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# Numerical Evaluation of the Forced Vibration Response of a Timoshenko Beam Subjected to a Moving Force Using the Modal Analysis Approach

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**Abstract**

*The dynamic response of a structure subjected to moving loads is an interesting and meaningful research subject in various engineering fields such as bridges, roadways, railways and aircrafts. A vast number of studies has employed the vibration model of the Euler-Bernoulli beam to calculate the dynamic response of a beam structure under the action of moving loads. The present paper deals with the dynamic response of an uniform Timoshenko beam subjected to a moving force. Properties of the natural frequencies and modes of the Timoshenko beam are discussed. Using the modal method, a procedure for calculating the forced vibrations of the Timoshenko beam has been proposed. It is shown that the solution of the forced vibration for the transverse displacement and the rotation of the cross section of the beam can be expressed in form of a sum of two infinite series. The numerical simulation result shows that the speed change of the moving force has little effect on the beam deflection, but its magnitude change greatly affects the beam deflection.*

Keywords: Timoshenko beam, moving force, vibrations, modal analysis.

## 1. Introduction

The dynamic response of a structure subjected to moving forces is an interesting and meaningful research subject in various engineering fields such as dynamics of bridges, roadways, railways, and runways as well as missiles and aircrafts. Different types of structures and girders such beams, plates, shells, and frames have been considered [1-3].

It would be interesting to study the problem of the dynamic response of Timoshenko beams subjected to moving forces. The governing equations for a Timoshenko beam of length $l$ can be written in the following forms [1-3].

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} = k^* G\frac{\partial}{\partial x}\left[A(x)\left(\frac{\partial w}{\partial x}-\psi\right)\right]+p(x,t) \quad (1)$$

$$\rho I(x)\frac{\partial^2 \psi}{\partial t^2} = k^* GA(x)\left(\frac{\partial w}{\partial x}-\psi\right)+E\frac{\partial}{\partial x}\left[I(x)\frac{\partial \psi}{\partial x}\right] \quad (2)$$

In the (1) and (2) *w(x, t)* is the transverse displacement, $\psi(x,t)$ is the rotation of the cross section due to bending, $p(x,t)$ is the external transverse force, $\tau(x,t)$ is the external bending moment, *A(x)* is the cross -section area, *I*(x) is the area moment of inertia, $E$ is Young's modulus, $G$ is the shear modulus, $k^*$ is the shear correction factor, and $\rho$ is the mass density.

Due to the complexity of equations (1) and (2), it is common to use approximation methods such as the Ritz method, the finite element method, to calculate the vibrations of the Timoshenko beam [1-3]. The calculating the free oscillations of the Timoshenko beam has been studied a lot. In contrast, the study of forced vibration of the Timoshenko beam is also an issue that many scientists are interested in researching.

The published documents for calculating the forced vibrations of Timoshenko beams by analytical methods are relatively few. The classical solution for the transverse displacement and the rotation of the cross section of a Timoshenko beam is expressed in a form of two infinite series, one of which represents the force vibrations and the other one free vibrations of the beam. By calculating the dynamic response of a Timoshenko beam to a moving force, Sniady [4] has been shown that one of the series can be presented in closed form. In [5], Majkut has proposed using the dynamic Green function for calculating forced vibrations of Timoshenko beam. Khoraskani et al. [6, 7] investigated the dynamic response of Timoshenko beam under a moving mass by using modal method. Kim et al. [8] calculated forced vibration of a Timoshenko beam subjected to moving loads in the steady-state using the analytical method. Zhdan [9] has investigated the dynamic behavior of Euler- Bernoulli

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and Timoshenko beams subjected to a moving concentrated load with two special form series. From the mathematical point of view, the calculation methods used in [4-7, 9] are not directly related to the modal method, but in the work [8], the modal method is applied to calculate the forced vibrations of the Timoshenko beam.

In the present study, we use the principles of mechanics to establish the formulas for calculating the forced vibration of the Timoshenko beam. The evaluation of the transient vibration for a simply supported Timoshenko beam subjected to a force moving with a constant velocity will then be presented in detail.

## 2. Analyzing Free Vibration of a Uniform Timoshenko Beam with the Constant Cross-Section

### *2.1. Natural Frequencies and Mode Shapes*

If the external force and the external bending moment do not exist and if $I(x)$ and $A(x)$ are assumed to be constant, (1) and (2) can be simplified so that free vibration is governed by

$$\rho A\frac{\partial^2 w}{\partial t^2}-k^*GA\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}-\psi\right)=0 \tag{3}$$

$$\rho I\frac{\partial^2\psi}{\partial t^2}-k^*GA\left(\frac{\partial w}{\partial x}-\psi\right)-EI\frac{\partial^2\psi}{\partial x^2}=0 \tag{4}$$

Equation (3) and (4) can be rewritten in matrix form as

$$\mathbf{M}\frac{\partial^2\mathbf{u}(x,t)}{\partial t^2}+\mathbf{K}\mathbf{u}(x,t)=\mathbf{0} \tag{5}$$

where

$$\mathbf{u}(x,t)=\begin{bmatrix} w(x,t)\\ \psi(x,t)\end{bmatrix}$$
$$\mathbf{M}=\begin{bmatrix}\rho A & 0\\ 0 & \rho I\end{bmatrix} \tag{6}$$
$$\mathbf{K}=\begin{bmatrix}-k^*GA\dfrac{\partial^2}{\partial x^2} & k^*GA\dfrac{\partial}{\partial x}\\ -k^*GA\dfrac{\partial}{\partial x} & k^*GA-EI\dfrac{\partial^2}{\partial x^2}\end{bmatrix}$$

