Density Operators in Quantum Continuous Feedback Control

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We give a pedagogical introduction to quantum feedback control. An external observer's knowledge of a system is given by a density matrix whose time-evolution obeys a stochastic master equation. We derive this master equation by analyzing the interaction of a quantum system with an external reservoir and relate that result to the most general dissipative solution. We show that continuous measurement necessary in feedback introduces system damping and noise, and that homodyne-mediated optical feedback can enhance squeezing from a single-mode cavity output.

Control theory, either classical or quantum, addresses the basic fact that dynamical systems do not always behave the way we want them to. We can control the state of such a system by combining it with a controller. In open-loop control, the system's output is not measured by controller, which assumes a dynamical model. Closed-loop control, however, can continuously monitor information from a physical system and then apply forces to control its dynamics. The advantages of the latter are significant, as a robust closed-loop controller can take into account variations in system parameters and environmental noise to produce a desired state.

The problem of measuring and controlling quantum mechanical states of light in experiments invites us to consider the possibility of controlling individual quantum systems, in real-time, using feedback. There are two distinct types of quantum feedback based on the manner in which we perform measurement [1]. In the first, a system interacts with an ancilla quantum system, which yields classical information through projective measurement. A classical controller processes that information and alters the system Hamiltonian's parameters within its own coherence time. The second method allows the ancilla system to interact with the system through an interaction potential. While these two approaches are conceptually quite different, it has been shown these two methods are actually equivalent in their outcomes [2].

In Section 1, we will first describe *quantum trajectories* applicable to both cases, namely, how a quantum state specified by a density matrix changes while measurement is taking place continuously through an ancilla reservoir. The form of our result from this example will motivate our derivation of the master equation frequently used in feedback and continuous measurement calculations. There are two examples that are important in understanding the physical behavior of this formalism. Section 2 discusses the Gaussian-weighted positive operator-valued measures (POVMs) relevant to understanding weak measurement and partial wavefunction collapse. In Section 3, we will demonstrate that using homodyne detection for direct phase-locking feedback of a pumped laser will enable perfect squeezing at its output.

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1. MASTER EQUATIONS

Analyzing feedback requires that we first understand the formalism describing the time-evolution of a system conditioned upon continuous measurement. It is possible to view this measurement process as an interaction with an environment (*reservoir*) where we are continuously performing measurement to obtain information. A *master equation* is an equation of motion for a density matrix describing such an open quantum system, much like the Heisenberg equation of motion for the evolution of a closed quantum system. We will first derive a master equation for such open systems and relate that to the most general stochastic master equation that is dissipative, trace-preserving, and completely positive.

1.1. System–Reservoir Interactions

Following Carmichael [3], we start with an open quantum system described by the total Hamiltonian

$$H = H_S \otimes I_R + I_S \otimes H_R + V, \tag{1.1}$$

where H_S and H_R are the Hamiltonians for the system (S) and the reservoir (R), respectively, and V is the interaction potential between them. The system's total density operator, ρ_{tot}, satisfies

$$\dot{\rho}_{\text{tot}} = \frac{1}{i\hbar}\left[H, \rho_{\text{tot}}\right]. \tag{1.2}$$

We want an equation that describes the mean values of Hermitian observables in S given by the reduced density operator over the Hilbert space of R:

$$\rho_S = \text{tr}_R\left[\rho_{\text{tot}}\right]. \tag{1.3}$$

We first transform into an interaction picture that separates the dynamics of $H_S \otimes I_R + I_S \otimes H_R$ from the slower dynamics by V:

$$\tilde{\rho}_{\text{tot}} = e^{(1/\hbar)(H_S+H_R)t}\rho_{\text{tot}}\left(t\right)e^{-(1/\hbar)(H_S+H_R)t}, \tag{1.4}$$

which obeys the equation of motion

$$\dot{\tilde{\rho}}_{\text{tot}} = \frac{1}{i\hbar}\left[H, \tilde{\rho}_{\text{tot}}\right] \tag{1.5}$$

where

$$\tilde{V}(t) = e^{(1/\hbar)(H_S+H_R)t} V e^{-(1/\hbar)(H_S+H_R)t}. \quad (1.6)$$

Integrating Equation 1.5 and re-substituting the result yields

$$\tilde{\rho}_{\text{tot}} = \frac{1}{i\hbar}\left[\tilde{V}(t), \rho_{\text{tot}}(0)\right] - \frac{1}{\hbar^2}\int_0^t dt' \left[\tilde{V}(t'), \left[\tilde{V}(t'), \tilde{\rho}_{\text{tot}}(t')\right]\right]. \quad (1.7)$$

