MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.007 – Applied Electromagnetics

Fall 2009

Tutorial Schedule

Tutorial notes written by William Herrington and Bhaskar Mookerji, Fall 2009.

Week Number	Date	Topic	PS Due
0	07 September	(Registration Day)	
1	14 September	Math Review	1
2	21 September	Electrostatics and Magnetostatics	2
3	28 September	Magnetic Materials, Circuits, and Actuators	3
4	05 October	Exam Review	
Midterm I			
5	13 October (Tuesday)	Magnetic Circuits and Actuators	4
6	19 October	Electric Machines	5
7	26 October	Electromagnetic Waves and Polarization	6
8	3 November	Electromagnetic Waves at Boundaries	7
9	9 November	Exam Review	
Midterm II			
10	16 November	Electromagnetic Waves at Oblique Incidence	8
11	23 November	Quantum Mechanics in 1-D Potentials	—
12	30 November	Tunneling and Flash Memory	9
13	7 December	Band Structure and Semiconductors	
Final			

Table 1: Schedule of tutorials this term.

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Source Material

These tutorial notes draw on the great wealth of course notes and textbooks from device engineering and physics classes in Course 6.

1. **6.013 Electromagnetics and Applications and 6.632 Electromagnetic Wave Theory** are great follow-up courses in electromagnetics. 6.013 has a different applications focus than 6.007, and 6.632 has a fairly thorough coverage of birefringent media and metamaterials (dielectrics with negative indexes of refraction!). Several problems and expositions on electromagnetic waves, polarization, and phase matching at boundaries are sourced from this material.

2. **6.453 Quantum Optical Communications** covers the intersection of quantum mechanics, optics, and linear systems (6.011), with a particular emphasis on communications applications of quantum mechanics (such as quantum teleportation and cryptography). Some discussion on quantum mechanics and phasor notation are borrowed.

3. **6.730 Physics for Solid-State Applications and 6.720 Integrated Microelectronic Devices** are graduate classes in solid state physics and its applications to modern semiconductor devices, such as integrated circuits. These classes are great if you're interested in understanding 6.007's later topics, such as band structure, carrier transport, transistor design, etc. 6.728 and 6.012 are definitely functional prerequisites. The last tutorial on PN-junctions and III-V doping borrows some exposition from here.

4. **6.161 Modern Optics Project Laboratory** is course 6's optics lab, which will definitely be useful if you do any optics, photonics, or laser related research at MIT (UROP, MEng, and beyond!). If you enjoyed lab 4 (the optics and absorption lab), working with lasers, or having Bill as your TA, you should check it out.

Textbooks

1. Jin A. Shen, Liang C.; Kong. *Applied Electromagnetism.* PWS, Cambridge, Mass., 1987.

2. David Griffiths. *Introduction to Electrodynamics.* 2nd edition, 1999.

3. David Griffiths. *Introduction to Quantum Mechanics.* 2nd edition, 1999.

4. J. A. Kong. *Electromagnetic Wave Theory.* EMW, Cambridge, Mass., 2000.

5. Paul Lorrain and Dale R Corson. *Electromagnetic fields and waves.* W. H. Freeman San Francisco, 2nd edition, 1970.

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Errata

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1. Written by William Herrington and Bhaskar Mookerji. Some obvious bugs removed in QM and device sections.

"Given enough eyeballs, all bugs are shallow."

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Tutorial 1: Work, Units, and Math Review

Some pointers:

1. Labs are on Monday and Tuesday evening (7–9PM), with pizza at 6:30PM. Make sure your pre-labs are done before lab.
2. The first problem set is due this Thursday (September 17) in lecture.
3. Office hours will be held on Wednesday in a room TBD.
4. Notes will be handed out in tutorial as an incentive for attendance. Let your TA know if you have conflicts.

1 Work and Power: Through and Across Variables

- **'Across' Variables.** Generally answers 'how hard we are pushing?'
- **'Through' Variables.** Generally answers 'how much stuff is flowing?'
- **Power:**
  - The product of a through and an across variable.
  - The rate energy is being expended.
  - Units of Watts.

Example 1: **Through and Across Variables**

Match the following *across* variables with the correct *through* variables.

Units	Across	Through	Units
________	Force	Current	________
________	Voltage	Flow	________
________	Torque	Velocity	________
________	Pressure	Angular Velocity	________

2 Units

On the homework set, Problem 1.2 relies heavily on the manipulation of units.

- What are the SI units for the **'Through'** and **'Across'** variables in the table above?
- The SI unit for Power is the Watt which is equal to ______________________
- What is a Joule?

Example 2: **Batteries**

- Referencing problem 1.2, how much energy is stored in a 1 $\mathrm{W} \cdot \mathrm{hr}$ battery?
  ____
- What about a 1 amp hour battery?
  ____

Example 3: **Application of Units to energy conversion**

- What is the question: 'How high would your Prius rise if the battery's energy were used to propel it skyward?' really asking?
- At what ______________________ is the ______________________ equal to the battery's initial energy?

3 Differential Operations and Integration

3.1 The ∇ Operator

The ∇ ('nabla' or 'del') operator quite a lot when you're thinking about electrodynamics, particularly plane waves.

What is the ∇ operator?

- ∇ is a vector operator defined as $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$, in the normal cartesian coordinate system.
- This operator can be applied to both scalar and vector fields.
  1. **Option 1.** Application to a scalar field $f(x, y, z)$:
     - $\nabla f =$ ____

- What is this operation called? ____________________
- What does it represent? ____________________

2. **Option 2: Dot Product.** Application to a vector field $\vec{v} = \hat{x}v_x + \hat{y}v_y + \hat{z}v_z$
   - $\nabla \bullet \vec{v} =$ ____________________
   - What is this operation called? ____________________
   - What does it represent? ____________________
3. **Option 3: Cross Product.**
   - $\nabla \times \vec{v} =$
   - What is this operation called? ____________________
   - What does it represent? ____________________

3.2 Vector Integration

The usage of integration in class will focus on line and surface integral solutions of Maxwell's equations. In the following, we'll outline a few example cases which will show up frequently:

1. **Potentials.**

$$\int d\mathbf{r} \cdot \mathbf{E} = \int_a^b dr\hat{r} \cdot E(r)\,\hat{r} = \int_a^b dr E(r) \tag{1}$$

2. **Surface integrals (using Gaussian surfaces).**

$$\int d\mathbf{A} \cdot \mathbf{E} = \int dA\hat{r} \cdot E(r)\,\hat{r} = E(r) \int dA \tag{2}$$

3. **Line integrals (using Faraday loops).**

$$\int d\mathbf{r} \cdot \mathbf{E} = \int dr\hat{r} \cdot E(r)\,\hat{r} = E(r) \int dr \tag{3}$$

4 Complex Numbers and Phasors

Real physical quantities that vary periodically with time, such as alternating-current (AC) voltage or the electric fields of lasers, are called *time-harmonic*. Analyzing these quantities with complex representations (or *phasors*) will simplify our analysis of them later in the course.

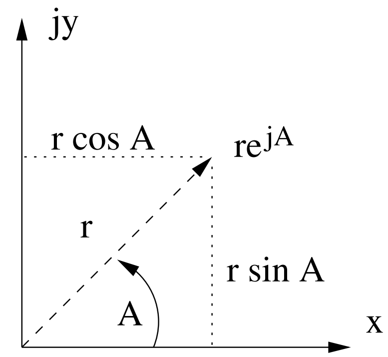

Figure 1: Euler's Theorem in the complex plane.

4.1 Complex Arithmetic

Complex numbers speak in two dialects: rectangular and polar form. A complex number $\tilde{z} = x + jy$[1] can be represented as a phasor using Euler's formula:

$$\tilde{z} = x + jy = re^{j\theta} = r\cos\theta + jr\sin\theta, \tag{4}$$

where

$$r = \text{mag}\,(\tilde{z}) = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \arctan\left(\frac{y}{x}\right). \tag{5}$$

This relation is polar, so you can imagine it geometrically, as in Figure 1. Also remember that the real and imaginary parts are given by

$$x = \text{Re}\,[\tilde{z}] = \frac{z + z^*}{2} \qquad \text{and} \qquad y = \text{Im}\,[\tilde{z}] = \frac{z - z^*}{2j}. \tag{6}$$

Example 4: **Phasors**

What are the phasor representations of

$$-\frac{1}{2} + j\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \sqrt{2}\,(1 + j)\,? \tag{7}$$

[1] You may have noticed that we're using 'j' (the Course 6 convention) instead of 'i' (the Course 8 convention). We'll be using 'j' in the EM-waves part of 6.007, and 'i' when looking at quantum mechanics (a handout on the reasons why later in the course). In the literature, the relation $i \to -j$ usually holds, although it's not always the case.

Example 5: **Sums of Phasors**

We're given two complex numbers z_1 and z_2 with valid phasor representations. What has to be true about them for $z_1 + z_2$ to have a valid phasor representation?

4.2 Phasors and Time-Harmonic Vectors

Using the relation, $\tilde{z} = x + jy = re^{j\theta}$. There are three equivalent representations of a real-valued sinusoid $z(t)$ of frequency ω:

1. Amplitude and phase representation: $z(t) = r\cos(\omega t - \theta)$.
2. Phasor (complex-amplitude) representation: $z(t) = \mathrm{Re}\left[\tilde{z}e^{j\omega t}\right]$.
3. Quadrature-component representation: $z(t) = x\cos(\omega t) + y\sin(\omega t)$.

Example 6: **Decaying Plane Wave**

Simplify $\mathrm{Re}\left[e^{j(\omega t - kz)}\right]$ into its amplitude and phase representation, where $k = k_R + jk_I$.

Example 7: **Time-Harmonic Vector**

Let $\mathbf{C} = j\hat{x} + (1 + j)\,\hat{y} + (3 - j4)\,\hat{z}$. What is $\mathbf{C}(t)$?

Example 8: **Time Harmonic Vector in the Plane**

Let $\mathbf{A} = \hat{x} + j\hat{y}$. How does the tip of $\mathbf{A}$ move as a function of time?

Example 9: **Orthogonality of Time Harmonic Vectors**

Let $\mathbf{B} = j\mathbf{A}$. Is $\mathbf{A}(t) \times \mathbf{B}(t) = 0$ or $\mathbf{A}(t) \times \mathbf{B}(t) \neq 0$?

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Tutorial 2: Electrostatics and Magnetostatics

More pointers:

1. Lab writeups are due on Tuesday (September 22).
2. The second problem set is due this Thursday (September 24) in lecture.

1 Questions about problem set 1?

2 Practice with Electrostatics and Magnetostatics

1. Terminology: **B** vs. **H**?
2. Static Maxwell's Equations:

$$\begin{aligned} \text{Gauss:} \quad & \oint\!\!\!\oint \epsilon_0 d\mathbf{A}\cdot\mathbf{E} = \iiint dV\,\rho_{\text{free}} = Q_{\text{enc}} \qquad \text{Faraday:} \quad \oint d\mathbf{l}\cdot\mathbf{E} = 0 \\ \text{Ampere:} \quad & \oint d\mathbf{l}\cdot\mathbf{H} = \oint\!\!\!\oint d\mathbf{A}\cdot\mathbf{J} = I_{\text{enc}} \end{aligned} \tag{1}$$

Example 1: **Short Questions on Maxwell's Equations**

1. Draw a Faraday (Ampere) loop for finding $\mathbf{B}(\mathbf{r})$ from a line and planar source. Draw a Gaussian surface for finding $\mathbf{E}(\mathbf{r})$ from a point, line, and plane sources.