The solution of (3) and (4) is assumed to be in the following form [7, 8]

$$w(x,t)=W(x)T(t),\quad \psi(x,t)=\Psi(x)T(t) \tag{7}$$

Substituting (7) into (3), (4) and performing some mathematical transformations, we get the following equations:

$$\ddot{T}(t)+\omega^2T(t)=0 \tag{8}$$

$$\begin{aligned}k^*GAW''(x)-k^*GA\Psi'(x)\\ +\omega^2\rho AW(x)=0\end{aligned} \tag{9}$$

$$EI\Psi''(x)+k^*GAW'(x)+(\omega^2\rho I-k^*GA)\Psi(x)=0 \tag{10}$$

From (8) we obtain

$$T(t)=e^{i\omega t} \tag{11}$$

where $\omega$ is the eigenfrequency of a Timoshenko beam.

We assume that the solutions of (9) and (10) are in the following form:

$$W(x)=ae^{sx},\ \Psi(x)=be^{sx} \tag{12}$$

where $s$ denotes the wavenumber. Substitution of (12) into (9) and (10) yields the algebraic equations as follows:

$$\begin{bmatrix}-(k^*GAs^2+\rho A\omega^2) & k^*GAs\\ -k^*GAs & k^*GA-EIs^2-\rho I\omega^2\end{bmatrix}\begin{bmatrix}\mathrm{a}\\ b\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{13}$$

For the existence of nontrivial solutions of linear algebra equations according to (13), the determinant of the coefficient matrix in (13) must vanish at certain values of $s$, that is, at eigenvalues. From this condition, a dispersion equation is obtained as follows:

$$\begin{aligned}s^4+\omega^2\left(\frac{\rho I}{EI}+\frac{\rho A}{k^*GA}\right)s^2\\ +\omega^2\left(\frac{\rho I}{EI}.\frac{\rho A}{k^*GA}\omega^2-\frac{\rho A}{EI}\right)=0\end{aligned} \tag{14}$$

In order to obtain the four eigenvalues, the above quartic equation (14) can be reduced to a quadratic equation by replacing $s^2$ with $\xi$

$$\begin{aligned}\xi^2+\omega^2\left(\frac{\rho I}{EI}+\frac{\rho A}{k^*GA}\right)\xi\\ +\omega^2\left(\frac{\rho I}{EI}.\frac{\rho A}{k^*GA}\omega^2-\frac{\rho A}{EI}\right)=0\end{aligned} \tag{15}$$

By solving the quadratic equation (15), we can obtain four eigenvalues as follows:

$$s_1=i\beta,\ s_2=-i\beta \tag{16}$$

$$s_3=-s_4=\alpha\ \ (if\ \omega\le\omega_c) \tag{17}$$

$$s_3=-s_4=i\alpha'\,(if\ \omega\ge\omega_c) \tag{18}$$

where $\beta,\alpha,$ and $\alpha'$ are always real numbers and $\omega_c$ is the cutoff frequency defined by

$$\omega_c=\sqrt{\frac{k^*GA}{\rho I}} \tag{19}$$

$$\beta = \frac{1}{\sqrt{2}}\sqrt{\begin{array}{l}(\frac{\rho I}{EI}+\frac{\rho A}{k*GA})\omega^2 \\ +\sqrt{(\frac{\rho I}{EI}-\frac{\rho A}{k*GA})^2\omega^4+4\frac{\rho A}{EI}\omega^2}\end{array}} \tag{20}$$

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{\begin{array}{l}\sqrt{(\frac{\rho I}{EI}-\frac{\rho A}{k*GA})^2\omega^4+4\frac{\rho A}{EI}\omega^2} \\ -(\frac{\rho I}{EI}+\frac{\rho A}{k*GA})\omega^2\end{array}} \tag{21}$$

$$\alpha' = \frac{1}{\sqrt{2}}\sqrt{\begin{array}{l}(\frac{\rho I}{EI}+\frac{\rho A}{k*GA})\omega^2 \\ -\sqrt{(\frac{\rho I}{EI}-\frac{\rho A}{k*GA})^2\omega^4+4\frac{\rho A}{EI}\omega^2}\end{array}} \tag{22}$$

By using the four eigenvalues given by (16), (17) and (18), the general solutions of (9) and (10) can be written as follows [8]:

1) When $0 < \omega \le \omega_c$

$$\begin{aligned}\mathrm{W}(x) = C_1\cosh\alpha x + C_2\sinh\alpha x \\ + C_3\cos\beta x + C_4\sin\beta x\end{aligned} \tag{23}$$

$$\begin{aligned}\Psi(x) = C_1\,g_\alpha\sinh\alpha x + C_2\,g_\alpha\cosh\alpha x \\ - C_3\,g_\beta\sin\beta x + C_4\,g_\beta\cos\beta x\end{aligned} \tag{24}$$