We can apply some reasonable approximations to this expression before continuing with our analysis:

1. *Born Approximation.* Because R is a large system, its state should be unaffected by its coupling to S, allowing us to neglect higher-order terms of V:
$$\tilde{\rho}_{\text{tot}}(t) \approx \rho_S(t) \otimes \rho_R(0) = \rho_S(t) \otimes \rho_R. \quad (1.8)$$

2. *Weak-Coupling Assumption.* We assume that S and R are initially uncorrelated, such that the total density operator factorizes as
$$\rho(0)_{\text{tot}} = \rho_S(0) \otimes \rho_R. \quad (1.9)$$
The reservoir operators coupling to S will have zero mean in the state ρ_R, we have
$$\frac{1}{i\hbar}\text{tr}_R\left[\tilde{V}(t), \rho_{\text{tot}}(0)\right] = 0 \quad (1.10)$$
because $\text{tr}_R\left[\tilde{V}\rho_R(0)\right] = 0$.

3. *Markov Approximation.* Our integrated expression implies that the evolution of $\rho_{\text{tot}}(t)$ depends on its past history through integration of $\rho_{\text{tot}}(t')$. Because of the Born approximation, we do not assume that the reservoir keeps a memory of system states long enough to effect the state of the system. With this in mind, time evolution is dependent on the current state of the system such that
$$\rho_{\text{tot}}(t') \approx \rho_{\text{tot}}(t). \quad (1.11)$$

Using these assumptions and applying Equation 1.3 to Equation 1.7, we finally have

$$\dot{\tilde{\rho}}_S = -\frac{1}{\hbar^2}\int_0^t dt' \text{tr}_R\left[\tilde{V}(t'), \left[\tilde{V}(t'), \tilde{\rho}_S(t) \otimes \rho_R\right]\right]. \quad (1.12)$$

1.2. Optical Master Equation for a Damped Harmonic Oscillator

For some physical context, let's examine the system-reservoir interaction master equation given by 1.12 when applied to a single mode cavity depicted in Figure 1. The action of probing by an external reservoir of harmonic oscillators ('thermal light') will provide some intuition

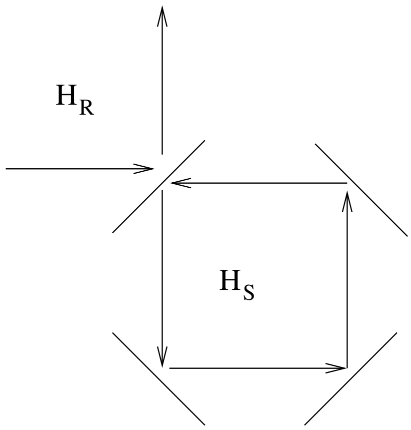

FIG. 1: Damping of a cavity by a bath of harmonic oscillators. The cavity mode specified by H_S is coupled to an external reservoir H_R, which can be thought of as a probing laser.

for measurement damping that we will consider later on in Section 1.3.

The depicted ring cavity (S) couples to the probing reservoir (R) through a dichroic mirror. The composite Hamiltonian in Equation 1.1 is described by

$$\begin{aligned} H_S &= \hbar\omega_c a^\dagger a, & (1.13)\\ H_R &= \sum_j \hbar\omega {b_j}^\dagger b_j, & (1.14)\\ V &= \sum_j \hbar\left(\kappa_j a {b_j}^\dagger + \kappa_j a^\dagger b_j\right) & \\ &= \hbar\left(a\Gamma^\dagger + a^\dagger\Gamma\right). & (1.15)\end{aligned}$$

Here, the cavity mode system S is assumed (for simplicity) to have a zero energy ground state and a characteristic frequency ω_c. The reservoir R is a collection of harmonic oscillators of frequency ω_j, and couples to S via a weak coupling constant κ_j. Furthermore, the reservoir is in thermal equilibrium at temperature T is specified by the mixed state

$$\rho_R = \prod_j e^{-\hbar\omega {b_j}^\dagger b_j/k_B T}\left(1 - e^{-\hbar\omega_j/k_B T}\right), \quad (1.16)$$

with mean population

$$\bar{n}(\omega_j, T) = \text{Tr}\left(\rho_R {b_j}^\dagger b_j\right) = \frac{e^{-\hbar\omega_j/k_B T}}{1 - e^{-\hbar\omega_j/k_B T}}. \quad (1.17)$$