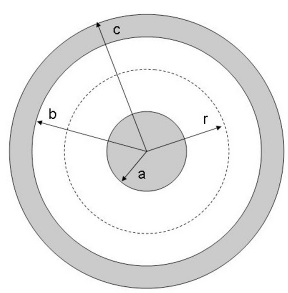
2. What is $\mathbf{E}(\mathbf{r})$ from an infinite sheet of charge density σ?

3. What is $\mathbf{E}(\mathbf{r})$ from an infinite line of charge density λ?

4. What is $\mathbf{H}(\mathbf{r})$ for a long solenoid with coil density $n = N/L$, radius a, and current I? What is $\mathbf{H}(\mathbf{r})$ if the solenoid is made into a toroidal coil with radius b?

Example 2: **Gauss Law: Spherical Capacitor**

Consider a capacitor that consists of a solid conducting sphere of radius a inside a con-



centric conducting spherical shell. The shell has an inner radius b and an outer radius c. A charge of $+q$ coulombs is placed on the inner conductor, and a charge of $-q$ is placed on the outer conductor .

1. For each of the following indicate which statement is correct and explain why.
   - The positive charge on the inner conductor is distributed uniformly ...

     · throughout the spherical volume. · on the surface of the sphere.

- The negative charge on the spherical shell is distributed uniformly . . .

  · on the inner surface. · throughout the shell's volume.
  · on the outer surface. · with $q/2$ on the inner surface and $q/2$ on the outer surface.

2. Setting $\Phi = 0$ at infinity find the potential Φ (r) at:

   - $r = c$
   - $b < r < c$
   - $a < r < b$
   - $r = a$

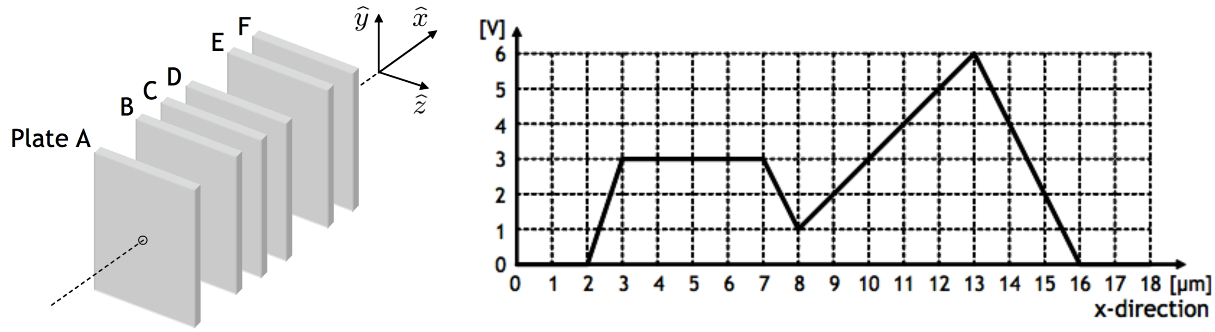
3. What is the expression for voltage between the inner and the outer sphere?

4. What is the expression for capacitance of this capacitor?

Example 3: Charge, Fields, and Electrostatic Potentials (Midterm Spring 2008)

An unusual capacitor structure consisting of a series of six parallel metal plates, each holding a fixed amount of charge in shown in Figure 1. The electrostatic potential within the capacitor is plotted below as a function of the x-direction.

1. From the above plot of the electrostatic potential as a function of x, determine at what x-positions are the six plates located.



(a) An unusual capacitor. (b) Electrostatic potential through the capacitor.

Figure 1: Electrostatic potential in a weird capacitor.

Plate_A	Plate_B	Plate_C	Plate_D	Plate_E	Plate_F
________	________	________	________	________	________

2. What is the electric filed magnitude between each pair of the plates?

$\mathbf{E}_{AB}$	$\mathbf{E}_{BC}$	$\mathbf{E}_{CD}$	$\mathbf{E}_{DE}$	$\mathbf{E}_{EF}$
________	________	________	________	________

3. What is the charge Q on each plate?

Q_A	Q_B	Q_C	Q_D	Q_E	Q_F
________	________	________	________	________	________

4. What two plates could we swap for this capacitor to have a higher electrostatic potential? What's the *new* maximum voltage in the capacitor?

5. Air's breakdown voltage is 3.6×10^6V/m. Will the capacitor experience the breakdown (yes/no)? If so, will the maximum electrostatic potential be higher or lower?

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Tutorial 2: Electrostatics and Magnetostatics

Example 1: **Short Questions on Maxwell's Equations**

1. What is $\mathbf{H}(\mathbf{r})$ for a long solenoid with coil density $n = N/L$, radius a, and current I? What is $\mathbf{H}(\mathbf{r})$ if the solenoid is made into a toroidal coil with radius b?

The goal of this document is to make some clarifications regarding the Ampere's Law questions in the second tutorial. I thought my explanations in class were a bit confusing, so the derivations below will discuss them in greater detail.

We have Ampere's Law, which states that:

$$\text{Ampere:} \quad \oint d\mathbf{l} \cdot \mathbf{H} = \oiint d\mathbf{A} \cdot \mathbf{J} = I_{\text{enc}} \tag{1}$$

and the two examples in question shown in Figure 1. Solving $\mathbf{H}$-field problems typically involves four steps:

1. Define a Faraday Loop to use (there are really only two).
2. Use symmetries of the system (planar, cylindrical, rotational) to determine the direction of $\mathbf{H}$.
3. Determine the enclosed current.
4. Determine the length of the loop that overlaps the field from part 2.

Solenoid

The solenoid is the first example we had where it was possible to solve the problem without truly understanding the full mechanics of the problem. As we discussed in tutorial, two components of the field—H_r and H_ϕ—were 0 because of the cylindrical symmetry of the solenoid. If you remember Prof. Ram's lecture on the planar current sheet in class, you can think a solenoid as a planar current sheet rolled into a cylinder, leaving only H_z (the field in the direction perpendicular to current flow). This leaves two remaining points when we apply the Faraday loop shown in Figure 1.

1. **What exactly is the length of overlap?** Figure 1 has two loops to consider. The first one isn't labeled well, so let's say that its inner edge is r_1 from the origin, and

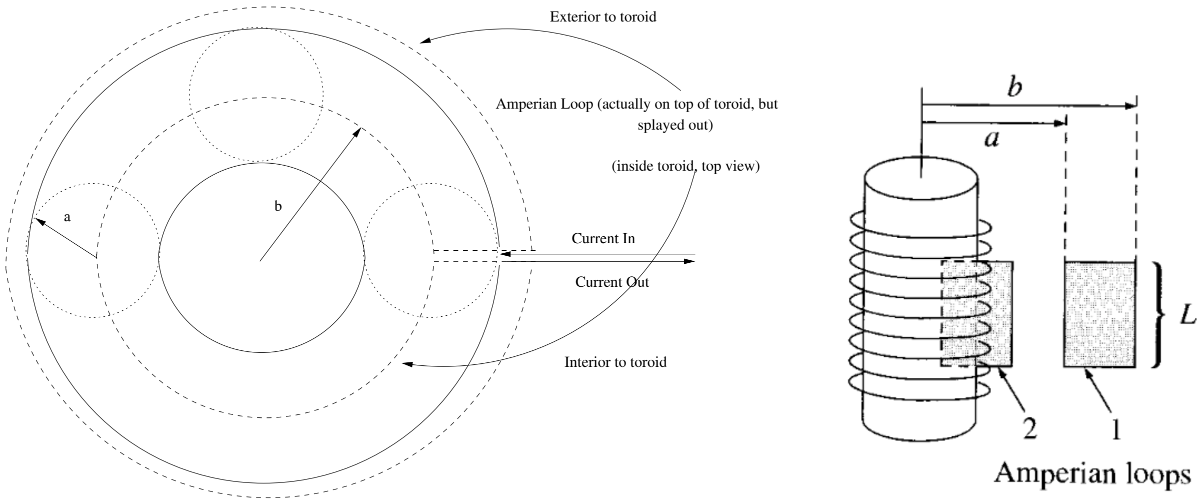

Figure 1: Toroid and solenoid, spread out and detailed view. Second figure taken from David Griffiths' *Introduction to Electrodynamics* (highly recommended, by the way).

that its outer edge is r_2. Each side is of length L. Then for the inner loop:

$$[H_z(r_1) - H_z(r_2)]\, L = I_{\text{enc}} \cdot N \tag{2}$$

and for the outer loop:

$$[H_z(a) - H_z(b)]\, L = 0. \tag{3}$$

This detail wasn't address in tutorial: careful accounting of overlaps is important, and both of these expressions use field overlaps at two different locations. It's easy to forget to do this sometimes, as **H** can sometimes be 0 . . .

2. **What's H outside the solenoid?** The magnetic field outside the solenoid is simply 0. Why is this true? Consider the second Ampere's law expression listed above. The enclosed charge is 0, and therefore the resulting field is zero. An even more formal argument is that $H_z(a) = H_z(b)$ because $I_{\text{enc}} = 0$, which implies that H_z is constant outside the solenoid. This is mighty strange, considering that $H_z(r \to \infty)$ should be 0. Therefore, as $H_z = \text{constant}$, then $H_z = 0$ since $H_z(r \to \infty) = 0$.

Again, this leaves us with the interior of the solenoid:

$$H_z \cdot L = I_{\text{enc}} \cdot N \qquad \to \qquad H_z = nI. \tag{4}$$

Toroid

The toroid is also shown in Figure 1. I had mentioned in tutorial that the toroid was a solenoid wrapped into a circle, providing the most superficial means of doing the problem, namely, that we could set $L = 2\pi b$ and finding that $H_\phi = NI/2\pi b$. Thinking about the problem from scratch is particularly useful, because it points out some features of the toroid that I didn't mention in class,

1. Like the solenoid, the toroid has entry and exit points for the current. The figure I drew from class *incorrectly* showed that Faraday loop was *just* the circle of radius $2\pi b$ running through the center of the toroid. Figure 1 shows a top down view of the solenoid with the actual Faraday loop. Here, imagine that the portion of the loop exterior to the toroid has been stretched to the side to show the full path of the loop (normally, the inner and the exterior portions would overlap in the top-down view). The interior and external paths are joined by two line segments, just like the horizontal line segments in the solenoid's Faraday loop. As the entry and exit windings for the toroid can be fairly close to each other, we can bring the two edges of the toroid's Faraday loop *infinitesimally* close to each other, and as a result, can validly say that $L \approx 2\pi b$.