2) When $\omega \ge \omega_c$

$$\mathrm{W}(x) = C_1\cos\alpha' x + C_2\sin\alpha' x + C_3\cos\beta x + C_4\sin\beta x \tag{25}$$

$$\begin{aligned}\Psi(x) = -C_1\,g_{\alpha'}\sin\alpha' x + C_2\,g_{\alpha'}\cos\alpha' x \\ - C_3\,g_\beta\sin\beta x + C_4\,g_\beta\cos\beta x\end{aligned} \tag{26}$$

where

$$g_\alpha = \frac{1}{\alpha}(\alpha^2 + \frac{\omega^2\rho A}{k^*GA}) \tag{27}$$

$$g_\beta = \frac{1}{\beta}(\beta^2 - \frac{\omega^2\rho A}{k^*GA}) \tag{28}$$

$$g_{\alpha'} = \frac{1}{\alpha'}(\alpha'^2 - \frac{\omega^2\rho A}{k^*GA}) \tag{29}$$

To obtain the natural frequencies in analytical closed forms for specific boundary conditions, we considered three frequency ranges separately:

(a) $0 < \omega < \omega_c$, (b) $\omega > \omega_c$, and (c) $\omega = \omega_c$.

### *2.2. Orthogonality of Mode Shapes*

For the modal analysis of the forced vibration of Timoshenko beam we must derive the orthogonality properties of the mode shapers of the beam. The equations (9) and (10) can be written in the following matrix form:

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{U}(x) = \mathbf{0} \tag{30}$$

where

$$\mathbf{U}(x) = \begin{bmatrix} W(x) \\ \Psi(x)\end{bmatrix} \tag{31}$$

From (30) we get the $j^{th}$ and $k^{th}$ sets of natural frequencies and mode shapes which must satisfy the following two equations separately as follows:

$$\mathbf{KU}_j(x) = \omega_j^2\mathbf{MU}_j(x) \tag{32}$$

$$\mathbf{KU}_k(x) = \omega_k^2\mathbf{MU}_k(x) \tag{33}$$

From (32) and (33), we obtain

$$\mathbf{U}_k^T\mathbf{KU}_j = \omega_j^2\mathbf{U}_k^T\mathbf{MU}_j \tag{34}$$

$$\mathbf{U}_j^T\mathbf{KU}_k(x) = \omega_k^2\mathbf{U}_j^T\mathbf{MU}_k \tag{35}$$

Because $\mathbf{M}$ is a symmetric matrix, from (35) we have:

$$\begin{aligned}(\mathbf{U}_j^T\mathbf{KU}_k)^T &= \omega_k^2(\mathbf{U}_j^T\mathbf{MU}_k)^T \\ \Rightarrow \mathbf{U}_k^T\mathbf{K}^T\mathbf{U}_j &= \omega_k^2\mathbf{U}_k^T\mathbf{MU}_j\end{aligned} \tag{36}$$

From (34) and (36), we obtain

$$\begin{aligned}\int_0^l \mathbf{U}_k^T\mathbf{KU}_j dx - \int_0^l \mathbf{U}_k^T\mathbf{K}^T\mathbf{U}_j dx \\ = (\omega_j^2 - \omega_k^2)\int_0^l \mathbf{U}_k^T\mathbf{MU}_j dx\end{aligned} \tag{37}$$

Taking simple mathematical transformations, the right-hand side of (37) can be rewritten as

$$\begin{aligned}&(\omega_j^2 - \omega_k^2)\int_0^l \mathbf{U}_k^T\mathbf{MU}_j dx \\ &= (\omega_j^2 - \omega_k^2)(\rho A\int_0^l W_k W_j dx + \rho I\int_0^l \Psi_k\Psi_j dx)\end{aligned} \tag{38}$$

From (38), the orthogonality property of the mode shapes with respect to the matrix $\mathbf{M}$ can be derived as follow [8]:

$$\int_0^l \mathbf{U}_k^T\mathbf{MU}_j dx = \begin{bmatrix} 0 \quad (if\,\omega_k \ne \omega_j) \\ \ne 0 \quad (if\,\omega_k = \omega_j)\end{bmatrix} \tag{39}$$

Using (33) and (38), we can derive the orthogonality property of mode shapes with respect to the matrix $\mathbf{K}$ as follows:

$$\int_0^l \mathbf{U}_k^T\mathbf{KU}_j dx = \begin{bmatrix} 0 \quad (if\,\omega_k \ne \omega_j) \\ \ne 0 \quad (if\,\omega_k = \omega_j)\end{bmatrix} \tag{40}$$

From the boundary conditions of the simply supported Timoshenko beam, the mode shapes have the following form

$$W_k(x) = \sin\frac{\pi k}{l}x\,,\ \Psi_k(x) = g_{\beta(k)}\cos\frac{k\pi x}{l} \tag{41}$$

where

$$g_{\beta(k)} = (\frac{k\pi}{l} - \frac{\omega_k^2 l}{k^* GAk\pi}) \tag{42}$$

It can be seen that the mode shapes satisfy the condition according to (39).