Applying the interaction potential and the reservoir state specified above, it can be shown that its state follows the equation of motion

$$\begin{aligned}\dot{\rho} &= -i\omega_0\left[a^\dagger a, \rho\right] + \tfrac{\kappa}{2}\bar{n}\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right)\\ &\quad + \tfrac{\kappa}{2}(\bar{n}+1)\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right). \quad (1.18)\end{aligned}$$

The first term of this master equation is the standard equation of motion for a closed quantum system, and two additional terms describe the damping of the cavity mode while weakly interacting with an external reservoir. This damping is not particularly surprising, considering that the simple interaction potential implies a likelihood that a cavity mode can lose energy by creating reservoir quanta (or vice versa). In the next section, we will see that *measurement backaction* terms in the optical master equation are characteristic of stochastic evolution in continuously measured quantum systems.

1.3. General Form of the Stochastic Master Equation

While our consideration of the system-reservoir interactions and the damping of the quantum harmonic oscillator accurately described the backaction that might result from a continuous measurement process on an open quantum system, it omitted a term of Gaussian *observation noise* introduced in the measurement process. Following Jacob and Steck [4], we will derive the most general master equation accounting for Gaussian noise. We will see the necessity of formally accounting for Gaussian observation noise in Section 3, as homodyne-mediated optical feedback introduces extra photocurrent shot noise into our observed system. For completeness, we've included an appendix on the stochastic calculus needed for understanding white noise processes in our derivations.

Our goal in the following is to consider the infinitesimal evolution of a quantum state ρ under measurement backaction and noise. In an infinitesimal time interval dt, a closed quantum state ρ undergoes unitary time evolution following Schrodinger's wave equation:

$$\begin{aligned}\rho + d\rho &= \left(1 - i\frac{H}{\hbar}dt\right)\rho\left(1 + i\frac{H}{\hbar}dt\right) \\ &= \rho + \frac{1}{i\hbar}[H,\rho]\,dt. \end{aligned} \quad (1.19)$$

More generally, any physical operation on a density operator must preserve the *positivity* of the density operator: the property that density operators have nonnegative eigenvalues indicating probabilities. From Nielsen and Chuang [5], the most general form of a completely positive transformation is given by

$$\rho' = \sum_n E_n \rho E_n^\dagger, \quad (1.20)$$

where $\{E_n\}$ are some operators mapping the input Hilbert space of ρ to the output Hilbert space of ρ'.

Adding a proportional white noise term dW to the infinitesimal unitary transformation in Equation 1.19 yields a general stochastic operator

$$E = 1 - i\frac{H}{\hbar}dt + bdt + cdW, \quad (1.21)$$

where b and c are operators. Substituting this operator into Equation 1.20 yields

$$d\rho = \frac{1}{i\hbar}[H,\rho]\,dt + \{b,\rho\}dt + c\rho c^\dagger dt + \left(c\rho + \rho c^\dagger\right)dW \quad (1.22)$$

where $\{A, B\} = AB + BA$ denotes the anticommutator. Taking the ensemble average of Equation 1.22 over all Wiener processes (denoted by $\langle\langle\ \rangle\rangle$) gives

$$d\langle\langle\rho\rangle\rangle = \frac{1}{i\hbar}[H,\langle\langle\rho\rangle\rangle]\,dt + \{b,\langle\langle\rho\rangle\rangle\}dt + c\langle\langle\rho\rangle\rangle c^\dagger dt. \quad (1.23)$$

Because $\langle\langle\rho\rangle\rangle$ is an average over valid density operators, it too is a density operator for which $\mathrm{Tr}\left[\langle\langle\rho\rangle\rangle\right] = 1$. Taking the trace of this expression, and using the cyclic property of the trace, we find that

$$\mathrm{Tr}\left[\langle\langle\rho\rangle\rangle\left(2b + c^\dagger c\right)\right], \quad (1.24)$$

which will only hold if $b = -c^\dagger c/2$. If we perform the substitution for b and temporarily ignore the Gaussian noise term, we find an expression remarkably similar to damped harmonic oscillator master equation. Letting $\mathcal{L}_0\rho = -i/\hbar\,[H,\rho]$, we've provided an argument for what is known as the *Lindblad form* of the master equation

$$\dot{\rho} = \mathcal{L}_0\rho + \mathcal{D}\left[c\right]\rho, \quad (1.25)$$

where $\mathcal{D}\left[c\right]$ is the standard measurement damping ('backaction') term

$$\mathcal{D}[c] = \mathcal{J}[c] - \mathcal{A}[c], \quad (1.26)$$

for which

$$\begin{aligned}\mathcal{J}[c]\rho &= c\rho c^\dagger, & (1.27)\\ \mathcal{A}[c]\rho &= \frac{1}{2}\left(c^\dagger c\rho + \rho c^\dagger c\right). & (1.28)\end{aligned}$$