2. The symmetries to this problem are remarkably similar to solenoid's. Using Faraday loops or the more formal argument described for the solenoid, the field external to the solenoid is 0. The remaining field is then,

$$H_\phi \cdot 2\pi b = I_{\text{enc}} \cdot N \qquad \rightarrow \qquad H_\phi = \frac{NI}{2\pi b}. \tag{5}$$

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Tutorial 3: Magnetic Materials, Actuators, and Circuits

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1. The third problem set is due this Thursday (October 1) in lecture. Coil guns lab next Monday/Tuesday (October 5/6).
2. First midterm quiz on Thursday, October 8. Quiz review on energy conversion/conservation, electromagnetic fields and forces, magnetic materials, and magnetic circuits next Monday (October 5). Problem set 4 due *after* Columbus Day weekend.

1 Questions about problem set 2?

2 Magnetic Materials

Matter is magnetized in the presence of a magnetic field. The tiny atomic dipoles constituting matter form a net alignment (or *polarization*) in two ways:

1. **Paramagnetism.** Dipoles associated with the spins of unpaired electrons experience a torque that ultimately alines them *parallel* to the applied field.
2. **Diamagnetism.** The orbital speed of electrons is altered so that the orbital dipole moment is aligned in a direction *opposite* to the applied field.

Some substances (**ferromagnets**) retain magnetization even after the external field is removed—such materials' magnetization is determined by the entire 'magnetic memory' of the material (*hysteresis*).

Whatever the cause, the macroscopic magnetic polarization is described by the vector *magnetization* $\mathbf{M}$, the magnetic dipole moment per unit volume. The resulting field $\mathbf{B}$ is related to $\mathbf{M}$ and $\mathbf{H}$ approximately linearly by the *magnetic susceptibility* χ_m,

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M}\right) = \mu_0 \left(1 + \chi_m\right) \mathbf{H}. \tag{1}$$

This expression can be shown in two ways: (i) using the magnetic 'monopole' analogy described in class, or (ii) considering the 'bound' and 'free' currents flowing through matter.

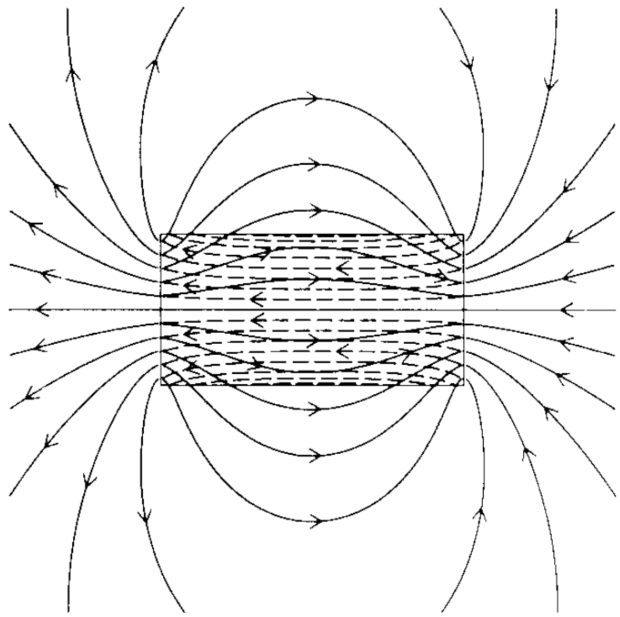
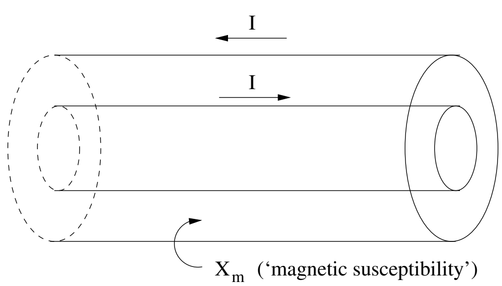


Figure 1: **H**, **B**, and **M** in a uniformly magnetized cylinder ('bar magnet').

Example 1: **Coax Cable with Linear Insulation**

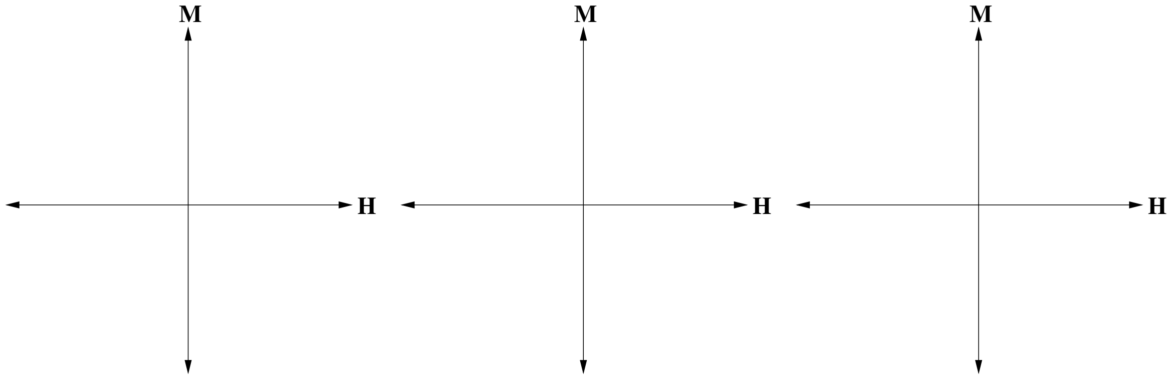
A coax cable is made of two conducting cylinders separated by a linear insulating material of magnetic susceptibility χ_m. With uniform distribution, a current I flows along the inner conductor and returns along the outer one.



1. Find the fields **H** and **B** between the cylinders.

2. Using part 1, calculate the magnetization **M** in the material.

Example 2: Hysteresis Curves and Magnetization



(a) Diamagnetic. (b) Paramagnetic. (c) Ferromagnetic.

Figure 2: Draw the magnetization **M** as a function of **H** as **H** increases from 0 to ∞, decreases to $-\infty$, and returns to 0.

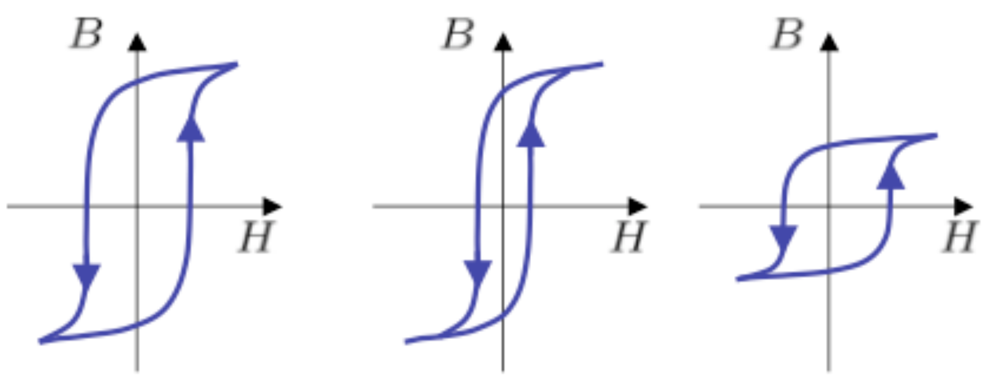


Figure 3: Circle the hysteresis curve that represents a ferromagnet with the strongest remnant field. Circle the easiest magnet to change from N-S to S-N.

3 Faraday's Law

Faraday's law tells us that there are two types of electric fields: those sourced directly from electric charges, and those associated with changing magnetic fields. This law quantifies this second type as follows,

$$V_{\text{emf}} = \oint d\mathbf{l} \cdot \mathbf{E} = -\frac{\partial}{\partial t} \oiint d\mathbf{A} \cdot \mu_0 \mathbf{H} = -\frac{\partial}{\partial t}\lambda \tag{2}$$

Example 3: Simple Electromotive Force: Bar Actuator

A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l

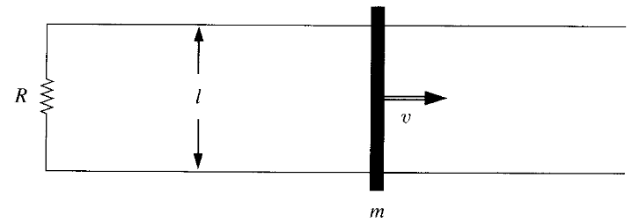


Figure 4: Moving bar of mass m in magnetic field.

apart (Figure 4). A resistor R is connected across the rails and a uniform magnetic field $\mathbf{B} = \mu_0 \mathbf{H}$, pointing into the page, fills the entire region.

1. If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

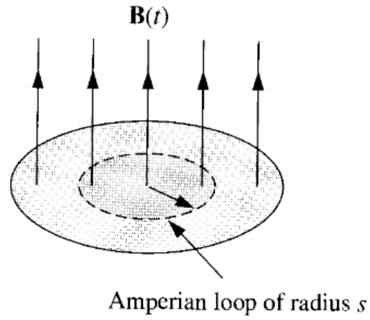
2. What is the magnetic force on the bar? In what direction?

3. If the bar starts at $v\,(t = 0) = v_0$, and is left to slide, what is its speed at a later time t_0?

4. What is the energy delivered to the resistor as a function of m and v_0?

Example 4: Inducting Heating

A uniform magnetic field $\mathbf{B}$, pointing up, fills the shaded circular region in the figure below. If $\mathbf{B} = \mathbf{B}\,(t)$, what is the induced electric field? Does it run clockwise or counter-clockwise?



4 Magnetic Circuits (for next time)

I don't believe these were covered in lecture last week, so I won't introduce a new concept during tutorial. We'll cover strategies for solving magnetic circuit problems prior to exam review next week.

5 Questions about problem set 3?

6 Whitespace Filler of the Week

Valentine by a Telegraph Clerk

The tendrils of my soul are twined
With thine, though many a mile apart.
And thine in close coiled circuits wind
Around the needle of my heart.

Constant as Daniel, strong as Grove.
Ebullient throughout its depths like Smee,
My heart puts forth its tide of love,
And all its circuits close in thee.

O tell me, when along the line
From my full heart the message flows,
What currents are induced in thine?
One click from thee will end my woes.

Through many a volt the weber flew,
And clicked this answer back to me;
I am thy farad staunch and true,
Charged to a volt with love for thee.
— *James Clerk Maxwell*

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Tutorial 3: Magnetic Materials, Actuators, and Circuits

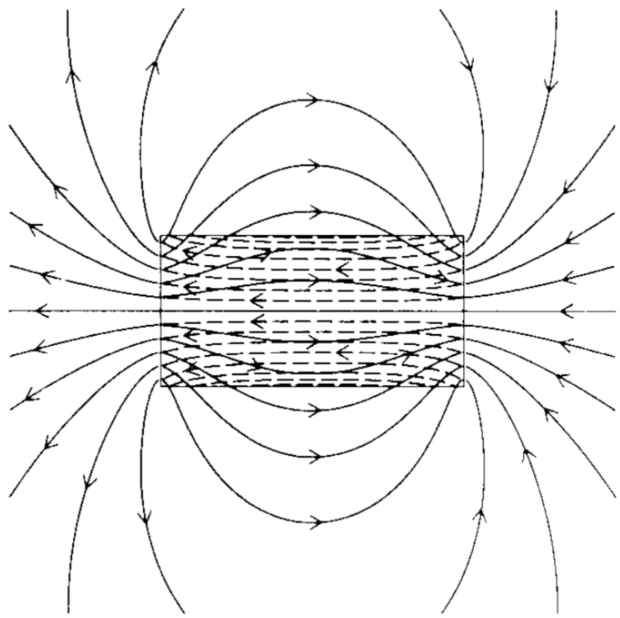

Figure 1: **H**, **B**, and **M** in a uniformly magnetized cylinder ('bar magnet').

