slow variable z. As indicated in Fig.1, in the bistable regime the fast subsystem exhibits two steady states: the lower steady state corresponds to the resting state and the upper one corresponds to a periodic spiking. The transition from the resting state to the periodic spiking is caused by a saddle-node bifurcation (the stable node and the saddle point of Eq. (4) approach each other as z is decreased). In contrast, the transition from the periodic spiking to the resting state is caused by a homoclinic bifurcation (the limit cycle becomes a homoclinic orbit to the saddle point as z changes).

Now consider the slow subsystem in Eq. (5). It is noted that the direction of change of the slow variable z plays a crucial role in the generation of burst.

Fig.2 shows the magnification of the bistable regime in Fig.1 with the z-nullcline of the slow subsystem ($\dot{z} = 0$), along with the phase portrait of the bursting oscillation.

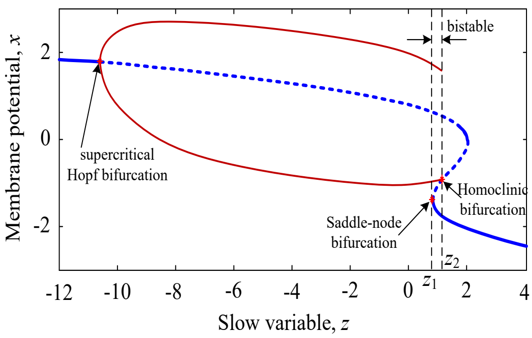


Fig. 1. Bifurcation diagram of the fast subsystem described in Eq. (4).

The bursting mechanism is explained as follows. When x is in the lower steady state, it can be seen from Fig.2 that $\dot{z}$ is negative, therefore z is depleted slowly. As z approaches the z-nullcline in the left hand side, $\dot{z}$ changes its sign become positive and x is transited to the upper state via a saddle-node bifurcation. Then the fast subsystem will generate a periodic spiking. While x is in the upper state, z is increased slowly until it approaches the z-nullcline in the right hand side, therefore $\dot{z}$ becomes negative which results in the transition of x from the periodic spiking to the resting state via a homoclinic bifurcation. Following the classification proposed in [17], this type of bursting mechanism is called as a fold/homoclinic bursting. The corresponding time courses of x and z are shown in Fig.3.

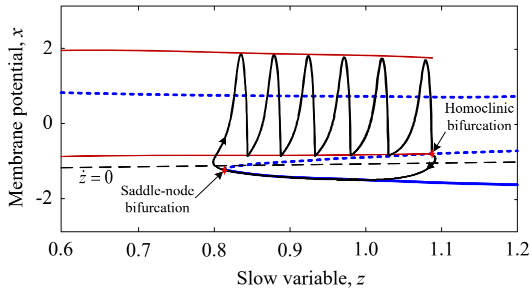


Fig. 2. Fold/homoclinic bursting mechanism in the HR neuron model.

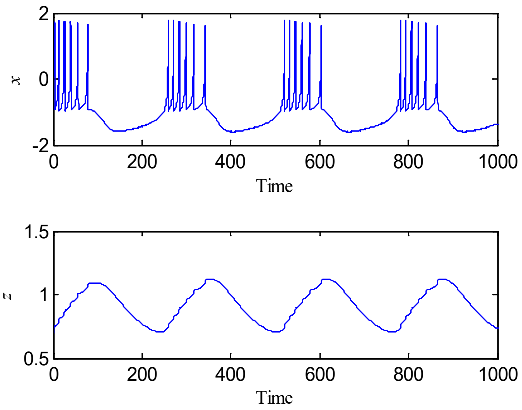


Fig. 3. Time responses (x – membrane potential, z – recovery variable) of bursting behavior in the HR neuron model.

2.3 Bifurcation diagram

In order to convey more information about dynamic behaviors of a single HR neuron under varying amplitude of the applied current, we investigate the bifurcation of the interspike intervals (ISIs) as a function of the applied current I, as shown in Fig.4. Fig.4 reveals that for small values of the applied current $I < 1.15$, the neuron is in the quiescent state. When the applied current is increased beyond $I = 1.15$, the period-one firing patterns appear and this behavior is maintained for the current up to $I \approx 1.41$. The period-two, -three, and -four firing patterns can be found in the regions $1.41 \le I < 1.98$, $1.98 \le I < 2.49$, and $2.49 \le I < 2.75$, respectively. It is obvious from Fig.4 that the HR neuron exhibits chaotic bursting for the values of the applied current in the region $2.75 \le I < 3.25$. After that, the HR neuron exhibits again the period-two and -one firing patterns with $3.25 \le I < 3.32$ and $I \ge 3.32$, respectively. The time course of the membrane potential that shows the chaotic behavior of the HR neuron for $I = 3.1$ is illustrated in Fig.5.