There are several alterations to be made to Equation 1.25 before we can fully account for the Gaussian noise term $\left(c\rho + \rho c^\dagger\right)dW$. The first is that trace preservation of the density operator $d\rho$ implies that the operator c is constrained by the condition

$$\mathrm{Tr}\left[\rho\left(c + c^\dagger\right)\right] = 0. \quad (1.29)$$

This constraint can be accounted for by explicitly including the constraint as part of the noise term $\mathcal{H}\left[c\right]dW$ in the full master equation

$$d\rho = \mathcal{L}_0\rho dt + \mathcal{D}[c]\rho dt + \mathcal{H}\left[c\right]\rho dW, \quad (1.30)$$

such that

$$\mathcal{H}\left[c\right]\rho = c\rho + \rho c^\dagger - \mathrm{Tr}\left[c\rho + \rho c^\dagger\right] \quad (1.31)$$

represents the information gain due to the measurement process. Our remaining change is to include a efficiency constant η accounting for the fraction of measured signal reaching a defector and the inherent quantum efficiency of the detector. Including that loss for a single measurement channel, dividing through by dt, and letting $\mathcal{L} = \mathcal{L}_0 + \mathcal{D}$, the general form of the master equation governing the completely positive evolution of a quantum state in the presence of real white noise $\xi\left(t\right) = dW/dt$ is

$$\dot{\rho} = \left(\mathcal{L}\left[c\right]dt + \sqrt{\eta}\xi\left(t\right)\mathcal{H}\left[c\right]\right)\rho. \quad (1.32)$$

2. WEAK MEASUREMENT

The master equation given in Equation 1.32 has several features that are worth interpreting in the context of physical observables and measurement. In particular, we are interested in finding out how stochastic state evolution from continuous measurement is related to the projective and POVM formalism we discussed in class. This relationship will help us understand the partial wavefunction collapse that occurs during quantum non-demolition (QND) measurements.

Our prior POVM formalism described measurement in terms of operators Π_m acting as projectors onto a Hilbert space. Instead, we can define our measurement operators as a weighted sum of projectors onto the eigenstates $|n\rangle$, each centered around the eigenvalue of an eigenstate of that observable. Assuming, without loss of generality, that we can label the countable eigenvalues of an observable O by the integer number states n, our weak measurement is described by the Gaussian POVM

$$\Pi_m = \frac{1}{\mathcal{N}} \sum_n e^{-k(n-m)^2/4} |n\rangle \langle n| , \tag{2.1}$$

where $\mathcal{N}$ is a normalization constant chosen such that $\{\Pi_m\}$ resolve the identity. The initial state of a system for which we have no information is a mixed state proportional to I. The result m immediately following Gaussian POVM measurement of that state is

$$\rho_f = \frac{\Pi_m \rho \Pi_m^\dagger}{\mathrm{Tr}\left[\Pi_m \rho \Pi_m^\dagger\right]} = \frac{1}{\mathcal{N}} \sum_n e^{-k(n-m)^2/4} |n\rangle \langle n| , \tag{2.2}$$

which has a peak centered about the eigenvalue m and a spread given by $1/\sqrt{k}$. Therefore, the uncertainty in our measurements is inversely proportional to k. Measurements with large k are called *strong*, whereas those with small k called *weak*. This formalism allows us to consider *continual* weak measurements in a sequence of infinitesimal intervals dt. We will find that Equation 1.32 is equivalent to a continuous sequence of Gaussian POVMs.