I have a clarification on the bar magnet example, that I explained poorly in class. Amusingly, a textbook I have (the source for the figure) was wrong as well.

Either way, we had that relation $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ which was true inside ($\mathbf{M} \neq 0$) and outside ($\mathbf{M} = 0$) the material. Outside, **B** and **H** both pointed in the same direction since $\mathbf{B} = \mu_0\mathbf{H}$.

In the magnet, **M** was pointing from right to left (this is the $\hat{z}$- axis). The field *opposing* was **H**, a fact which has to do with Gauss' law for **B** and the divergences of **B**, **H**, and **M**.

First, remember that $\nabla \cdot \mathbf{B} = 0$ (always true). In the case of the bar magnet, $\nabla \cdot \mathbf{M} \neq 0$, as **M** is a position-dependent magnetization vector pointing from one end of the bar magnet to another (if you don't delieve me, consider $\nabla \cdot \mathbf{M}$ for $\mathbf{M} = \mathbf{M}(z)$). If $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{M} \neq 0$, this means that $\nabla \cdot \mathbf{H} \neq 0$, because $\nabla \cdot \mathbf{B} = 0 = \mu_0(\nabla \cdot \mathbf{H} + \nabla \cdot \mathbf{M})$, implying that $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$. Therefore, for $\nabla \cdot \mathbf{B} = 0$ everwhere, **H** and **M** must point in different directions through a given surface in the bar magnet, thereby canceling out.

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Tutorial 4: Exam Review

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1. Energy and power (definitions and units)

$$\text{Power} = \text{Through} \cdot \text{Across}$$

2. DC Motors (τ, K, I, R, β, current and voltage source operation, etc.)
3. Maxwells equations in free space
   (a) Electrostatics and magnetostatics
      i. Calculate fields to due charges (Gauss Law) and currents (Amperes Law)

$$\text{Gauss:} \quad \oiint d\mathbf{A} \cdot \epsilon_0 \mathbf{E} = \iiint dV \rho_{\text{free}} = Q_{\text{enc}} \qquad \text{Faraday:} \quad \oint d\mathbf{l} \cdot \mathbf{E} = 0$$

$$\text{Ampere:} \quad \oint d\mathbf{l} \cdot \mathbf{H} = \oiint d\mathbf{A} \cdot \mathbf{J} = I_{\text{free}}$$

      ii. Lorentz Force Law—forces on charges and currents (e.g., wires, point charges)

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = q\mathbf{E} + I\mathbf{l} \times \mathbf{B}$$

   (b) Calculating capacitance of air-filled structure $Q = CV$.
   (c) Scalar potential definition (in units of voltage).

$$\mathbf{E} = -\nabla V \qquad\qquad V(b, a) = -\int_a^b d\mathbf{l} \cdot \mathbf{E}$$

(d) Boundary conditions on electric and magnetic fields.

$$\hat{n} \cdot (\mathbf{E}_{\text{inside}} - \mathbf{E}_{\text{outside}}) = \sigma_{\text{surface}}/\epsilon_0 \qquad \hat{n} \times (\mathbf{E}_{\text{inside}} - \mathbf{E}_{\text{outside}}) = 0$$
$$\hat{n} \cdot (\mu_0 \mathbf{H}_{\text{inside}} - \mu_0 \mathbf{H}_{\text{outside}}) = 0 \qquad \hat{n} \times (\mathbf{H}_{\text{inside}} - \mathbf{H}_{\text{outside}}) = \mathbf{K}$$

(e) Energy density of electric and magnetic fields (equation, not full derivation)—remember that it is a density, so total energy must be an integral over volume of the density.

$$N\Phi = \lambda = Li \qquad w_m = \frac{1}{2}\mathbf{B} \cdot \mathbf{H} \qquad W_m = \frac{1}{2}Li^2 = \frac{\lambda^2}{2L}$$
$$Q = CV \qquad w_e = \frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E} \qquad W_e = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$

4. Response of materials—magnetization
   (a) Origin of magnetization **B**, **H**, **M** fields and their relationship

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \rightsquigarrow \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H}$$

   (b) Magnetic susceptibility types of magnetic response (*dia-*, *para-*, *ferro-*)
   (c) Hysteresis—coercivity and remnance.

5. Magnetoquasistatics
   (a) Faradays law—electromotive force (EMF)

$$\text{Faraday:} \quad V_{\text{emf}} = \oint d\mathbf{l} \cdot \mathbf{E} = -\frac{\partial}{\partial t} \oiint d\mathbf{A} \cdot \mu_0 \mathbf{H} = -\frac{\partial}{\partial t}\lambda \qquad \text{Joule:} \quad \mathbf{J} = \sigma \mathbf{E}$$

   (b) Calculating inductance

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Tutorial 4: Exam Review Problems

Example 1: **Electrostatics: Capacitance of concentric spheres**

Two concentric spherical metal shells of radius a and b ($b > a$) form a capacitor, with charges $+q$ and $-q$ on the inner and outer shells, respectively.

1. What is the field $\mathbf{E}$ between the two spheres?

2. What is the potential difference between them?

3. What is the capacitance of the spherical capacitor?

Example 2: Inductance of a Coax Cable (From Griffiths)

A long coax cable carries a current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b) as shown in Figure 1.

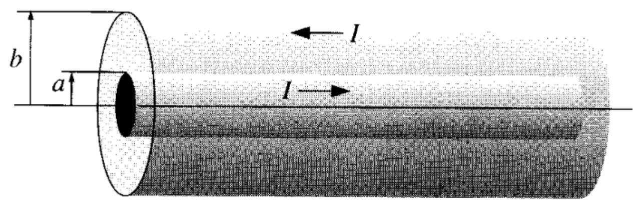

Figure 1: Coax cable.

1. What are **H** and **B** between the two cylinders? In what direction do they point?

2. What is the energy density (per unit volume) between the two cylinders?

3. What is the total energy between the two cylinders over a length l?

4. What is the self-inductance L of the cable per unit length l?

Example 3: Inductance of a Filled Toroid (From Griffiths)

Find the self-inductance L of a toroidal coil with rectangular cross-section (inner radius a, outer radius b, height h—as shown in Figure 2), which carries a total of N turns. The toroid is filled with a ferromagnetic material of magnetic susceptibility χ_m—i.e., permeability $\mu = \mu_0 (1 + \chi_m)$.

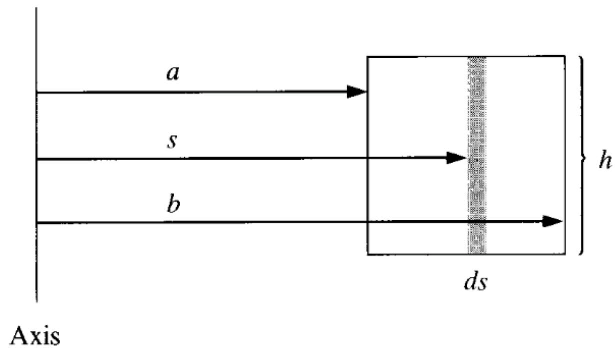

Figure 2: Toroid cross section.

1. A DC current I is applied to the toroidal coil. What are **H** inside and outside the toroid? What are **B** inside and outside the toroid?

2. What is the total flux linkage through the toroid?

3. Using part 2, what is the self-inductance of the toroid?

Example 4: Magnetic Levitation (From Shen and Kong)

A magnetoplane is a high-speed train that uses magnetic levitation to fly in the air above a conducting track. Each vehicle has beneath it a superconducting coil with current I_v levitating above the conducting surface. Using the method of images, we know that we can replace the conducting surface with a mirror current distribution with current $I_c = I_v$, as shown in Figure 3.

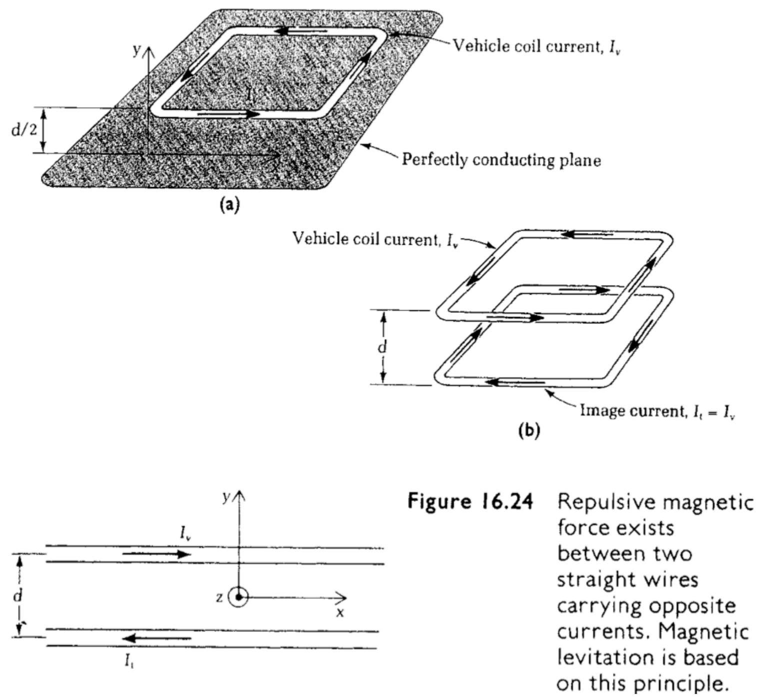

Figure 3: Magnetic levitation using superconducting coils.

1. The equivalent current distribution shown in Figure 3 creates an equivalent field $\mathbf{H}_c$ exerting a force on the vehicle coil located a distance d away. What is $\mathbf{H}_c$ (magnitude and direction)?

2. The side-view of a single segment of the coils is shown in Figure 3; each side has a length l. What is the force experienced by the top segment from the field found in part 1? What is the total magnetic force?

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Tutorial 5: Magnetic Circuits and Actuators

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1. The fourth problem set is due this Thursday (October 15) in lecture. Coil gun lab report due Wednesday (October 16),
2. First midterm back in lecture tomorrow.

1 Magnetic Circuits

A magnetic circuit representation is an engineering tool, a simplification of Faraday's Law that allows us to find out how much flux linkage (i.e., stored magnetic energy) will result from a structure containing permanent magnets, ferromagnetic materials, electromagnets, and even air.