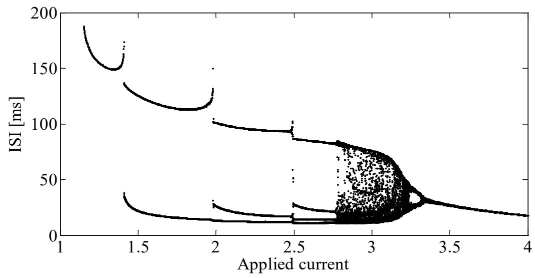
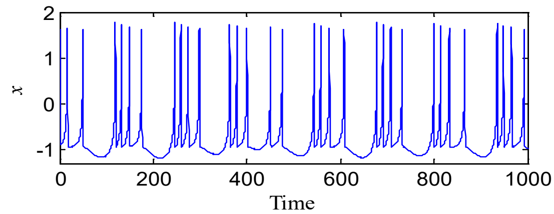
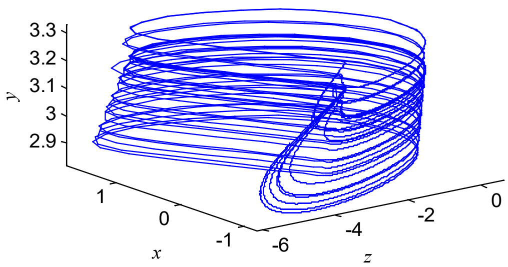


Fig. 4. Bifurcation diagram of interspike intervals vs. the applied current of a single HR neuron model.



(a)



(b)

Fig. 5. Chaotic behavior of the HR neuron model: (a) membrane potential, (b) x-y-z phase portrait.

3. Synchronization of two uncoupled chaotic HR neuron

Based on Eqs. (1)-(3), the two uncoupled HR neurons can be described as a master-slave system as follows.

$$\begin{cases} \dot{x}_1 = y_1 - a x_1^3 + b x_1^2 - z_1 + I, \\ \dot{y}_1 = c - d x_1^2 - y_1, \\ \dot{z}_1 = \varepsilon \left\{ s\left(x_1 - x_e\right) - z_1 \right\}, \end{cases} \tag{6}$$

$$\begin{cases} \dot{x}_2 = y_2 - ax_2^3 + bx_2^2 - z_2 + I + u(t), \\ \dot{y}_2 = c - dx_2^2 - y_2, \\ \dot{z}_2 = \varepsilon\left\{s\left(x_2 - x_e\right) - z_2\right\}, \end{cases} \tag{7}$$

where x_i, y_i, z_i (i = 1, 2) are the state variables and $u(t)$ is the control signal. Let $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, and $e_3 = z_2 - z_1$ be the error signals between the states of the system described in Eqs. (6)-(7).

Definition: *The two HR neurons described in Eqs.* (6)-(7) *are said to be globally asymptotically synchronized if, for all initial conditions* ($x_1(0)$, $y_1(0)$, $z_1(0)$, $x_2(0)$, $y_2(0)$, $z_2(0)$), $\lim_{t\to\infty} e_i(t) = 0$ (i = 1, 2, 3).

From Eqs. (6)-(7), the error dynamics can be obtained as follows.

$$\dot{e}_1 = \left\{ b(x_1 + x_2) - a(x_1^2 + x_1x_2 + x_2^2) \right\} e_1 + e_2 - e_3 + u, \tag{8}$$

$$\dot{e}_2 = -d(x_2 + x_1)e_1 - e_2, \tag{9}$$

$$\dot{e}_3 = \varepsilon\left(se_1 - e_3\right). \tag{10}$$

Let define the state-dependent terms in Eq. (8) and Eq. (9) as follows:

$$h_1(x_1, x_2, e_1) = b(x_1 + x_2) - a(x_1^2 + x_1x_2 + x_2^2), \tag{11}$$

$$h_2(x_1, x_2) = -d(x_2 + x_1). \tag{12}$$

Then Eqs. (8)-(10) are reduced to

$$\dot{e}_1 = h_1(x_1, x_2, e_1)e_1 + e_2 - e_3 + u, \tag{13}$$

$$\dot{e}_2 = h_2(x_1, x_2)e_1 - e_2, \tag{14}$$

$$\dot{e}_3 = \varepsilon\left(se_1 - e_3\right). \tag{15}$$

The synchronization problem is now replaced by finding a suitable control law u such that the error dynamics described in Eqs. (13)-(15) are globally asymptotically stable at the origin.