To see why this might be true, let's consider infinitesimal observation of a continuous spectrum of eigenstates $|x\rangle$, where $\langle x'|x\rangle = \delta(x - x')$. The Gaussian POVM for this measurement can be labeled continuously by α such that,

$$A(\alpha) = \left(\frac{4k\Delta t}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} dx e^{-2k\Delta t(x-\alpha)^2} |x\rangle \langle x| \tag{2.3}$$

which yields a probability distribution

$$P(\alpha) = \mathrm{Tr}\left[A(\alpha)^\dagger A(\alpha) |\psi\rangle \langle\psi|\right] \tag{2.4}$$

such that $\langle\alpha\rangle = \langle x\rangle$. When Δt is sufficiently small, the Gaussian weight is much broader than $\psi(x)$, which can then be approximated by a delta function centered at $\langle\alpha\rangle$, giving

$$\begin{aligned} P(\alpha) &= \sqrt{\frac{4k\Delta t}{\pi}} \int_{-\infty}^{\infty} dx |\psi(x)|^2 e^{-2k\Delta t(x-\alpha)^2} \\ &\approx \sqrt{\frac{4k\Delta t}{\pi}} e^{-2k\Delta t(\alpha - \langle x\rangle)^2}, \end{aligned} \tag{2.5}$$

which is a Gaussian probability distribution centered at the result.

We can derive the equation of motion for the system by calculating the first-order change by the application of a single continuous measurement $A(\alpha)$. If we write α as a stochastic variable

$$\alpha \to \langle X\rangle + \frac{\Delta W}{\sqrt{8k}\Delta t}, \tag{2.6}$$

take the first-order approximation with $\Delta t \to dt$ and $\Delta W \to dW$, and apply the Ito rule $dW^2 = dt$, this new state is given by

$$\begin{aligned} |\psi(t+dt)\rangle &\propto A(\alpha) |\psi(t)\rangle \\ &\approx \Big[1 - \left(kx^2 - 4kx\langle x\rangle\right) dt \\ &\qquad + \sqrt{2k} x dW\Big] |\psi(t)\rangle . \end{aligned} \tag{2.7}$$

Writing $|\psi(t+dt)\rangle = |\psi(t)\rangle + d|\psi(t)\rangle$, we find that

$$d|\psi\rangle = \left[-k(x - \langle x\rangle)^2 dt + \sqrt{2k}(x - \langle x\rangle) dW\right] |\psi(t)\rangle \tag{2.8}$$

and therefore,

$$d\rho = -k[x, [x, \rho]] dt + \sqrt{2k}(x\rho + \rho x - 2\langle x\rangle\rho) dW, \tag{2.9}$$

as $\rho(d + dt) = \rho(t) + d\rho$. This expression is precisely of the form in Equation 1.32 if $x = c/\sqrt{2k}$, c is Hermitian, and $\xi(t) = dW/dt$. Therefore, the most general stochastic evolution of a system experiencing noise is, conditionally, equivalent to performing a continual weak measurement through Gaussian-weighted POVMs.

3. OPTICAL FEEDBACK VIA HOMODYNE DETECTION

As an application of this density matrix formalism for continuous observation, we will now consider the dynamics of quantum-limited photocurrent feedback on a source cavity, as depicted in Figure 2. Our discussion is based on the thesis work of H.M. Wiseman, who carried out much of the early work applying quantum continuous measurement towards feedback in familiar optical systems like homodyne detection and quantum non-demolition measurement [2, 6, 7, 8]. The usefulness of feedback in our example will be evaluated in the context of quadrature squeezed light of a laser.

First, we'll have to find a master equation describing feedback linear in photocurrent. In what follows, we'll

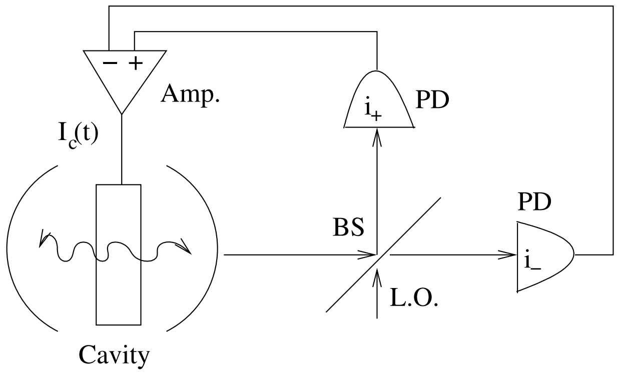

FIG. 2: Homodyne-mediated feedback of an optical cavity. The cavity has some output spectrum, such as pumped laser light, measured by a homodyne detector. The abbreviations are: photodetector (PD), local oscillator (LO), and beam-splitter (BS).