As Faraday's law and charge conservation were used with KVL and KCL circuit analysis in 8.02, Ampere's and Gauss' laws can be applied to the branches of a magnetic circuit (Figure 1). The voltage source is replaced by a *magnetomotive* force, related to the turns N,

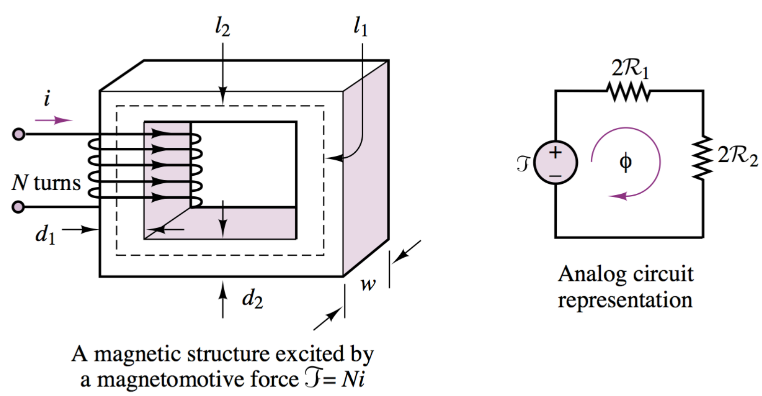

Figure 1: Analogy between magnetic and electric circuits.

applied current I, and fields of branches m of length l_m and cross-sectional area A_m:

$$\mathcal{F} = \sum_m H_m l_m = NI \qquad \sum_m B_m A_m = 0. \tag{1}$$

$\mathcal{F}$ and Φ can be related to form a magnetic Ohm's law ('Hopkinson's law'),

$$\mathcal{R} = \frac{l}{\mu A} \qquad \mathcal{F} = NI = \Phi\mathcal{R}. \tag{2}$$

Finally, the inductance of a structure is related to its reluctance,

$$L = \frac{N^2}{\mathcal{R}}. \tag{3}$$

Unlike EMF, which drives electrical charge in circuits, the magnetomotive force $\mathcal{F}$ (MMF) is definitely a misnomer, it may 'drive' flux, but not real physical quantity. No power is actually flowing in the circuit. Furthermore, magnetic circuit representation involves several assumptions and simplifications:

1. All of the magnetic flux is linked by all the turns of the coil.
2. The flux is confined exclusively within the magnetic core.
3. The density of the flux is uniform across the cross-sectional area of the core.

The full analogies are in Table 1

Electric		Magnetic	Units
Current	I	Magnetic Flux	
Current Density	$\mathbf{J}$	Magnetic Flux Density	
Conductivity	σ	Permeability	
Applied Voltage	V	Magnetomotance	
Electric Field Strength	$\mathbf{E}$	Magnetic Field Strength	
Resistance	R	Reluctance	
Conductance	ρ	Permeance	

Table 1: Corresponding quantities in electric and magnetic circuits.

Example 1: Magnetic Circuit with an Air Gap

We'll analyze the electromagnet and air gap in Figure 2 using Ampere's and Gauss' laws. A current I is applied to two windings. Denote the iron core as f and air gap as g. Let L_i $(i = f, g)$ be measured along the center of the center of the cross-section around the yolk.

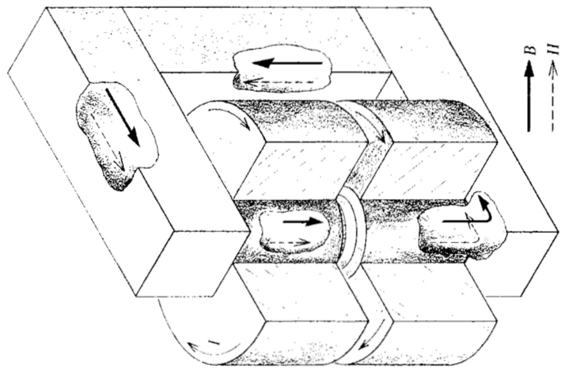

Figure 2: Electromagnet with a soft-iron yolk. Each winding has $NI/2$ A-turns.

1. Using Ampere's Law, what's the magnetomotance $\mathcal{F}$? What is the Gauss' law relation for the iron core and the air gap?

2. Calculate the flux through the air gap using part 1.

3. Draw the equivalent magnetic circuit.

Example 2: Variable Reluctance Motion Sensor

An iron core electromagnet forms the sensor: a tabbed steel disk connected to a rotating shaft rotates between the pole pieces of the sensor. The area of the tab is assumed equal to the area of the cross section of the pole pieces and is equal to a^2.

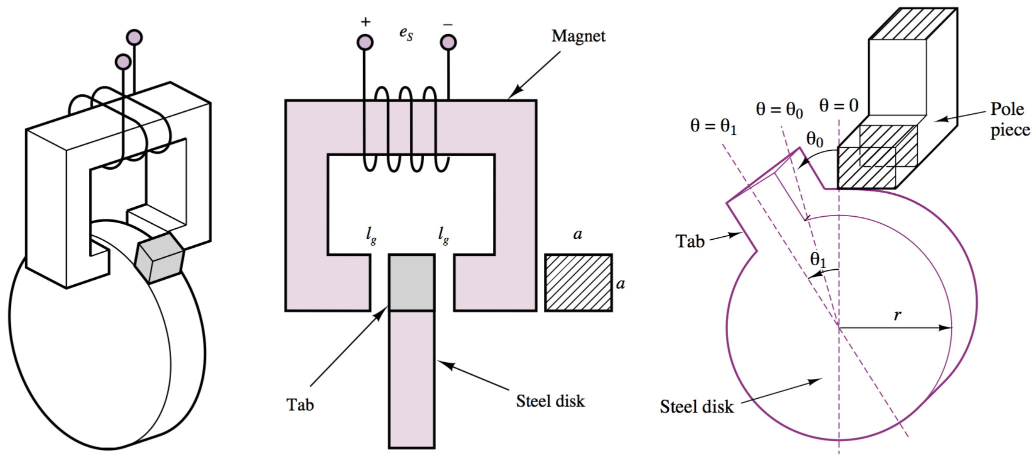

Figure 3: Variable reluctance sensor for measuring angular position.

1. The principle of operation is that an electromotive force, V_{emf}, is induced across the coil by the passage of the tab between the pole pieces when the disk rotates with speed ω. Draw the shape of the waveform $V_{\mathrm{emf}} = e_s$ measured across the coil.

2. Using Figure 3(a), draw the equivalent magnetic circuit.

3. What is the equivalent reluctance $\mathcal{R}_{\mathrm{gap}}$?

4. What is the flux $\phi(\theta)$? What is the peak V_{emf}?

2 Energy Method

The energy method is an amazing application of fundamental ideas in thermodynamics to the understanding of physical actuators. The first law of thermodynamics states that both work and heat are forms of energy, and that the total energy is conserved. Generally, we can write this as

$$dQ = dE - dW. \tag{4}$$

Ignoring heat (dQ), we can expand the work done on a system into a set of generalized displacements $d\mathbf{x}$ and generalized forces $\mathbf{J}$ such that any infinitesimal transformation can be given by

$$dW = \sum_i J_i dx_i. \tag{5}$$

Note that the displacement is usually an extensive quantity, i.e. proportional to system size, while the forces are intensive and independent of size.

3 Magnetic Actuators and Systems

With this in mind, work in a magnetic system can be expanded in terms of its state variables as,

$$dW_m = id\lambda - F_x dx \tag{6}$$

Differentiating by dt relates the power flow in the system as,

$$\frac{dW_m}{dt} = i\frac{d\lambda}{dt} - F_x\frac{dx}{dt}, \tag{7}$$

and we can integrate over a constant contour of λ to solve for the force:

$$F_x = \frac{-dW_m}{dx} \qquad W_m = \frac{\lambda^2}{2L}, \tag{8}$$

which is typically our quantity of interest.

Example 3: Air Gap Revisited

Consider the toroidal loop with a gap that we studied in Problem Set 3, shown below. In this case, the toroidal loop has a rectangular cross-section with area $A = ab$ as shown. The wire wound around the toroid is a short circuited superconductor with initial current I. For this problem, assume that all of the energy is in the magnetic field of the gap.

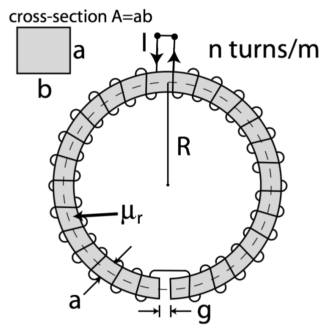

Figure 4: Look! Another toroid!

1. What is the stored magnetic energy in the gap in terms of H_g? What is H_g in terms of i? In terms of λ and L?

2. What is the inductance L of the toroid? What is H_g in terms of the area of the gap?

3. If we make $g = g_0 - x$ (earlier $x = 0$), what is the stored energy in terms of H_g? What is the force f_x in the left face of the gap?

4. If we alter the area of overlap so that $A_{\text{new}} = a\,(b - d)$, what is the new inductance L of the gap? H_g? Stored magnetic energy and f? (In what direction does f act?)

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Tutorial 6: Magnetic Actuators (cont.) and Electric Machines

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1. The fourth problem set is due this Thursday (October 22) in lecture.

1 The Energy Method

Last tutorial, we ended on electromagnetic energy conservation. Recall that for magnetic systems,

$$dW_m = id\lambda - F_x dx. \tag{1}$$

Differentiating by dt relates the power flow in the system, and we can integrate over a constant contour of λ to solve for the force

$$F_x = -\frac{dW_m}{dx}\bigg|_\lambda \qquad W_m = \frac{\lambda^2}{2L}, \tag{2}$$

Similarly, for electric systems,

$$dW_e = VdQ - F_x dx \tag{3}$$

which yields the force,

$$F_x = -\frac{dW_e}{dx}\bigg|_Q \qquad W_e = \frac{Q^2}{2C}. \tag{4}$$

2 Electric Fields in Materials — Dielectrics and Polarization

Within good approximation, most everyday objects are either conductors or insulators (*dielectrics*). Conductors are substances that contain an effectively unlimited supply of *free* charges: valence electrons of a given atom permeate the entire material. Electrons in dielectrics, by contrast, are *bound* to specific atoms or molecules and can only move within that atom or molecule. In the presence of an electric field $\mathbf{E}$, the charge distributions of atoms can change two ways:

1. **'Stretching' neutral atoms (or non-polar molecules).** The center of charge of the electron orbital is displaced from the center of charge of the nucleus, leaving the atom or molecule *polarized* with an induced dipole moment $\mathbf{p} = \alpha\mathbf{E}$, where α is called the atomic polarizability.

2. **'Rotating' polar molecules.** Polar and ionic molecules with permanent dipole moments experience a torque from $\tau = \mathbf{p} \times \mathbf{E}$ and are aligned parallel to $\mathbf{E}$.

The collective dipole alignment of the material is called the **polarization P**, the dipole moment per unit volume. For a given dielectric material, the field arising from this polarization can be calculated from the bound volume (ρ_b) and surface (σ_b) charges:

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \sigma_b = \mathbf{P} \cdot \hat{n}, \tag{5}$$

where $\mathbf{P}$ is the polarization in the material and $\hat{n}$ is the surface normal enclosing the dielectric.

