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American Mathematics Competitions

Solutions Pamphlet

MAA American Mathematics Competitions

36th Annual

AIME I

American Invitational Mathematics Examination I

Tuesday, March 6, 2018

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

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AIME Chair

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1. **ANSWER (600):**

   The factoring condition is equivalent to the discriminant $a^2 - 4b$ being equal to c^2 for some integer c. Because $b \geq 0$, the equation $4b = (a-c)(a+c)$ shows that the existence of such a b is equivalent to $a \equiv c \pmod{2}$ with $0 \leq c \leq a$. Thus the number of ordered pairs is

$$S = \sum_{a=1}^{100} \left\lceil \frac{a+1}{2} \right\rceil = 2600.$$

   The requested remainder is 600.

2. **ANSWER (925):**

   The problem is equivalent to finding a solution to the system of Diophantine equations $196a + 14b + c = 225a + 15c + b$ and $225a + 15c + b = 216a + 36c + 6a + c$, where $1 \leq a \leq 5$, $0 \leq b \leq 13$, and $0 \leq c \leq 5$. Simplifying the second equation gives $b = 22c - 3a$. Substituting for b in the first equation and simplifying then gives $a = 4c$, so $a = 4$ and $c = 1$, and the base-10 representation of n is $222 \cdot 4 + 37 \cdot 1 = 925$. It may be verified that $b = 10 \leq 13$.

3. **ANSWER (157):**

   Assume without loss of generality that the first card laid out is red. Then the arrangements that satisfy Kathy's requirements are RRRRR, RRRRG, RRRGG, RRGGG, and RGGGG. The probability that Kathy will lay out one of these arrangements is

$$\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{1}{6}+\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{5}{6}+\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{5}{7}\cdot\frac{4}{6}+\frac{4}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}+\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}=\frac{31}{126}.$$

   The requested sum is $31 + 126 = 157$.

OR

   Assume without loss of generality that the first card laid out is red. The probability that k of the four remaining laid out cards are red, where $0 \leq k \leq 4$, is

$$\frac{\binom{4}{k}\binom{5}{4-k}}{\binom{9}{4}}.$$

   Given that there are exactly k red cards, the probability that they are laid out at the start is $\frac{1}{\binom{4}{k}}$. Hence the required probability is

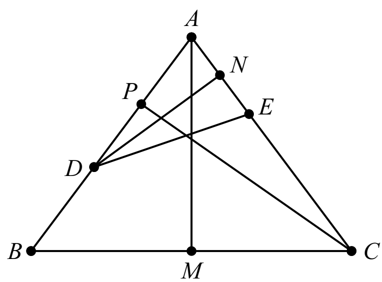
$$\sum_{k=0}^{4} \frac{\binom{5}{4-k}}{\binom{9}{4}} = \frac{2^5-1}{\binom{9}{4}} = \frac{31}{126}.$$

4. **ANSWER (289):**

Let M be the foot of the perpendicular from A to $\overline{BC}$, and let N be the foot of the perpendicular from D to $\overline{AC}$. Because $\triangle ABC$ and $\triangle ADE$ are isosceles, M and N are the midpoints of $\overline{BC}$ and $\overline{AE}$, respectively. Thus $CM = 6$, so if $x = AD = DE = EC$, then $AN = \frac{10-x}{2}$. Let $\theta = \angle CAM$. Because $AM = \sqrt{10^2 - 6^2} = 8$, it follows that $\cos\theta = \frac{4}{5}$. Note that $\cos(\angle BAC) = \cos(2\theta) = 2\cos^2\theta - 1 = 2 \cdot \left(\frac{4}{5}\right)^2 - 1 = \frac{7}{25}$. Thus

$$\cos(\angle DAN) = \frac{\left(\frac{10-x}{2}\right)}{x} = \frac{7}{25}\,.$$

Solving this equation yields $x = \frac{250}{39}$. The requested sum is $250 + 39 = 289$.



OR

By the Law of Cosines in $\triangle ABC$,

$$\cos A = \frac{10^2 + 10^2 - 12^2}{2 \cdot 10 \cdot 10} = \frac{7}{25}.$$

Let $x = AD = DE = EC$. Then $AE = 10 - x$. Because

$$\cos(\angle ADE) = \cos(\pi - 2\angle A) = -\cos(2A) = 1 - 2\cos^2 A,$$

the Law of Cosines in $\triangle ADE$ gives

$$\begin{aligned} x^2 + x^2 &= (10-x)^2 + 2x^2(1 - 2\cos^2 A) \\ 2x^2 &= (10-x)^2 + 2x^2\left(1 - 2\cdot\frac{7^2}{25^2}\right) \\ 4x^2 \cdot \frac{7^2}{25^2} &= (10-x)^2 \\ \frac{14x}{25} &= 10 - x, \end{aligned}$$

and $x = \frac{250}{39}$.