## 3. Forced Vibration Analysis of Timoshenko Beam Using Modal Analysis Method

### *3.1 The Formula for Determining the Solution*

The governing equations for a uniform Timoshenko beam with constant cross-section can be written in the following form

$$\rho A\frac{\partial^2 w}{\partial t^2} - k^* GA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) = p(x,t) \tag{43}$$

$$\rho I\frac{\partial^2 \psi}{\partial t^2} - k^* G\,A\left(\frac{\partial w}{\partial x} - \psi\right) - EI\frac{\partial^2 \psi}{\partial x^2} = \tau(x,t) \tag{44}$$

Equation (43) and (44) can be written in matrix form as:

$$\mathbf{M}\frac{\partial^2 \mathbf{u}(x,t)}{\partial t^2} + \mathbf{K}\mathbf{u}(x,t) = \mathbf{f}(x,t) \tag{45}$$

where

$$\mathbf{M} = \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix},\quad \mathbf{K} = \begin{bmatrix} -k^* GA\dfrac{\partial^2}{\partial x^2} & k^* GA\dfrac{\partial}{\partial x} \\ -k^* GA\dfrac{\partial}{\partial x} & k^* GA - EI\dfrac{\partial^2}{\partial x^2} \end{bmatrix} \tag{46}$$

$$\mathbf{u}(x,t) = \begin{bmatrix} w(x,t) \\ \psi(x,t) \end{bmatrix},\quad \mathbf{f} = \begin{bmatrix} p(x,t) \\ \tau(x,t) \end{bmatrix} \tag{47}$$

The solution of (43) and (44) is assumed to be

$$\begin{aligned} w(x,t) &= \sum_{j=1}^{\infty} W_j(x) q_j(t) \\ \psi(x,t) &= \sum_{j=1}^{\infty} \Psi_j(x) q_j(t) \end{aligned} \tag{48}$$

In (48) $W_j(x),\ \Psi_j(x)$ are the mode shapes of the beam, and $q_j(t)$ is the function to be determined.

If we use the notation

$$\mathbf{U}_j(x) = \begin{bmatrix} W_j(x) \\ \Psi_j(x) \end{bmatrix} \tag{49}$$

then the solution of (48) can be rewritten as

$$\mathbf{u}(x,t) = \sum_{j=1}^{\infty} \mathbf{U}_j(x) q_j(t) \tag{50}$$

From (50), we have

$$\frac{\partial^2 \mathbf{u}(x,t)}{\partial t^2} = \sum \mathbf{U}_j(x)\ddot{q}_j(t) = \begin{bmatrix} \sum W_j(x)\ddot{q}_j(t) \\ \sum \Psi_j(x)\ddot{q}_j(t) \end{bmatrix} \tag{51}$$

$$\mathbf{K}\mathbf{u}(x,t) = \begin{bmatrix} -k^* GA\dfrac{\partial^2}{\partial x^2} & k^* GA\dfrac{\partial}{\partial x} \\ -k^* GA\dfrac{\partial}{\partial x} & k^* GA - EI\dfrac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} \sum W_j(x) q_j(t) \\ \sum \Psi_j(x) q_j(t) \end{bmatrix} \tag{52}$$

Substituting (51) and (52) into (45) one obtains:

$$\begin{aligned} &\rho A\sum_{j=1}^{\infty} W_j(x)\ddot{q}_j(t) - k^* GA\sum_{j=1}^{\infty} W_j''(x) q_j(t) \\ &+ k^* GA\sum_{j=1}^{\infty} \Psi_j'(x) q_j(t) = p(x,t) \end{aligned} \tag{53}$$

$$\begin{aligned} &\rho I\sum_{j=1}^{\infty} \Psi_j(x)\ddot{q}_j(t) - k^* GA\sum_{j=1}^{\infty} W_j'(x) q_j(t) \\ &+ k^* GA\sum_{j=1}^{\infty} \Psi_j(x) q_j(t) - EI\sum_{j=1}^{\infty} \Psi_j''(x) q_j(t) = \tau(x,t) \end{aligned} \tag{54}$$

Using properties of the normal mode shapes, from (53) and (54) we get:

$$\rho A\sum_{j=1}^{\infty} W_j(x)[\ddot{q}_j(t) + \omega_j^2 q_j(t)] = p(x,t) \tag{55}$$

$$\rho I\sum_{j=1}^{\infty} \Psi_j(x)[\ddot{q}_j(t) + \omega_j^2 q_j(t)] = \tau(x,t) \tag{56}$$

Multiplying both sides of (55) by $W_j(x)$ and integrating over the length of the beam, we have

$$\begin{aligned} &\rho A\sum_{j=1}^{\infty} [\ddot{q}_j(t) + \omega_j^2 q_j(t)]\int_0^l W_j(x) W_k(x) dx \\ &= \int_0^l p(x,t) W_k(x) dx \end{aligned} \tag{57}$$

It follows from (56) that

$$\begin{aligned} &\rho I\sum_{j=1}^{\infty} [\ddot{q}_j(t) + \omega_j^2 q_j(t)]\int_0^l \Psi_j(x)\Psi_k(x) dx \\ &= \int_0^l \tau(x,t)\Psi_k(x) dx \end{aligned} \tag{58}$$

It follows from (57), (58) that

$$\sum_{j=1}^{\infty}[\ddot{q}_j(t)+\omega_j^2 q_j(t)]\int_0^l[\rho A W_j(x)W_k(x)+\rho I\Psi_j(x)\Psi_k(x)]dx$$
$$=\int_0^l p(x,t)W_k(x)dx+\int_0^l \tau(x,t)\Psi_k(x)dx \tag{59}$$

Using the orthogonality conditions of the normal mode shapes, from (59) we obtain the modal equations as follows:

$$\ddot{q}_k(t)+\omega_k^2 q_k(t)=\frac{\int_0^l p(x,t)W_k(x)dx+\int_0^l \tau(x,t)\Psi_k(x)dx}{\int_0^l[\rho A W_k^2(x)+\rho I\Psi_k^2(x)]dx}$$
$$=h_k(t),\ \ k=1,2,3,... \tag{60}$$

It should be noted that the cutoff frequency is generally very large, so that the forced vibration of the Timoshenko beam in the frequency domain less than or equal to the cutoff frequency of the beam $(\omega\le\omega_c)$ will be investigated in practice.

