Maths Beyond Limits Qualifying Quiz 2018

The following Qualifying Quiz is a set of 7 demanding problems. Try to solve as many as you can and submit your solutions in PDF via our application system at mathsbeyondlimits.eu. The deadline for submitting solutions is **April 30th at 23:59 (UTC +2) 2018**.

Problem **1**. Prove that for every natural number $n > 1$ there exist n consecutive natural numbers such that their product is divisible by all primes less than $2n+2$, but it is not divisible by any other prime.

Problem **2**. There are 130212 distinct points (no three points collinear), every two of which are connected by a line segment. Marta and Ania take turns erasing line segments, so that Marta is allowed to erase only one line segment per turn, and Ania is allowed to erase two or three line segments per turn. The person after whose move there is a point connected to no other points loses. Marta makes the first move. Who has the winning strategy?

Problem **3**. Let H_A, H_B, H_C be feet of altitudes from vertices A, B, C of triangle ABC, respectively. Line parallel to CA passing through B intersects line H_BH_C at point X. Point M is the middle of segment AB. Show that

$$\angle ACM = \angle XH_AB.$$

Problem **4**. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \leqslant 12.$$

Problem **5**. Show that for each prime $p \geqslant 7$ there exists a positive integer n and integers x_i, y_i $(i = 1, \ldots, n)$ such that x_i, y_i are not divisible by p for each $i = 1, \ldots, n$ and

$$\begin{cases} x_1^2 + y_1^2 \equiv x_2^2 \pmod{p} \\ x_2^2 + y_2^2 \equiv x_3^2 \pmod{p} \\ \vdots \\ x_n^2 + y_n^2 \equiv x_1^2 \pmod{p}. \end{cases}$$

Problem **6**. An invisible rabbit Szymon is moving along a straight line on the Euclidean plane by making identical jumps every minute. Paweł — the hunter — sets his Trap every hour as he wishes to catch Szymon. The Trap catches rabbit when hunter puts it precisely at the moment when rabbit lands on the place of the Trap.

a) Assume Paweł's Trap is a unit square that can be put on the plane only in such a way its vertices are lattice points (i.e. they have integer coordinates). Moreover, he knows that Szymon starts at the origin (but of course neither direction nor length of Szymon's jump is known to the hunter). Can Paweł catch Szymon in finite time?

b) Now, Paweł's Trap is just a point, but it can be put at any place. He doesn't know where Szymon starts, but he knows that at each minute Szymon jumps from a lattice point to lattice point. Can Paweł catch Szymon in finite time?

Problem **7**. Let P be a point in the plane of triangle ABC such that

$$\frac{AP}{BC} = \frac{BP}{CA} = \frac{CP}{AB}.$$

Prove that P lies on the Euler line of triangle ABC.