OR

By the Pythagorean Theorem the altitude of $\triangle ABC$ to $\overline{BC}$ has length 8. Let points N and P lie on $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{DN} \perp \overline{AC}$ and $\overline{CP} \perp \overline{AB}$. Because $2[ABC] = 12 \cdot 8 = CP \cdot AB$, it follows that $CP = \frac{48}{5}$ and $AP = \sqrt{AC^2 - CP^2} = \frac{14}{5}$. Because $DA = DE$, it follows that

$$AN = NE = \frac{AC - CE}{2} = \frac{10 - AD}{2}.$$

Right triangles ADN and ACP are similar to each other. Because $\frac{AC}{AP} = \frac{AD}{AN}$,

$$\frac{10}{\left(\frac{14}{5}\right)} = \frac{AD}{\left(\frac{10-AD}{2}\right)},$$

from which it follows that $AD = \frac{250}{39}$.

5. **ANSWER (189):**

Because $x^2 + xy + 7y^2 = \left(x + \frac{y}{2}\right)^2 + \frac{27}{4}y^2 > 0$, the right side of the first equation is real. It follows that the left side of the equation is also real, so $2x + y > 0$ and

$$\log_2(2x + y) = \log_{2^2}(2x + y)^2 = \log_4(4x^2 + 4xy + y^2).$$

Thus $4x^2+4xy+y^2 = x^2+xy+7y^2$, which implies that $0 = x^2+xy-2y^2 = (x+2y)(x-y)$. Therefore either $x = -2y$ or $x = y$, and because $2x + y > 0$, x must be positive and $3x + y = x + (2x + y) > 0$. Similarly,

$$\log_3(3x + y) = \log_{3^2}(3x + y)^2 = \log_9(9x^2 + 6xy + y^2).$$

If $x = -2y \neq 0$, then $9x^2 + 6xy + y^2 = 36y^2 - 12y^2 + y^2 = 25y^2 = 3x^2 + 4xy + Ky^2$ when $K = 21$. If $x = y \neq 0$, then $9x^2 + 6xy + y^2 = 16y^2 = 3x^2 + 4xy + Ky^2$ when $K = 9$. The requested product is $21 \cdot 9 = 189$.

6. **ANSWER (440):**

If z satisfies the given conditions, there is a $\theta \in [0, 2\pi)$ such that $z = e^{i\theta}$ and $e^{720\theta i} - e^{120\theta i}$ is real. This difference is real if and only if either the two numbers 720θ and 120θ represent the same angle or the two numbers represent supplementary angles. In the first case there is an integer k such that $720\theta = 120\theta + 2k\pi$, which implies that θ is a multiple of $\frac{\pi}{300}$. In the second case there is an integer k such that $720\theta + 120\theta = (2k + 1)\pi$, which implies that θ is $\frac{\pi}{840}$ plus a multiple of $\frac{\pi}{420}$. In the interval $[0, 2\pi)$ there are 600 values of θ that are multiples of $\frac{\pi}{300}$, there are 840 values that are $\frac{\pi}{840}$ plus a multiple of $\frac{\pi}{420}$, and there are no values of θ that satisfy both of these conditions. Therefore there must be $600 + 840 = 1440$ complex numbers satisfying the given conditions. The requested remainder is 440.

7. **ANSWER (052):**

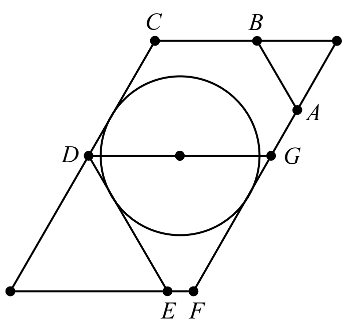
There are 2 equilateral and 6 other isosceles triangles on each base, providing a total of 16 triangles when all 3 vertices are chosen from the same base.