### *3.2 Application Example*

We consider now the transient vibration of a simply supported Timoshenko beam of finite length $l$ subjected to a force $P_0$ moving with a constant velocity $v_P$ (Fig. 1). Using the Delta Dirac function, the density $p(x,t)$ can be written in the following form

$$p(x,t)=L(x_P)P_0\delta(x-x_P) \tag{61}$$

where $L(x_P)$ is a logical signal function

$$L(x_P)=\begin{cases}1 & if\ \ 0\le x_P\le L\\ 0 & if\ \ x_P\ge L\end{cases} \tag{62}$$

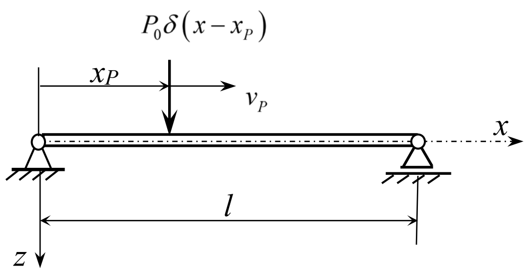


Fig. 1. The simply supported Timoshenko beam.

If $EI(x)$ and $A(x)$ are assumed to be constant, it follows from (43) and (44) that

$$\rho A\frac{\partial^2 w}{\partial t^2}-k^*GA\frac{\partial^2 w}{\partial x^2}+k^*GA\frac{\partial\psi}{\partial x}$$
$$=L(x_P)P_0\delta(x-v_Pt) \tag{63}$$

$$\rho I\frac{\partial^2\psi}{\partial t^2}-k^*G\,A\frac{\partial w}{\partial x}+k^*G\,A\psi-EI\frac{\partial^2\psi}{\partial^2 x}=0 \tag{64}$$

Using the modal analysis method, the forced vibration responses defined by (63) and (64) can be represented as follows [8]:

$$w(x,t)=\sum_{k=1}^{\infty}W_k(x)q_k(t),\ \ \psi(x,t)=\sum_{k=1}^{\infty}\Psi_k(x)q_k(t) \tag{65}$$

where $W_k(x)$ and $\Psi_k(x)$ are the mode shapes of the beam, $q_k(t)$ are generalized coordinates that are determined to satisfy initial conditions. For the simply supported Timoshenko beam, the mode shapes have the form as (41) and (42). Since $\tau(x,t)=0$, the function $q_k(t)$ is defined by the following equation

$$\ddot{q}_k(t)+\omega_k^2 q_k(t)=$$
$$\frac{\int_0^l W_k(x)p(x,t)\,dx}{\rho A\int_0^l W_k^2(x)\,dx+\rho A\int_0^l\Psi_k^2(x)\,dx}=h_k(t) \tag{66}$$
$$(k=1,2,...)$$

It should be noted that

$$\rho A\int_0^l W_k^2(x)\,dx=\rho A\int_0^l[\sin(\frac{k\pi x}{l})]^2dx$$
$$=\rho A\frac{l}{2}$$
$$\rho I\int_0^l\Psi_k^2(x)\,dx=(g_{\beta(k)})^2\rho I\int_0^l[\cos(\frac{k\pi x}{l})]^2dx$$
$$=(g_{\beta(k)})^2\rho I\frac{l}{2} \tag{67}$$
$$\int_0^l W_k(x)p(x,t)\,dx=\int_0^l\delta(x-vt)\sin(\frac{k\pi x}{l})dx$$
$$=P_0\sin\frac{k\pi v}{l}t$$

Substitution of (67) into (66) leads to

$$\ddot{q}_k+\omega_k^2 q_k=\frac{P_0}{m^*}\sin\Omega_k t,\ \ (k=1,2,...) \tag{68}$$

where

$$m_k^*=\rho A\frac{l}{2}+(g_{\beta(k)})^2\rho I\frac{l}{2},\ \ \Omega_k=\frac{k\pi v}{l} \tag{69}$$

$$\omega_k^2 = k^* GA\left(\left(\frac{\pi k}{l}\right)^2 - g_{\beta(k)}\frac{\pi k}{l}\right) \tag{70}$$

Substituting (70) into (68) we obtain

$$\ddot{q}_k + k^* GA\left(\left(\frac{\pi k}{l}\right)^2 - g_{\beta(k)}\frac{\pi k}{l}\right)q_k = \frac{P_0}{m_k^*}\sin\Omega_k t \quad , \left(k = 1, 2, \ldots\right) \tag{71}$$

Applying the initial condition on the displacement applied to (68) we obtain

$$t_0 = 0, q_k(0) = 0, \dot{q}_k(0) = 0 \;, (k=1, 2, \ldots) \tag{72}$$

Solving (71) with the initial conditions according to (72) we get the functions $q_k(t)$

$$q_k = -\frac{P_0\Omega_k}{m_k^*\omega_k(\omega_k^2 - \Omega_k^2)}\sin\omega_k t + \frac{P_0}{m_k^*(\omega_k^2 - \Omega_k^2)}\sin\Omega_k t \;, \left(k = 1, 2, \ldots\right) \tag{73}$$