simplify our analysis by allowing the feedback delay time to be negligible compared to the system's response time. The photocurrent from the extracavity homodyne measurement of the a_1 quadrature includes the local oscillator shot noise $\xi(t)$ such that

$$I(t) = \eta\langle a + a^\dagger\rangle(t) + \sqrt{\eta}\xi(t), \tag{3.1}$$

where the first term is an ensemble average over the condition quantum state ρ. The effect of this continuous homodyne measurement is given precisely by Equation 1.32. The simplest way we can introduce feedback into our system is by adding a feedback Hamiltonian H_{fb} to the total system Hamiltonian H discussed originally in Section 1. In our continuous observation formalism, this is tantamount to adding a density operator evolution term

$$[\dot{\rho}(t)]_{\text{fb}} = -i[H_{\text{fb}}, \rho(t)]. \tag{3.2}$$

A feedback circuit would use the photocurrent to control some electro-optic or electromechanical device to influence the source cavity, and so we may assume, by construction, that the feedback term we are adding is linear in photocurrent,

$$[\dot{\rho}]_{\text{fb}} = \left[\langle a + a^\dagger\rangle(t) + \xi(t)/\sqrt{\eta}\right]\mathcal{K}\rho, \tag{3.3}$$

where $\mathcal{K}$ is some superoperator based on the particular behavior of cavity we are choosing[1].

Adding Equation 1.32 to Equation 3.3 is more complicated than it sounds. From our discussion in Appendix A, we know that former is an Ito equation, whereas the latter is an Stratonovich equation. Converting Equation 1.32 into the Stratonovich form gives

$$\dot{\rho}_c^{(S)} = \left[\mathcal{L} + \sqrt{\eta}\xi(t)\mathcal{H} - \frac{1}{2}\eta\mathcal{H}^2\right]\rho_c \tag{3.4}$$

[1] The construction of the operator $\mathcal{K}$ is outside of the scope of this paper. We will assume a particular form later.

which added to Equation 3.3 yields

$$\begin{aligned}\dot{\rho}_c^{(S)} &= \left[\mathcal{L} - \tfrac{1}{2}\eta\mathcal{H}^2 + \langle a + a^\dagger\rangle(t)\mathcal{K}\right]\rho \\ &\quad + \sqrt{\eta}\xi(t)\left(\mathcal{H} + \eta^{-1}\mathcal{K}\right)\rho.\end{aligned} \tag{3.5}$$

Applying the definition of $\mathcal{H}$ in Equation 1.31 and converting back to the Ito form

$$\begin{aligned}\dot{\rho}_c^{(I)} &= \mathcal{L}\rho_c + \mathcal{K}\left(a\rho_c + \rho_c a^\dagger\right) + \tfrac{1}{2\eta}\mathcal{K}^2\rho_c \\ &\quad + \sqrt{\eta}\xi(t)\left(\mathcal{H} + \eta^{-1}\mathcal{K}\right)\rho\end{aligned} \tag{3.6}$$

and averaging over the noise $\xi(t)$ by $\rho = E[\rho_c]$ gives

$$\dot{\rho} = \mathcal{L}\rho + \mathcal{K}\left(a\rho + \rho a^\dagger\right) + \frac{1}{2\eta}\mathcal{K}^2\rho, \tag{3.7}$$

which is the general equation for homodyne-mediated feedback.

We can now evaluate this feedback scheme in the context of a useful application: using the phase-locking of a regularly-pumped laser to produce a near-minimum uncertainty squeezed states. From our previous class discussions, we know that the equilibrium state of a laser is a mixture of equal amplitude coherent states with Poisson-distributed photon number statistics. On the other hand, a more regularly pumped laser can have a nonclassical equilibrium state with a photon distribution narrower than the classical Poisson distribution. In other words, it is possible to start with an equilibrium state that has a sub-shot noise intensity spectrum that is far from minimum uncertainty. We can use a homodyne-mediate feedback loop to take advantage of the nonclassical behavior of such a pumped laser by stabilizing its phase noise relative to a local oscillator, allowing us to pick a preferred quadrature to be squeezed. We can achieve this phase-locking by using the feedback homodyne current to alter the effective optical length of the cavity using an elecro-optic modulator.