To count the rest of the isosceles triangles, there are 2 ways to choose the base that will contain 2 vertices. Assume that 2 vertices have been chosen from the bottom base. Then there are no isosceles triangles if the 2 vertices on the bottom base are adjacent vertices of the hexagon, $6 \cdot 2 = 12$ isosceles triangles if the 2 vertices on the bottom base have 1 vertex between them on the hexagon, and $3 \cdot 2 = 6$ isosceles triangles if the 2 vertices on the bottom base have 2 vertices between them on the hexagon.

The total number of isosceles triangles is $16 + 2(12 + 6) = 52$.

8. **ANSWER (147):**

Because each interior angle of an equiangular hexagon is $120°$, attaching two equilateral triangles to two opposite sides of the hexagon gives a parallelogram with $60°$ and $120°$ angles as shown in the diagram.



The fact that the opposite sides in the parallelogram must be congruent implies that the sum of the lengths of any two adjacent sides of the hexagon is the same as the sum of the lengths of the two opposite sides. From $AB+BC = DE+EF$ it follows that $EF = 2$, and from $FA + AB = CD + DE$ it follows that $FA = 16$.

The fact that the parallelogram has $60°$ angles implies that the distance between two opposite parallel sides of the hexagon is $\frac{\sqrt{3}}{2}$ times the sum of the lengths of the two adjacent sides connecting them. The shortest distance between the parallelogram's opposite sides is thus the distance from line AF to line CD, which is $\frac{\sqrt{3}}{2}(6+8) = 7\sqrt{3}$. Let G lie on $\overline{AF}$ so that $\overline{DG} \parallel \overline{EF}$. Because $\overline{DG}$ is the bisector of $\angle CDE$, the midpoint of $\overline{DG}$ is $\frac{7}{2}\sqrt{3}$ from each of the lines CD, DE, and AF. Its distance from line EF is $6\sqrt{3} > \frac{7}{2}\sqrt{3}$, its distance from line BC is $5\sqrt{3} > \frac{7}{2}\sqrt{3}$, and its distance from line AB is $\frac{11}{2}\sqrt{3} > \frac{7}{2}\sqrt{3}$. This shows that the midpoint of $\overline{DG}$ is the center of a circle with diameter $7\sqrt{3}$ that fits inside the hexagon, and no larger circle will fit between $\overline{CD}$ and $\overline{AF}$. The requested square of this diameter is $(7\sqrt{3})^2 = 147$.

9. **ANSWER (210):**

There are two types of $\{a, b, c, d\} \subseteq \{1, 2, 3, 4, \ldots, 20\}$ that have the needed property. There is either an assignment of distinct values for a, b, c, and d such that $a + b = 16$ and $c + d = 24$ or an assignment such that $a + b = 16$ and $a + c = 24$. These two types are mutually exclusive because $c + d = 24$ and $a + c = 24$ imply that $a = d$. For the first type, there are 7 choices for $\{a, b\}$, namely $\{1, 15\}$, $\{2, 14\}$, $\{3, 13\}$, $\{4, 12\}$, $\{5, 11\}$, $\{6, 10\}$, and $\{7, 9\}$, and there are 8 choices for $\{c, d\}$, namely $\{4, 20\}$, $\{5, 19\}$, $\{6, 18\}$, $\{7, 17\}$, $\{8, 16\}$, $\{9, 15\}$, $\{10, 14\}$, and $\{11, 13\}$. Thus a four-element subset of the first type can be formed by taking the union of one of 7 two-element subsets with one of 8 two-element subsets as long as those two subsets are disjoint. There are 10 such pairings that are not disjoint out of the $7 \cdot 8 = 56$ pairings, so there are $56 - 10 = 46$ subsets of the first type.

For subsets of the second type, there are 10 choices for a value of a (4, 5, 6, 7, 9, 10, 11, 13, 14, 15) such that $b = 16-a$ and $c = 24-a$ can be two other elements of the subset. Note that in each of these cases, $c - b = (24-a) - (16-a) = 8$. For each of these, there are $20 - 3 = 17$ other values that can be chosen for the element d in the subset. But $10 \cdot 17 = 170$ counts some subsets more than once. In particular, a subset is counted twice if $b + d = 24$ or $c + d = 16$. In such cases either $d = a + 8$ or $d = a - 8$. There are exactly 6 subsets where the role of a can be played by two different elements of the set. They are $\{1, 7, 9, 15\}$, $\{2, 6, 10, 14\}$, $\{3, 5, 11, 13\}$, $\{5, 11, 13, 19\}$, $\{6, 10, 14, 18\}$, and $\{7, 9, 15, 17\}$. Thus there are $170 - 6 = 164$ subsets of the second type.