Lastly, as the field $\mathbf{H}$ characterized the magnetic field arising from free currents in the material, the displacement field $\mathbf{D}$ arises from free charges, and is related to the polarization and electric field vectors through the constitutive relation,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \left(1 + \chi_e\right) \mathbf{E} = \epsilon \mathbf{E} \tag{6}$$

where Gauss's law still holds for ρ_{free}:

$$\oint\!\!\!\oint d\mathbf{A} \cdot \mathbf{D} = \rho_{\text{free}}. \tag{7}$$

Example 1: **Air Gap Revisited**

Consider the toroidal loop with a gap that we studied in Problem Set 3, shown below. In this case, the toroidal loop has a rectangular cross-section with area $A = ab$ as shown. The wire wound around the toroid is a short circuited superconductor with initial current I. For this problem, assume that all of the energy is in the magnetic field of the gap.

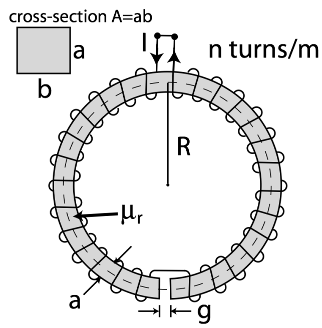

Figure 1: Look! Another toroid!

1. What is the stored magnetic energy in the gap in terms of H_g? What is H_g in terms of i? In terms of λ and L?

2. What is the inductance L of the toroid? What is H_g in terms of the area of the gap?

3. If we make $g(x) = g_0 - x$ (earlier $x = 0$), what is the stored energy in terms of H_g? What is the force f_x in the left face of the gap?

4. If we alter the area of overlap so that $A_{\text{new}}(d) = a(b - d)$, what is the new inductance L of the gap? H_g? Stored magnetic energy and f? (In what direction does f act?)

Example 2: Capacitor with a Spring

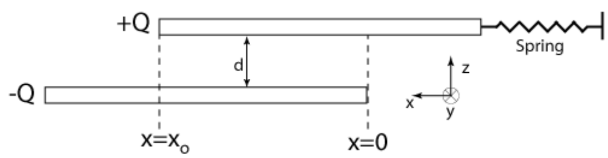

Figure 2: Two charged Plates, one attached to a spring.

Consider the capacitive structure shown in Figure 2. Two conducting plates are positioned near each other. The top plate is attached to a spring, so that the spring exerts no force when the overlap distance of the two plates is $x_\circ$ as shown above. Assume the plates extend a distance w into the page along the $\hat{y}$ direction. Also assume that the spring exerts a force of $f_s = K(x - x_\circ)$ $[N]$, where x is the position of the left edge of the top plate.

After assembling the system shown, we then place a charge of $+Q$ on the top plate and $-Q$ on the bottom plate and watch how the system moves.

1. What is the force acting on the top plate in the $\hat{z}$ direction (as a function of Q)?

2. What is the force acting on the top plate in the $\hat{x}$ direction (as a function of Q)?

3. Ignoring the spring, if we allow the top plate to move in the $\pm\hat{x}$ direction how will the voltage change as the plates move?

4. If we connect a voltage source with voltage V across the two plates and let the top plate move in the $\pm\hat{x}$ direction

   (a) for what value of overlap x will the system reach equilibrium?

   (b) what is the capacitance of the parallel plate structure at this point?

   (c) what is the total charge Q at this point?

Example 3: iPhone Accelerometer — Linear electrostatic actuator

The iPhone uses an accelerometer with a detached silicon mass to detect its orientation. The mass is connected to dielectric arms and the capacitance of the structure depends on the position of the mass. We can detect orientation relative to the gravitational field by measuring the flow of charge onto or off of the capacitors as their capacitance changes. We will consider the one dimensional version of this device shown in Figure 3.

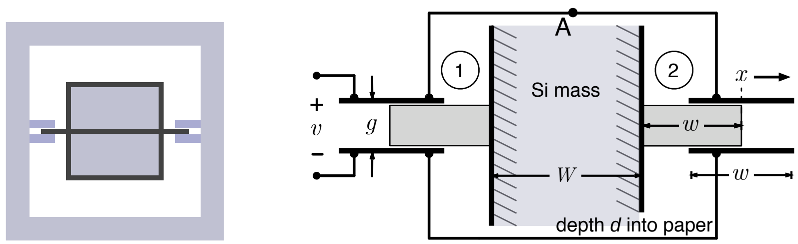

Figure 3: One dimensional version of the iPhone accelerometer. Capacitors 1 and 2 are connected in parallel to bias voltage v. Each has plate width w, separation g, and an adjustable dielectric of length w. The dielectric completely fills the capacitor in the vertical dimension and moves without fiction. At $x = 0$, each capacitor is half filled with dielectric; thus the valid range of x is $-w/2 < x < w/2$. The structure extends depth d into the paper.

1. The capacitors of our 1D accelerometer are connected in parallel as shown. Find $q_1(v, x)$ and $q_2(v, x)$, where q_n is the charge on the top plate of capacitor n and v is the applied voltage.

2. Determine the total electrostatic force on the silicon mass as a function of position x.

3. In terms of the density of silicon ρ, what is the gravitational force on the free silicon mass? Find the steady state deflection when gravity points in the $+\hat{x}$ direction. Note the allowed range of x is $-w/2 \leq x \leq w/2$.

4. Assume $g = 1\ \mu\text{m}$, $w = 10\ \mu\text{m}$, $W = 50\ \mu\text{m}$, $d = 10\ \mu\text{m}$, $V = 3.3\ \text{V}$, and $\rho = 2.3\ \text{g/cm}^3$. The dielectric is SiO_2 with $\varepsilon = 3\varepsilon_0$. The device is initially aligned so that gravity points in the $+\hat{x}$ direction. If the device flips orientation, how many electrons flow past point A in the figure?

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Tutorial 7: Electromagnetic Waves and Polarization

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1. The fifth problem set is due this Thursday (October 29) in lecture.
2. Liquid crystal display lab tonight and tomorrow!
3. Recommended reading this week: *The Lorentz Oscillator and its Applications*, Chapter 3 in Shen and Kong, and phasor manipulations from the first problem set and tutorial.

1 Electromagnetic Fields

An amazing thing happens when we solve Maxwell's Equations in a vacuum: electromagnetic fields can leave their sources and travel alone through space. Our primary interest for the next two weeks is to understand these plane wave solutions—

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\tilde{\mathbf{E}} e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}\right] \tag{1}$$

—in the context of polarization ($\tilde{\mathbf{E}}$) and propagation ($\mathbf{k}$) in free space and materials.

1.1 The Wave Vector k

The wave vector $\mathbf{k}$ specifies the direction of propagation (i.e., power flow of the wave) and the spatial variation of an electromagnetic wave. Generally,

$$\mathbf{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \qquad |\mathbf{k}| = \frac{2\pi}{\lambda}. \tag{2}$$

Maxwell's Equations for the plane wave solution in Equation 1 are given by,

$$\begin{aligned} &\text{Faraday: } \mathbf{k}\times\mathbf{E} = -\omega\mu\mathbf{H} &&\text{Gauss: } \mathbf{k}\cdot\mathbf{E} = 0 \\ &\text{Ampere: } \mathbf{k}\times\mathbf{H} = \omega\epsilon\mathbf{E} &&\text{Gauss: } \mathbf{k}\cdot\mathbf{H} = 0 \end{aligned} \tag{3}$$

from which it can be shown that

$$\omega^2\mu\epsilon = k^2 = \mathbf{k}\cdot\mathbf{k}, \tag{4}$$

which relates the spatial and temporal frequencies (known as a *dispersion relation*), and that

$$\eta = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\frac{\mu}{\epsilon}}, \tag{5}$$

which defines an electromagnetic impedance in a medium. In free space, $\eta = 377\Omega \approx 120\pi$.

The Poynting vector $\mathbf{S}$ specifies the density of power flow from $\mathbf{E}$ and $\mathbf{H}$ and is generally in the direction of $\mathbf{k}$. There are two formulae depending on whether the fields are real or written as phasors:

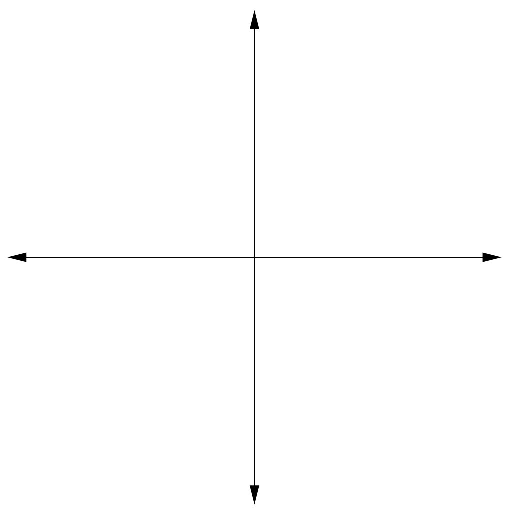
$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad \langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re} \left[\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right]. \tag{6}$$

1.2 Polarization

An electromagnetic wave's polarization describes its orientation as it propagates through space. Let's consider an expanded and simplified version of Equation 1,

$$\tilde{\mathbf{E}}(z) = \hat{x} E_x e^{-jkz} + \hat{y} E_y e^{-jkz} e^{j\phi}. \tag{7}$$

The polarization here is completely described by the ratio E_y/E_x and the phase difference ϕ between the two waves. Let's consider some special cases, There are as many ways to write



polarization as there are ways of manipulating phasors, so I recommend you review the manipulations from the first tutorial. Furthermore, the representation of the polarization of a wave is not unique. In general, any polarization can be broken down into a sum of orthogonal polarizations (think basis vectors in signals and systems).

2 Birefringence

Birefringent media change light polarization by changing ϕ. In a birefringent medium, the $\hat{x}$ and $\hat{y}$ directions have different indices of refraction n_x and n_y. By convention, we assume that $\hat{x}$ is aligned with the 'fast' (or *extraordinary*) axis and that $\hat{u}$ is aligned with the 'slow' (or *ordinary*) axis, resulting in $n_x > n_y$. Equation 1 becomes

$$\tilde{\mathbf{E}}(z) = \hat{x} E_x e^{-jk_x z} + \hat{y} E_y e^{-jk_y z} e^{j\phi}. \tag{8}$$

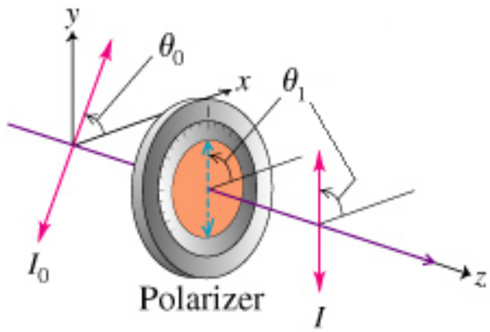


Figure 1: Perfect polarizer.

The $\hat{x}$ component develops a phase $k_x l$ after traveling a distance l in the medium. The resulting phase difference is given by

$$\Delta\phi = \phi(L) - \phi(0) = kl(n_x - n_y). \tag{9}$$

Lastly, when a perfect polarizer is placed in a polarized beam of light, the intensity, I, of the light that passes through is given by

$$I = I_0 \cos^2 \theta_i, \tag{10}$$

where I_0 is the initial intensity and θ_i is the angle between the light's initial polarization direction and the axis of the polarizer. This is shown in Figure 1 below, where $\theta_i = \theta_1 - \theta_0$.