An ideal laser obeying Poissonian statistics, with a pump rate $\mu \gg 1$ is given by a mixture of coherent states $\left|\sqrt{\mu}e^{i\phi}\right\rangle$ $(0 \leq \phi \leq 2\pi)$ whose dynamics are described by the master equation[9]

$$\dot{\rho} = -\frac{1+\nu}{4\mu}\left[a^\dagger a\left[a^\dagger a, \rho\right]\right], \tag{3.8}$$

where $\nu \geq 0$ represents the excess phase noise in the laser above Heisenberg uncertainty. For small changes in the path length of the optical cavity, the superoperator in Equation 3.3 is defined by

$$\mathcal{K}\rho = i\frac{\lambda}{2\sqrt{\mu}}\left[a^\dagger a, \rho\right] \tag{3.9}$$

which we substitute into Equation 3.7 to yield the feedback master equation

$$\dot{\rho} = -\frac{1+\nu+\lambda^2/2\eta}{4\mu}\left[a^\dagger a\left[a^\dagger a, \rho\right]\right] + i\frac{\lambda}{2\sqrt{\mu}}\left[a^\dagger a, a\rho + \rho a^\dagger\right]. \tag{3.10}$$

We can parametrize the solution of this expression in terms of the angle ϕ from the coherent state amplitude $\alpha = \sqrt{\mu}e^{i\theta}$. It can be shown that Equation 3.10 can be written in the $P(\alpha, \alpha^*)$-representation as

$$\dot{P}(\phi) = \left[-\frac{\partial}{\partial \phi}\lambda \cos\phi + \frac{1}{2}\frac{\partial^2}{\partial \phi^2}\frac{1+\nu+\lambda^2/2\eta}{2\mu} \right] P(\theta). \tag{3.11}$$

Let η be the fraction of emitted light used in the feedback process and q parametrize the quality of squeezing (e.g., $q = 0$ for coherent light and $q < 0$ if one quadrature is sub-shot noise). If the fraction θ of light available as output is at most $1 - \eta$, the noise spectral density is

$$\begin{aligned} S_{a_1}(\omega) &= 1 + \frac{\theta\left[\lambda^2/\eta + 2(1+\nu)\right]}{\lambda^2 + \omega^2} \\ S_{a_2}(\omega) &= 1 + \frac{\theta q}{1 + \omega^2}. \end{aligned} \tag{3.12}$$

Letting $\lambda \to \infty$, the optimum output noise reduction at zero frequency is given by

$$S_{a_1}(0) = 1/\eta, \qquad S_{a_2}(0) = \eta, \tag{3.13}$$

which exhibits the two properties of perfect squeezing on resonance by (i) satisfying a minimum-uncertainty relation $S_{a_1}(0) S_{a_2}(0) = 1$, and (ii) allowing zero noise to be attained in the a_2 quadrature by letting $\eta \to 1$. As a result, we can conclude that feedback is useful for turning nonclassical laser light with sub-Poissonian number statistics into perfect quadrature squeezing.

4. CONCLUSIONS AND RECOMMENDED READING

We've presented a density operator formalism which we can (i) use density operators to understand measurement processes in quantum feedback control, and (ii) actually analyze simple problems in quantum optical systems when using linear, homodyne-mediated feedback. This process has made as aware of two nonclassical effects when performing continuous measurement of a quantum system—measurement backaction (damping, really) and Gaussian stochastic noise—and has demonstrated that quantum feedback can enhance squeezing and noise reduction.

This noise reduction comes with a caveat, namely, that nonclassical states cannot be produced unless the cavity itself is already produced light that is at least partially squeezed. Unfortunately, it has been shown that incorporating finite time-delays and realistic loss into this feedback scheme ultimately removes the feedback advantage altogether and may even degrade what squeezing may have been present to begin with [8]. On a slightly ambivalent note, it was also shown that combining homodyne-mediated feedback with an *intracavity* QND measurement can enhance squeezing without degrading output, but that measurement rate of pump photons in the process would (rather unrealistically) need to be many orders of magnitude larger than the cavity linewidth.

The following sources were useful in building a context for this paper:

1. *An Open Systems Approach to Quantum Optics* (H. Carmichael) and *Quantum Noise* (C.W. Gardiner) are typically cited when referring to master equations, open quantum systems, and stochastic processes in quantum mechanics.

2. The thesis work of Howard Wiseman found at http://www.cit.gu.edu.au/~s285238/, which includes several of the papers mentioned in this paper, provides a cohesive view of continuous measurement and feedback in quantum optical systems.

3. There are many theoretical proposals using Wiseman's feedback formalism, although few of them have actually been experimentally implemented. Homodyne-mediated feedback was recently applied by Mabuchi, et al. for adaptive measurement of optical phase [10].