In all, there are $46 + 164 = 210$ subsets with the required property.

10. **ANSWER (004):**

Let X represent a step that is either counterclockwise or inward along a spoke, and let Y represent a step that is either clockwise or outward along a spoke. Then there is a one-to-one correspondence between the paths with 15 steps starting at point A and sequences of Xs and Ys in which the total number of Xs and Ys is 15. Furthermore, because there are five points on each circle, the bug ends at point A if and only if the last move is an X and the difference between the number of Xs and the number of Ys is a multiple of 5. Thus the bug ends at point A if and only if, among the first 14 moves, there are either 4, 9, or 14 Xs and the last move is an X. Therefore the number of paths beginning and ending at A is $\binom{14}{4} + \binom{14}{9} + \binom{14}{14} = \binom{14}{4} + \binom{14}{5} + \binom{14}{14} = \binom{15}{5} + \binom{14}{14} = 3004$. The requested remainder when 3004 is divided by 1000 is 4.

11. **ANSWER (195):**

The requested positive integer is the least value of $n > 0$ such that $3^n \equiv 1 \pmod{143^2}$. Note that $143 = 11 \cdot 13$. The least power of 3 that is congruent to 1 modulo 11^2 is $3^5 = 243 = 2 \cdot 11^2 + 1$. It follows that $3^n \equiv 1 \pmod{11^2}$ if and only if $n = 5j$ for some positive integer j.

The least power of 3 that is congruent to 1 modulo 13 is $3^3 = 27 = 2 \cdot 13 + 1$. It follows that $3^n \equiv 1 \pmod{13}$ if and only if $n = 3k$ for some positive integer k. Additionally, for positive integer k, the Binomial Theorem shows that $3^{3k} = (26+1)^k \equiv 26 \cdot k + 1 \pmod{13^2}$. In particular, $3^n = 3^{3k} \equiv 1 \pmod{13^2}$ if and only if $k = 13m$ for some positive integer m, that is, if and only if $n = 39m$.

Because 11^2 and 13^2 are relatively prime, $3^n \equiv 1 \pmod{143^2}$ if and only if $3^n \equiv 1 \pmod{11^2}$ and $3^n \equiv 1 \pmod{13^2}$. This occurs if and only if n is a multiple of both of the relatively prime integers 5 and 39, so the least possible value of n is $5 \cdot 39 = 195$.

12. **ANSWER (683):**

Let $A = \{1, 4, 7, 10, 13, 16\}$, $B = \{2, 5, 8, 11, 14, 17\}$, and $C = \{3, 6, 9, 12, 15, 18\}$. For $T \subseteq U$, let $a = |T \cap A|$, $b = |T \cap B|$, and $c = |T \cap C|$. Then $s(T)$ is divisible by 3 if and only if $|a - b| = 0$, 3, or 6. Because elements of $T \cap C$ do not influence whether 3 divides $s(T)$, it suffices to calculate the conditional probability that $s(T)$ is divisible by 3 given that $T \cap C = \emptyset$.

The number of subsets with $a - b = 0$ is

$$\sum_{k=0}^{6} \binom{6}{k}^2 = \sum_{k=0}^{6} \binom{6}{k}\binom{6}{6-k} = \binom{12}{6} = 924.$$

The number of subsets with $a - b = 3$ is

$$\sum_{k=0}^{3} \binom{6}{k+3}\binom{6}{k} = \sum_{k=0}^{3} \binom{6}{3-k}\binom{6}{k} = \binom{12}{3} = 220,$$

and the number with $b - a = 3$ is the same. There is one subset with $a - b = 6$ and one with $b - a = 6$. Hence the number of sets T for which $s(T)$ is divisible by 3 and $T \cap C = \emptyset$ is $924 + 2 \cdot 220 + 2 = 1366$. The required probability is $\frac{1366}{2^{12}} = \frac{683}{2^{11}}$. The requested numerator is 683.

OR

Defining A and B as above, associate with each $T \subseteq A \cup B$ the set $T^* = (T \cap A) \cup (B \backslash T)$. Then T has $a + b$ elements if and only if T^* has $a + (6 - b) = 6 + (a - b)$ elements, so $a - b$ is a multiple of 3 if and only if T^* contains 0, 3, 6, 9, or 12 elements. Therefore the number of sets T^* is $\binom{12}{0} + \binom{12}{3} + \binom{12}{6} + \binom{12}{9} + \binom{12}{12} = 1366$.