3 The Lorentz Oscillator Model (Introduction)

Our experiences in everyday life tell us that transparent optical materials have a non-uniform frequency response: prisms and raindrops disperse white light into rainbows, and deflect certain colors more than others. The Lorentz Oscillator model is a semi-classical model describing this frequency response in term so of material polarization and a driving electric field (light). This second-order model is given by,

$$\begin{aligned}\frac{d^2\mathbf{P}}{dt^2} + \gamma\frac{d\mathbf{P}}{dt} + \omega_0^2\mathbf{P} &= \frac{Nq^2}{m}\mathbf{E} \\ &= \epsilon_0\omega_p^2\mathbf{E}.\end{aligned} \tag{11}$$

Substituting $\tilde{\mathbf{P}} = \mathbf{P}e^{j\omega t}$ and $\tilde{\mathbf{E}} = \mathbf{E}e^{j\omega t}$ (or equivalently take its Fourier transform), the relative permittivity is given by

$$\tilde{\epsilon}_r(\omega) = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{\omega_p^2}{(\omega_0 - \omega)^2 + j\omega\gamma}, \tag{12}$$

and accordingly, the index of refraction is given by

$$\tilde{n} = \frac{c}{v_p} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\mu_r\tilde{\epsilon}_r} = n - j\kappa. \tag{13}$$

Example 1: Electromagnetic Waves

1. Matching waves:

E	Statement
$\hat{y}E_0e^{-jx+jz}$	1. Wave in a lossy medium
$(\hat{x}+j\hat{z})\,E_0e^{j2y}$	2. Right-circularly polarized wave
$\hat{z}E_0e^{y}e^{-j2x}$	3. Left-circularly polarized wave
$(\hat{y}-j\hat{z})\,E_0e^{-j2x-j3z}$	4. Evanescent wave in lossless medium
$\hat{y}E_0e^{-jx-\alpha x}$	5. Wave propagating at $\pi/4$ to $\hat{x}$
$\hat{x}E_0e^{-jy}e^{-2z}$	6. Impossible **E**!
	7. None of the above!

Table 1: Match each wave to *one* statement that describes it.

2. What is the **H** fields that accompanies $\mathbf{E} = \hat{x}E_0\cos(\omega t - kz)$ and $\mathbf{E} = \hat{y}E_0e^{-jx+jz}$?

3. What is the power flow ($\mathbf{S}$ and $\langle\mathbf{S}\rangle$) for each of these waves?

4. What is the propagation frequency ω of the first wave?

Example 2: Birefringence and Wave Plates

1. Your thesis is due tomorrow and you need to rotate laser light ($\lambda = 700\text{nm}$) polarized along the $\hat{y}$-axis to the $\hat{x}$-axis. Your have some birefringent glass with $\Delta n = 0.5$. How must the glass be aligned to achieve this rotation? How thick should it be?

2. An EM wave described by $\mathbf{E}(\mathbf{r}, t) = \hat{x}E_0 \cos(\omega t - kz) - \hat{y}E_0 \sin(\omega t - kz)$ is sent through an appropriately chosen 3/4 wave plate. What is the resulting EM wave?

Example 3: Waves in Conductors—Lorentz Oscillator

An $\hat{x}$-polarized electromagnetic wave $\mathbf{E}$ is traveling in conducting medium.

1. What is a general wave vector $\mathbf{k}$ describing $\mathbf{E}$'s in the medium?

2. What are $\mathbf{E}$, $\mathbf{H}$, $\mathbf{S}$, and $\langle \mathbf{S} \rangle$?

3. What is the penetration depth d in terms of λ and ω_p?

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.007 – Applied Electromagnetics
Fall 2009

Tutorial 8: Electromagnetic Waves at Boundaries

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1. Problem set 7 is due this Thursday (November 5) in lecture. LCD lab write-up due Tuesday in lecture.
2. Midterm 2 next Thursday, November 12.
3. Recommended reading this week: *The Lorentz Oscillator and its Applications.*

1 The Lorentz Oscillator Model

Our experiences in everyday life tell us that transparent optical materials have a non-uniform frequency response: prisms and raindrops disperse white light into rainbows, and deflect certain colors more than others. The Lorentz Oscillator model is a semi-classical model describing this frequency response in term so of material polarization and a driving electric field (light). This second-order model is given by,

$$\begin{aligned} \frac{d^2\mathbf{P}}{dt^2} + \gamma\frac{d\mathbf{P}}{dt} + \omega_0^2\mathbf{P} &= \frac{Nq^2}{m}\mathbf{E} \\ &= \epsilon_0\omega_p^2\mathbf{E}. \end{aligned} \tag{1}$$

Substituting $\tilde{\mathbf{P}} = \mathbf{P}e^{j\omega t}$ and $\tilde{\mathbf{E}} = \mathbf{E}e^{j\omega t}$ (or equivalently take its Fourier transform), the relative permittivity is given by

$$\tilde{\epsilon}_r(\omega) = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{\omega_p^2}{(\omega_0 - \omega)^2 + j\omega\gamma}, \tag{2}$$

and accordingly, the index of refraction is given by

$$\tilde{n} = \frac{c}{v_p} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\mu_r\tilde{\epsilon}_r} = n - j\kappa. \tag{3}$$

2 Reflection and Transmission at Boundaries

When an electromagnetic wave is normally incident at a boundary, $\mathbf{E}_i$ and $\mathbf{H}_i$ are parallel to the surface of the boundary. Applying boundary conditions for the incident, reflected, and transmitted waves, it can be shown that,

$$r_{12} = \frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{n_1 - n_2}{n_2 + n_1} \tag{4}$$

$$t_{12} = \frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2n_1}{n_2 + n_1}. \quad (5)$$

Reflection and transmission of power at boundaries with oblique ($\theta > 0$) angles of incidence is described by Fresnel's equations (covered in lecture this week).

3 Interference

Under some conditions we can add the electric fields from two or more waves. Depending on the relative phases of the individual waves the sum can be larger, smaller, or the same as the amplitude of one of the components. This effect is called interference, and the observation of interference patterns is one of the strong arguements for the wave nature of light. The truth, of course, is quite a bit stranger.

For now lets restrict ourselves to the case of two waves, at the same wavelength, traveling along z. Lets assume that wave 1 has zero phase at the position $z = 0$ and wave 2 has phase ϕ at $z = 0$. We can then write the expressions for each wave, and the sum.

Wave 1:

Wave 2:

Sum:

When the waves are both traveling in the same direction along z, what happens to the amplitude of the wave as ϕ is changed?

This gives us the following conditions for constructive and destructive interference.

$$\text{Constructive:}\phi = 2N\pi \quad (6)$$

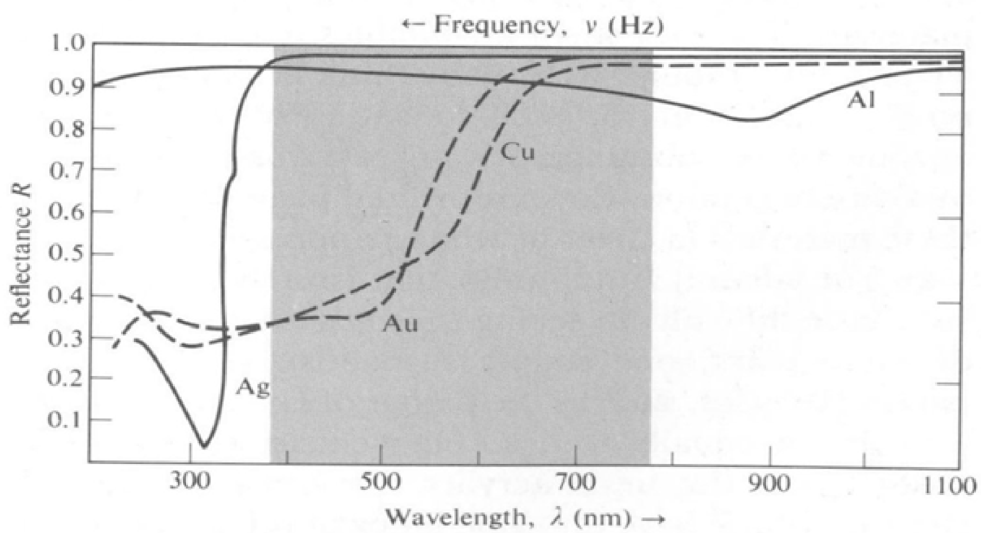
$$\text{Destructive:}\phi = (2N + 1)\pi \quad (7)$$

If one wave travels along $+\hat{z}$ and the other along $-\hat{z}$ what happens to the sum of the waves written above?

Why would we call this a standing wave pattern?

Example 1: **Metal Mirrors**

In this problem, we'll consider the power reflectivity ($|r|^2$) for metals using the Lorentz oscillator model. Consider power reflectivity for a variety of metals plotted below. The metal sample we have in mind is thick and has a plasma frequency ω_p dependent on the free electron density in the metal.



1. In the Lorentz oscillator model of a metal, which parameter(s) (ω_p, ω_0, γ) do we assume is negligible (i.e., zero)? What is $n(\omega)$?

2. Plot n and κ for a metal as a function of ω. For what range of frequencies would we expect a metal to be a good reflector?

3. Given an expression for the power reflectivity at an air/metal interface for $\omega > \omega_p$ and $\omega < \omega_p$.

4. Use the figure to give qualitative explanations of the following:
   - Why is copper yellow?
   - Why is silver used for mirrors instead of (cheaper) aluminum?

Example 2: Multiple Slit Grating

Draw the diagram used to compute the pattern for a two slit grating:

What assumptions do we need to make to get the far field pattern:

Draw the diagram you would use to compute a four slit pattern:

Write an expression for the electric field at some angle θ in the far field, assuming all slits are illuminated with the same amplitude and phase. .

We can extend this result to the case of N slits using the expression for the sum of the first N terms in a geometric series.

Using a similar logic, it's possible to solve the case of a double slit illuminated off axis. Write the electric field sum, taking into account the phase of the light at each aperature.

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Tutorial 9: Midterm 2 Review

Notes:

1. The second midterm this Thursday will be held in 36-156 from 7:30 - 9:30PM (the usual lecture room 37-212 is not available). It will cover material from problem sets 4-7 and labs 2-3 (lectures 11 - 28). You'll be allowed one page of notes, front and back.
2. Extra office hours will be held on Tuesday in 26-314 from 3-6PM. Regular office hours will also be held on Wednesday in 36-144 from 6-8PM (despite the holiday). Bill, Kevin, and I will be in lab on Monday/Tuesday in case there are extra questions.
3. Send me (`mookerji@mit.edu`) an email if you need to take a conflict exam.
4. Relevant readings: *The Lorentz Oscillator and its Applications*, Shen and Kong: 1 (complex phasor notation), 3 (uniform place waves), 4 (reflection/transmission of waves), 8 (plane waves in anisotropic media).