The transient bending vibration of the beam is then determined by the formulas

$$w(x,t) = \sum_{k=1}^{N} q_k(t)\sin\frac{k\pi}{l}x$$
$$\psi(x,t) = \sum_{k=1}^{\infty} q_k(t) g_{\beta(k)}\cos\frac{k\pi}{l}x \tag{74}$$

where

$$g_{\beta(k)} = \beta - \frac{\omega_k^2}{k^* GA\beta} = \frac{k\pi}{l} - \frac{\omega_k^2 l\rho A}{k^* GAk\pi} \tag{75}$$

## 4. Numerical Results and Discussion

For the numerical investigation, the geometric and material properties of the uniform simply supported Timoshenko are listed in Table 1. This parameter set has also been used by Azam et al. [6]. In this example we also need the dynamic parameters of the moving load. That is, the toving force

$$P_0 = mg = 3600 \times 9.81 = 35316\ (N)$$

and the speed of the force $v_p = 18\ km/h\ (= 5\ m/s)$.

The calculation results are listed in Table 2 and showed in Fig. 2 to Fig. 6. The number of eigenfunctions is chosen to be three ($N = 3$).

Table 1. Parameter used for numerical calculation

| Symbols | Values | Parameters of the beam |
|---|---|---|
| $l$ | 50.0 (m) | Length of beam |
| A | 2.0 ($m^2$) | Cross section area |
| I | 1.042 ($m^4$) | Inertial moment |
| E | 3.34 x $10^{10}$ ($N/m^2$) | Elastic modulus |
| G | 1.34 x $10^{10}$ ($N/m^2$) | Shear modulus |
| ρ | 2400 ($kg/m^3$) | Mass per unit volume |
| k* | 0.7 | Shear correction factor |

Table 2. Natural frequency of the beam [Hz]

| Mode number | Timoshenko beam | E-B beam [3] [Hz] | Calculation error TB/EB (%) |
|---|---|---|---|
| 1 | 1.684 | 1.692 | 0.475 |
| 2 | 6.644 | 6.767 | 1.85 |
| 3 | 14.629 | 15.227 | 4.08 |
| 4 | 25.279 | 27.070 | 7.08 |
| 5 | 38.186 | 42.296 | 10.76 |

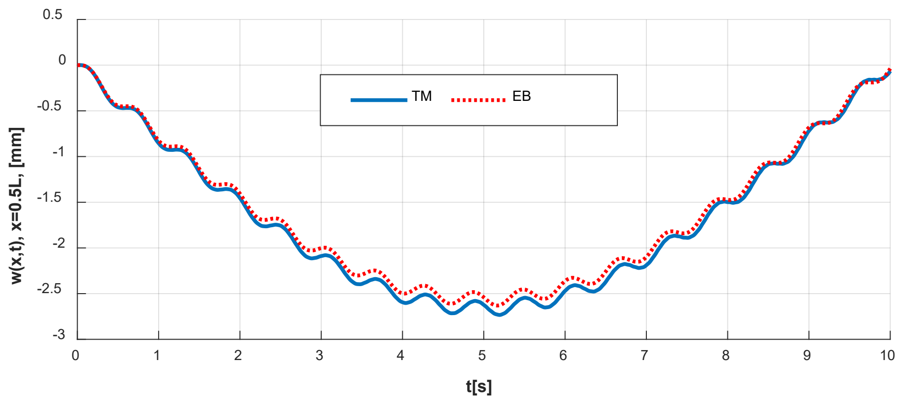


Fig. 2. Transverse displacement w($x$,$t$) of the beam at the position x=0.5$l$.

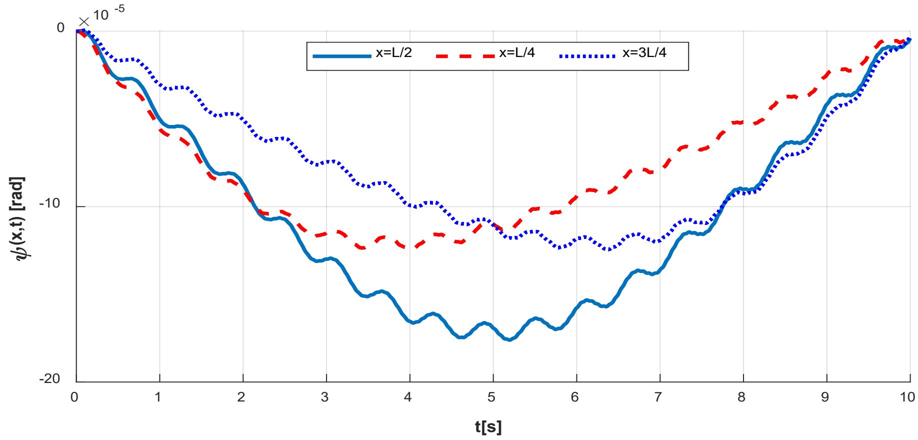


Fig. 3. Rotation angle of the beam cross section with respect to the vertical axis at the positions $x = 0.25l$; $x = 0.5l$, $x = 0.75l$.