OR

Consider the function

$$F(X) = \prod_{k=1}^{18}(1 + X^k) = \sum_{T \subseteq U} X^{s(T)}.$$

Note that if ω is a cube root of unity not equal to 1, then $1^k + \omega^k + \omega^{2k}$ equals 3 when k is divisible by 3, and it equals 0 otherwise. Then the number N of sets T with $s(T) \equiv 0 \pmod 3$ is

$$N = \frac{F(1) + F(\omega) + F(\omega^2)}{3}.$$

Now $F(1) = 2^{18}$, while

$$F(\omega) = F(\omega^2) = \big(2(1 + \omega)(1 + \omega^2)\big)^6 = 2^6.$$

Hence

$$N = \frac{2^{18} + 64 + 64}{3} = \frac{2^{11} + 1}{3} \cdot 2^7,$$

meaning $p = \frac{2^{11}+1}{3} = 683$.

13. **Answer (126):**

First note that

$$\angle I_1AI_2 = \angle I_1AX + \angle XAI_2 = \frac{\angle BAX}{2} + \frac{\angle CAX}{2} = \frac{\angle A}{2}.$$

This is a constant not depending on X. Let $a = BC$, $b = AC$, $c = AB$, and $\alpha = \angle AXB$. Note that

$$\angle AI_1B = 180^\circ - (\angle I_1AB + \angle I_1BA) = 180^\circ - \frac{1}{2}(180^\circ - \alpha) = 90^\circ + \frac{\alpha}{2}.$$

Applying the Law of Sines to $\triangle ABI_1$ gives

$$\frac{AI_1}{AB} = \frac{\sin(\angle ABI_1)}{\sin(\angle AI_1B)}, \quad \text{implying that} \quad AI_1 = \frac{c \sin \frac{B}{2}}{\cos \frac{\alpha}{2}}.$$

Analogously, $\angle AI_2C = 90^\circ + \frac{\angle AXC}{2} = 180^\circ - \frac{\alpha}{2}$ and

$$AI_2 = \frac{b \sin \frac{C}{2}}{\sin \frac{\alpha}{2}}.$$

Because the area of $\triangle I_1AI_2$ can be written as $\frac{1}{2}(AI_1)(AI_2)\sin(\angle I_1AI_2)$, it follows that

$$[\triangle AI_1I_2] = \frac{bc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{2 \cos \frac{\alpha}{2} \sin \frac{\alpha}{2}} = \frac{bc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \alpha}$$
$$\geq bc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2},$$

with equality when $\alpha = 90^\circ$, that is, when X is the foot of the perpendicular from A to $\overline{BC}$. In this case the desired area is $bc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$. To make this feasible to compute, note that

$$\sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{1 - \frac{b^2+c^2-a^2}{2bc}}{2}} = \sqrt{\frac{(a-b+c)(a+b-c)}{4bc}}.$$

Applying similar logic to $\sin \frac{B}{2}$ and $\sin \frac{C}{2}$ and simplifying yields a final answer of

$$bc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = bc \cdot \frac{(a-b+c)(b-c+a)(c-a+b)}{8abc}$$
$$= \frac{(32-34+30)(34-30+32)(30-32+34)}{8 \cdot 32} = 126.$$

14. **ANSWER (351):**

Define $A = \{S, P_1\}$, $B = \{P_2, P_5\}$, and $C = \{P_3, P_4\}$. Then the frog can reach vertex E on jump n only if it reaches a vertex in C on jump $n-1$. The frog can reach a vertex in C on jump n only if it reaches a vertex in B on jump $n-1$. The frog can reach a vertex in B on jump n only if it reaches a vertex in either A or C on jump $n-1$. Finally, the frog can reach a vertex in A on jump n only if it reaches a vertex in either A or B on jump $n-1$. Let a_n, b_n, and c_n be the number of paths of length n that end in set A, B, and C, respectively. Then $a_{n+1} = a_n + b_n$, $b_{n+1} = a_n + c_n$, and $c_{n+1} = b_n$. Replacing the b terms with c terms yields $a_{n+1} = a_n + c_{n+1}$ and $c_{n+2} = a_n + c_n$, which implies that $a_n = c_{n+2} - c_n$. Finally, this gives $c_{n+3} - c_{n+1} = c_{n+2} - c_n + c_{n+1}$, which implies that $c_{n+3} = c_{n+2} + 2c_{n+1} - c_n$. It is easy to see that $c_0 = 0$, $c_1 = 0$, and $c_2 = 1$. From the recursive formula, it is easy to complete the following table.