Chose the Lyapunov function as

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2\right) \geq 0. \tag{16}$$

The derivative of V along Eqs. (13)-(15) are given by

$$\begin{aligned} \dot{V} &= h_1(x_1, x_2, e_1)e_1 + e_1e_2 - e_1e_3 + ue_1 \\ &+ h_2(x_1, x_2)e_1e_2 - e_2^2 + \varepsilon se_1e_3 - \varepsilon e_3^2 \end{aligned}. \tag{17}$$

Then let chose

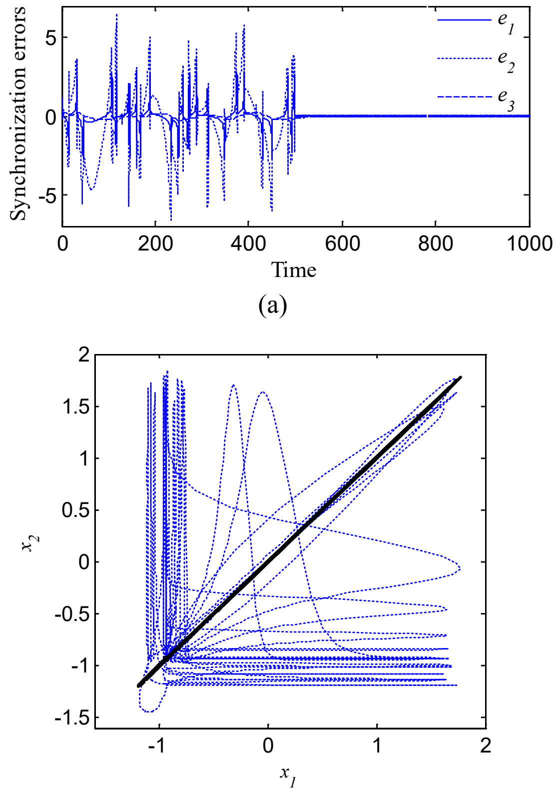
$$\begin{aligned} u(t) = &-h(x_1, x_2, e_1)e_1 - ke_1 - \left\{h_2(x_1, x_2) + 1\right\}e_2 \\ &+ (1 - \varepsilon s)e_3, \end{aligned} \tag{18}$$

where k is the positive constant. Substituting Eq. (18) into Eq. (17), we obtain

$$\dot{V} = -ke_1^2 - e_2^2 - \varepsilon e_3^2 \leq 0. \tag{19}$$

According to Lyapunov theory [18], the global asymptotic stability at the origin of Eqs. (13)-(15) holds, which is equivalent to the fact that two uncoupled HR neurons described in Eqs. (6)-(7) are globally asymptotically synchronized.

To demonstrate the effectiveness of the proposed control law, numerical simulations are performed. Here, we set I = 3.1 such that individual neurons exhibit chaotic behavior (see Fig.5). The initial conditions of the master and the slave neurons were chosen as ($x_1(0)$, $y_1(0)$, $z_1(0)$) = (0.3, 0.3, 3.0) and ($x_2(0)$, $y_2(0)$, $z_2(0)$) = (-0.3, 0.4, 3.2), respectively. The positive constant k in Eq. (18) is chosen as k = 0.2. The total simulation time is set as t = 1000. The control law in Eq. (18) is applied at time t = 500. As shown in Fig.6(a), the synchronization errors between two neurons, $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, and $e_3 = z_2 - z_1$, converge asymptotically to zero within a finite period of time after applying the control law. The phase portraits x_1–x_2 before (dashed line) and after (solid line) application of the control law are plotted in Fig.6(b).



(a)

(b)

Fig. 6. Synchronized dynamics of the uncoupled HR neurons with the proposed control law in Eq. (18) at t = 500: (a) synchronization errors, (b) x_1-x_2 phase portrait.

4. Conclusion

In this paper, we first studied dynamic behaviors of a single HR neuron. The bursting mechanism was analytically investigated by using the fast/slow dynamic analysis method. By varying the amplitude of the applied current, a full range of dynamic behaviors was reported. Second, we studied the synchronization of two uncoupled chaotic HR neurons. On that basis, we formulated a nonlinear Lyapunov function-based control law that guarantees the synchronization between two uncoupled chaotic HR neurons. The effectiveness of the proposed control law is verified through numerical simulations.

The obtained results of this study are promising to contribute a method for deep brain stimulation area as well as for synchronization of two practical chaotic systems.

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