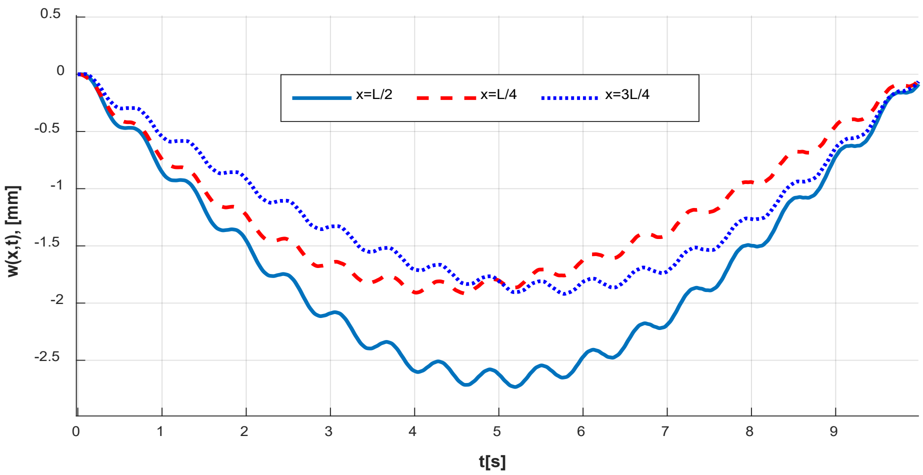


Fig. 4. Transverse vibration of beam at the positions $x = 0.25l$, $x = 0.5l$, $x = 0.75l$.

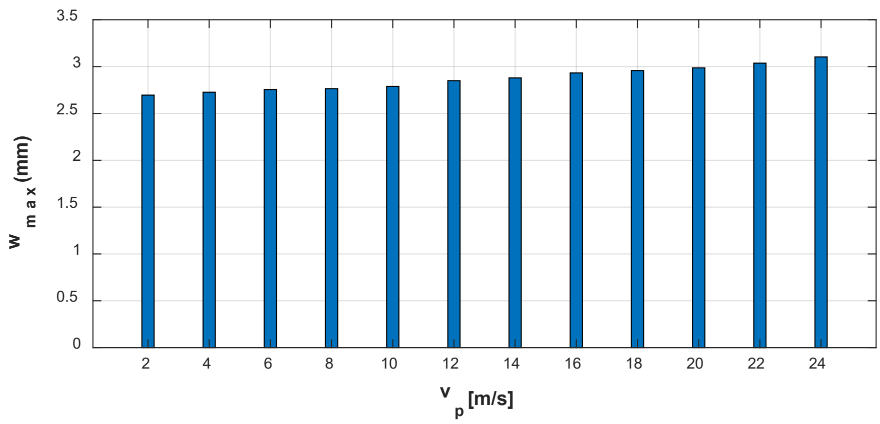


Fig. 5. Maximum transverse of beam at the positions $x = 0.5l$.

Table 3. Natural frequency of the beam [Hz]

| Mode number | TM beam | EB beam [3] | Calculation error TB/EB (%) |
|---|---|---|---|
| 1 | 10.279 | 10.574 | 2.87 |
| 2 | 38.186 | 42.296 | 10.76 |
| 3 | 77.813 | 95.167 | 22.30 |
| 4 | 124.155 | 169.186 | 36.27 |
| 5 | 174.073 | 264.354 | 51.86 |

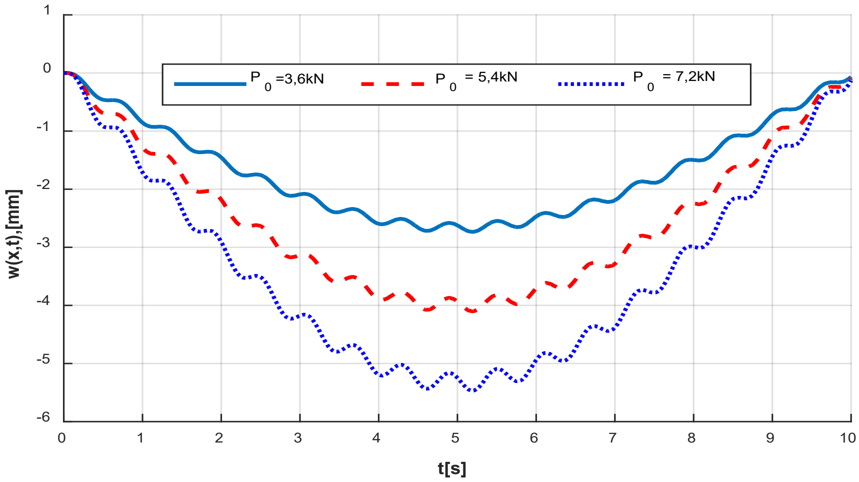


Fig. 6. Transverse displacement of the beam at the position x=0.5*l*.

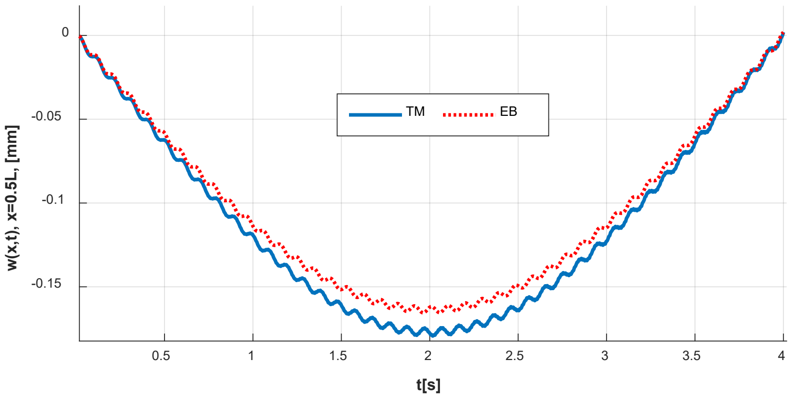


Fig. 7. Transverse displacement w(*x*,*t*) of the beam at the position x=0.5*l*.