5. APPENDIX A: QUANTUM STOCHASTIC CALCULUS

An intuition for the calculus of noise processes is important when analyzing the continuous measurement and feedback of quantum systems. Following Wiseman [2], and Jacobs and Sheck [4], we will provide some intuition for stochastic calculus and a summary of useful identities that are used in the paper. We will focus largely on the first of two different formulations of stochastic differential equations (SDEs)—Ito and Stratonovich quations—but provide a rudimentary intuition relating the two when necessary.

White noise in quantum continuous measurement is represented by a *Wiener process* $W(t)$, an ideal random walk with small, statistically-independent steps taken arbitrary often. In particular, any random walk $W(t)$ can be represented as a zero-mean Gaussian random variable following a probability density

$$P(W, t) = \frac{1}{\sqrt{2\pi t}} e^{-W^2/2t}. \tag{5.1}$$

Whereas a regular differential equation may be given by $dx/dt = \alpha$, an Ito stochastic differential equation carries an additional differential element dW such that

$$dx = \alpha dt + \beta dW. \tag{5.2}$$

The basic rule of applying a differential dW is that $[dW(t)]^2 = dt$, while $dt^2 = dtdW = 0$.

To see how a Wiener increment might change our rules of calculus, let's take a moment to the second-order Taylor expansion for a function $f(x)$:

$$df(x) = f'(x)\,dx + \frac{1}{2}f''(x)\,dx^2. \tag{5.3}$$

Including a Wiener increment and applying the rule that $[dW(t)]^2 = dt$ yields

$$df(x) = \left[f'(x) + \frac{1}{2}f''(x)\,b(x)^2\right]dt + f'(x)\,b(x)\,dW(t). \tag{5.4}$$

Calculating the differential dz for a simple exponential like $f(x) = z = e^x$ in the deterministic calculus gives $dz = z\alpha dt$, whereas the second-order expanion in the stochastic calculation yields,

$$dz = z\left(\alpha + \frac{1}{2}\beta^2\right)dt + z\beta dW. \tag{5.5}$$

These derivatives show that Ito SDEs are difficult and fairly different from our normal conception of calculus—not even the chain rule applies! Ito SDEs, however, are still useful because noise increments are statistically independent of (and commute with) physical observables, a fact that we will use in our derivation of a general stochastic master equation. An alternative stochastic equation can be given in the which this fact is not true is the Stratonovich form,

$$\dot{x} = \alpha(x) + \beta(x)\,\xi(t) \tag{5.6}$$

where $\alpha(x)$ and $\beta(x)$ can be arbitrary real functions, and $\xi(t)$ is a rapidly varying stochastic continuous function of time such that

$$dW(t) = \xi(t)\,dt. \tag{5.7}$$

In the case of homodyne detection, $\xi(t)$ can be idealized of as local oscillator shot noise, and as Gaussian white noise more generally. Because the average and correlation time for such a noise process is zero, we also expect that

$$E[\xi(t)\,\xi(t')] = \delta(t-t') \text{ and } E[\xi(t)] = 0. \tag{5.8}$$

In our derivations, taking ensemble averages over noise processes will be the equivalent to finding the equation of a system undergoing continuous measurement. This average will cause terms that are proportional to white noise terms to go to zero.

Applying differential calculus, it is possible to convert between the forms easily. The Ito form of the Stratonovich form in Equation 5.6 is

$$dx = \left[\alpha(x) + \frac{1}{2}\beta(x)\,\beta'(x)\right]dt + \beta(x)\,dW(t). \tag{5.9}$$

Similarly, the Ito equation

$$dx = a(x)\,dt + b(x)\,dW(t) \tag{5.10}$$

has the Stratonovich equivalent

$$\dot{x} = a(x) - \frac{1}{2}b(x)\,b'(x) + b(x)\,\xi(t). \tag{5.11}$$

Ultimately, these classical variables x will be replaced by density operators ρ, and we will want to derive the time-varying probability distributions. The time-evolution equation (also known as a Fokker-Planck) equation for the probability distribution P of a random variable x described by Equation 5.10 is

$$\dot{P}(x) = \left[-\partial_x a(x) + \frac{1}{2}\partial_x^2 b(x)^2\right]P(x). \tag{5.12}$$

This expression will be useful in deriving $P(\alpha, \alpha^*)$-representation of the master equation in our treatment of homodyne-mediated feedback.

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