n	0	1	2	3	4	5	6	7	8	9	10	11
c_n	0	0	1	1	3	4	9	14	28	47	89	155

The frog can end at E on jump n if it enters C on the jump $n-1$. Thus the number of paths of no more than 12 jumps that end at E is $0+0+1+1+3+4+9+14+28+47+89+155=351$.

15. **ANSWER (059):**

Let the four sticks have lengths a, b, c, and d. By renaming the sides and considering reflections, it can be assumed without loss of generality that $b > c$, quadrilateral A has sides a, b, c, d in that order, quadrilateral B has sides a, c, b, d in that order, and quadrilateral C has sides a, b, d, c in that order.

Let $WXYZ$ denote quadrilateral A with $WX = a$, $XY = b$, $YZ = c$, and $ZW = d$, shown in the figure below. Construct point Y' on arc $\overset{\frown}{XY}$ in the direction $W \to X \to Y \to Z$ so that $XY' = YZ$, and let P be the intersection of $\overline{XZ}$ and $\overline{WY'}$. Note that $WXY'Z$ has side lengths a, c, b, and d in that order, so it is congruent to quadrilateral B. Because $\angle WPZ$ is half the sum of the central angles of arcs $\overset{\frown}{XY'}$ and $\overset{\frown}{ZW}$, and $\angle WY'Y$ is half the sum of the central angles of arcs $\overset{\frown}{XY'}$ and $\overset{\frown}{ZW}$, it follows that $\angle WPZ = \angle WY'Y$. Because the angles φ_A, φ_B, and φ_C have the same sines as their supplements, it does not matter whether these angles are acute or obtuse. Hence set

$$\varphi_B = \angle WPZ = \angle WY'Y = \angle WXY.$$

This means that an angle with measure φ_B can actually be found in an interior angle of quadrilateral A. Similarly, $\angle ZWX = \varphi_C$.

Now by the Law of Sines,

$$\frac{WY}{\sin\varphi_B} = \frac{XZ}{\sin\varphi_C} = 2.$$

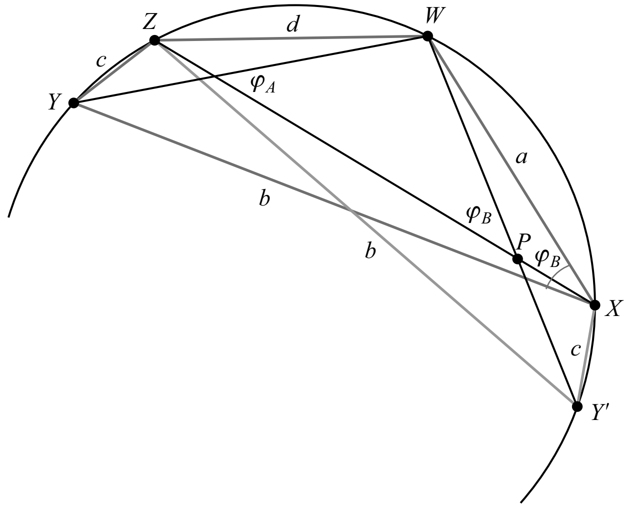
This means that

$$\frac{WY \cdot XZ}{\sin\varphi_B \cdot \sin\varphi_C} = 4.$$

But note that the requested area is $K = \frac{1}{2} WY \cdot XZ \cdot \sin\varphi_A$. Therefore

$$K = 2\sin\varphi_A \sin\varphi_B \sin\varphi_C = \frac{24}{35}.$$

The requested sum is $24 + 35 = 59$.



Note that this result is actually a generalization of a similar result for triangles, namely that

$$K = 2R^2 \sin\alpha \sin\beta \sin\gamma,$$

where R is the circumradius of the triangle.

This result also follows from the fact that the area of a triangle with sides x, y, and z and circumradius R is $\frac{xyz}{4R}$. The specifications in the problem are satisfied if the lengths of the sticks are approximately 0.32, 0.91, 1.06, and 1.82.

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