The cutoff frequency of this beam is

$$\omega_c = \sqrt{\frac{k * GA}{\rho I}} = 2738.9047\,(rad / s)$$

$$\approx 435.91\,(Hz)$$

In Table 2, the cutoff frequency is 435.91 Hz. Accordingly, the number of natural frequencies is $n_B = 21$. The obtained natural frequencies are the same as the results in work [6]. Fig. 2 shows transient vibrations of the Timoshenko beam and the Euler-Bernoulli beam at cross section $x = l/2$. Fig. 3 shows the results for the angle of rotation of the Timoshenko beam at the cross-sections $x = l/4$, $x = l/2$ and $x = 3l/4$. Fig. 4 shows the results calculated for transverse vibration of Timoshenko beam at the cross-sections $x = l/4$, $x = l/2$ and $x = 3l/4$. Fig. 5 shows a graph of the maximum deflection versus the velocity of the force on the beam. Fig. 6 shows the beam deflection at the mid-beam section depending on the magnitude of the moving force on the beam in which $P_0$ = 3.6 kN, 5.4 kN và 7.2 kN respectively. From Fig. 5 and 6, it can be seen that during the transition process the influence of the speed change of the moving force on the beam deflection is only little, while the influence of the magnitude of the moving force on the beam deflection is very large.

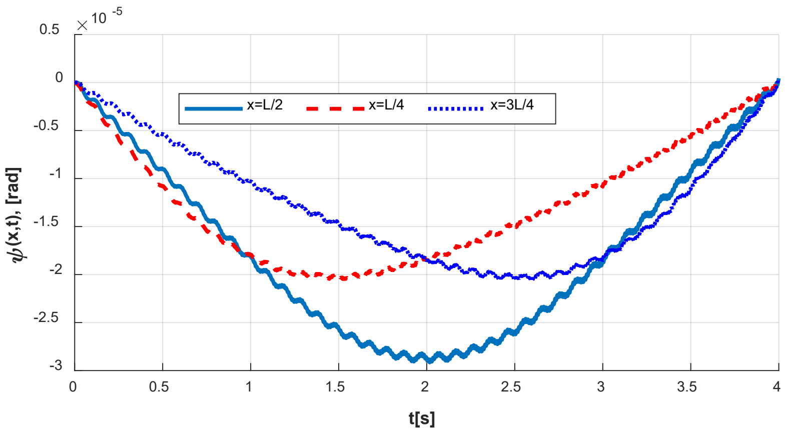


Fig. 8. Rotation angle of the beam cross section with respect to the vertical axis.

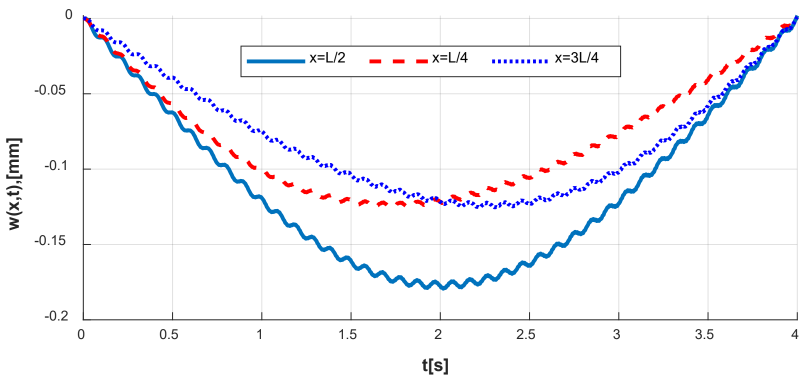


Fig. 9. Transverse vibration of beam at the positions $x = 0.25l$, $x = 0.5l$, $x = 0.75l$.

To study the influence of the length of the Timoshenko beam on the vibrational properties of the beam, the beam of length $l = 20$ m is taken into account. The other parameters are the same as in the first example. Some of the calculation results are listed in Tab. 3 and plotted in Fig. 7, 8, and 9.

From Table 2 and 3, it can be shown that the natural frequencies of the beam change very fast if ít length is relatively short. Fig. 2 and 7 show clearly the effects of shear deformation and rotatory inertia on the beam deflection.

## 5. Conclusion

In this paper, the calculation of forced vibration of the Timoshenko beam in the transition process by the modal analysis method was presented. Based on the obtained results, the following concluding remarks can be reached.

1. Using the modal method, a procedure for calculating the forced vibrations of the Timoshenko beam has been proposed.

2. From formulas (73) and (44) and from numerical simulation results as shown in Fig. 6, it can be concluded that during the transition process the speed change of the moving force has little effect on the beam deflection.

3. The numerical results of the transient vibration of a simply supported Timoshenko beam subjected to a force $P_0$ moving with a constant velocity $v_P$ show also that the magnitude change of the moving force greatly affects the beam deflection.

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