Notes:

1. Forces and the energy method
   (a) Magnetic circuits
   (b) Electric actuators and sensors
2. Quasi-static limit of Maxwell's equations
3. Maxwell's equations in differential form
   (a) Solving for Helmholtz wave equation, which gives waves in terms of angular frequency, ω, and spatial frequency, k or β, related by dispersion relation ($\omega = ck_0$ in free space, $\omega = \frac{c}{n}k$ in a material with index of refraction n).
   (b) Impedance, η, relates $\mathbf{E}$ and $\mathbf{H}$; in free space, $\eta_0 = \sqrt{\mu_0/\epsilon_0} \approx 377\Omega$.
   (c) Phasor notation for uniform plane waves: $\mathbf{E} = \mathrm{Re}\left[\tilde{\mathbf{E}} e^{j(\omega t - \beta z)}\right]$.
   (d) Power/Area = Intensity $\rightarrow I = |\mathbf{S}|$; Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, and for phasors $\left\langle \tilde{\mathbf{S}} \right\rangle = \frac{1}{2}\left\langle \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right\rangle$.
4. Waves in polarizable media

(a) Linear materials $\rightarrow \mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$

(b) Lorentz oscillator model

(c) Dissipation in non-magnetic materials: complex $\tilde{\epsilon}(\omega)$

i. Complex index of refraction: $\tilde{n} = \sqrt{\epsilon_r} = n - j\kappa$

ii. Complex wave number: $\tilde{k} = \tilde{n} k_0 = k - j\alpha$

iii. Complex impedance: $\tilde{\eta} = \sqrt{\mu/\tilde{\epsilon}}$

5. Polarization of fields

(a) Linear polarizers and intensity transmitted (Malus' Law)

(b) Circular/elliptical polarization, right-handed and left-handed

(c) Birefringence

i. Wave plates

ii. Uniaxial material with optic axis (has extraordinary index, n_e, for waves polarized in optic axis direction, and ordinary index, n_0, for waves *propagating* along optic axis)

(d) Liquid crystals and LCD displays.

6. Reflection/Transmission at boundaries

(a) Boundary conditions for $\mathbf{E}$ and $\mathbf{H}$

(b) Diffraction

(c) Normal incidence reflection and transmission

6.007 Midterm 2 - Review ⟹ B.M. (mookerji@mit.edu)

Magnetic & Electric Energy Method

→ state variables

$$dW_m = i\,d\lambda - F_x\,dx$$ } energy conservation

dW_m: stored magnetic energy; $F_x\,dx$: work in/out of system.

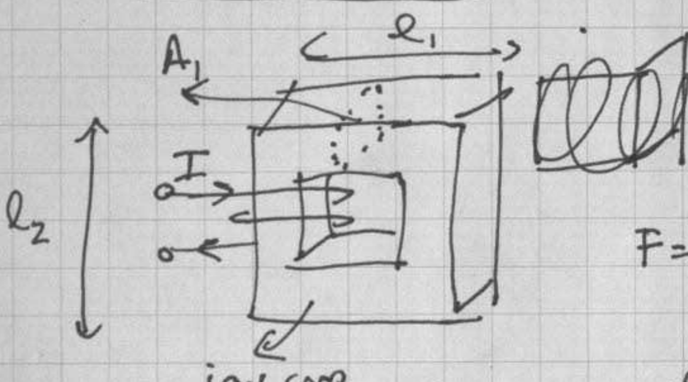
$$\boxed{F_x = -\frac{dW_m}{dx}\Big|_\lambda}\,, \quad W_m = \frac{1}{2}Li^2 = \frac{\lambda^2}{2L} \quad \text{} \} \text{ magnetic}$$

$$\boxed{F_x = -\frac{dW_e}{dx}\Big|_q} \quad W_e = \frac{1}{2}\frac{q^2}{2C} = \frac{1}{2}CV^2 \quad \} \text{ electric}$$

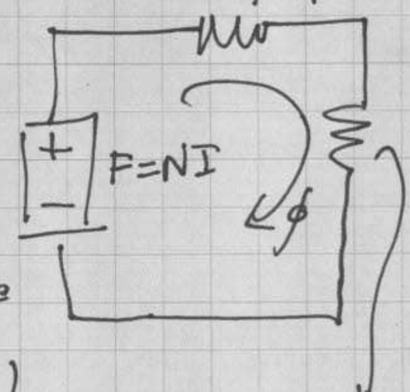
<u>Point</u> W_m / W_e will have same dependence on a spatial parameter, typically through inductance or capacitance, which are geometric parameters of system being analyzed.

See: toroid w/ variable airgap
capacitor w/ dielectric washer.

Magnetic circuits



electric circuit analogy ⟹



$R_1 = \frac{2l_1}{\mu A_1}$

$R_2 = \frac{2l_2}{\mu A_2}$

$F = NI$ ⟹ "magnetomotive force" (like voltage)

ϕ ⟹ flux (like current).

Relations:

mmf: $F = NI = \phi R$, $\quad R = \frac{l}{\mu A}$

Inductance: $L = \frac{N^2}{R_{eq}}$ $\quad R_{eq} = R_1 + R_2$ (in <u>series</u>) ⟹

⇒ also recall <u>variable reluctance</u>

w/ variable $A_{overlap}$, l

Remember $\boxed{R_{gap} \gg R_{core}}$ as $\mu \gg \mu_0$..

Differential Maxwell's Eqns for Plane waves

$$\left\{\begin{array}{ll} \text{Faraday} & \bar{\nabla}\times\bar{E} = -\frac{\partial}{\partial t}\bar{B} \\ \text{Ampere} & \bar{\nabla}\times\bar{H} = \frac{\partial}{\partial t}\bar{D} \end{array}\right\}$$

eliminate $\bar{D}, \bar{B}$ using constitutive relations

$$\Rightarrow \bar{B} = \mu_0(\bar{H}+\bar{M})$$

$$= \mu_0(1+\chi_m)\bar{H}$$ linear medium

$$\boxed{\bar{B} = \mu\bar{H}}$$

$$\bar{D} = \epsilon_0\bar{E} + \bar{P}$$

$$= \epsilon_0(1+\chi_e)\bar{E}$$

$$\boxed{\bar{D} = \epsilon\bar{E}}$$

Combine to get (in 1D)

$$\frac{\partial^2}{\partial z^2}\bar{E} = \mu_0\epsilon_0\frac{\partial^2\bar{E}}{\partial t^2}$$

$\rightarrow c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ = 'speed of light'.

in material $\rightarrow v_p = \frac{1}{\sqrt{\mu\epsilon}}$ 'phase velocity'. $\Rightarrow$

dispersion relation
$\omega = ck$
$\omega^2 = \mu\epsilon k^2$

Solution is a superposition of $\hat{z}$ & $-\hat{z}$ propagating waves:

(y or x polarized) $E_y = f_+\left(t - \frac{z}{c}\right) + f_-\left(t + \frac{z}{c}\right)$ → [rad/m]

$$= \frac{E_0}{2}\left[e^{j(\omega t - kz)} + e^{-j(\omega t - kz)}\right]$$

[V/m] [rad/s]

$$= \mathrm{Re}\{E_0 e^{j(\omega t - kz)}\}$$

→ wave vector → direction of propagation

~~Generall~~ Generally, $\boxed{\bar{E}(t) = \mathrm{Re}\{\tilde{E}\, e^{j(\omega t - \bar{k}\cdot\bar{r})}\}}$

polarization

$$\bar{K}\cdot\bar{r} = k_x x + k_y y + k_z z$$

$$= \underbrace{(k_x \hat{x} + k_y \hat{y} + k_z \hat{z})}_{\text{wavevector}} \cdot \underbrace{(x\hat{x} + y\hat{y} + z\hat{z})}_{\text{general displacement}}$$

$$|k| = \frac{2\pi}{\lambda}$$

For EXAMPLE,

$$e^{-j(3x+2z)} = e^{-j(3\hat{x}+2\hat{z})\cdot(x\hat{x}+y\hat{y}+z\hat{z})}$$

Substituting $\bar{E} = \tilde{E} e^{j(\omega t - \bar{k}\cdot\bar{r})}$ into Maxwell.

linear material

$$-j\bar{k}\times\bar{E} \quad \bar{\nabla}\times\bar{E} = -\frac{\partial}{\partial t}\bar{B} = -\mu\frac{\partial}{\partial t}H = -j\omega\mu H$$

$$\Leftarrow$$

$$-j\bar{k}\times\bar{H} \quad \bar{\nabla}\times\bar{H} = \frac{\partial}{\partial t}\bar{D} = +\epsilon\frac{\partial E}{\partial t} = j\omega\epsilon\bar{E}$$

As, $\frac{d}{dt}\left(e^{j(\omega t - \bar{k}\cdot\bar{r})}\right) = j\omega e^{j(\omega t - \bar{k}\cdot\bar{r})}$

$$\bar{\nabla}\times\left(e^{j(\omega t-\bar{k}\cdot\bar{r})}\right) = -j\bar{k}\times e^{j(\omega t-\bar{k}\cdot\bar{r})}$$

So $H = \frac{1}{\omega\mu}\bar{k}\times\bar{E} = \frac{1}{\omega\mu}(k\hat{k})\times(E\hat{E})$

$$= \frac{k}{\omega\mu}E(\hat{k}\times\hat{E}) = \frac{E}{\eta}\hat{k}\times\hat{E} = \bar{H}$$

$\frac{E}{\eta}$ — H magnitude

Impedance

$$\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} \overset{\text{vacuum}}{\Rightarrow} \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$$
$$\sim 120\pi\Omega.$$

Poynting vector

$\bar{S} = \bar{E}\times\bar{H}$, $\langle\bar{S}\rangle = \frac{1}{2}Re\{\bar{E}\times\bar{H}^*\}$ if $\bar{H}/\bar{E}$, are complex plane waves.

$\rightarrow$ power through a surface dA $\Rightarrow \left[\frac{W}{m^2}\right]$.

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Polarization

How $\bar{E}(t)$ traces in time...

$$\bar{E}(t) = Re\{\tilde{E}\, e^{j(\omega t - \bar{k}\cdot\bar{r})}\}$$

$\tilde{E}$ ↳ determines polarization

↳ $\hat{x}, \hat{y}, \hat{x}+\hat{y}$ all <u>linear</u>

$\hat{x}+j\hat{y}$, $\hat{x}-j\hat{y}$ <u>circular</u>

$\hat{x}+j\hat{y}$ → RC, $\hat{x}-j\hat{y}$ → LC

if $\hat{x}, \hat{y}$ are weight → <u>elliptical</u>.

For example, $\hat{z}$-propagating wave $\theta = \omega t - kz$

$\bar{E}(t)$

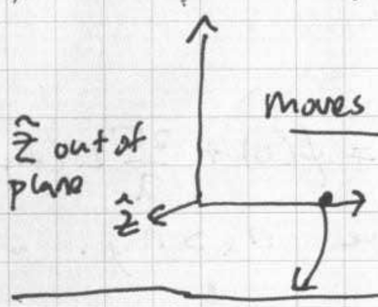
$Re\{(\hat{x}+j\hat{y})\, e^{j\theta}\}$

~~Re{(x̂+j~~

$\hat{x}\, Re\{e^{j\theta}\} + \hat{y}\, Re\{je^{j\theta}\}$

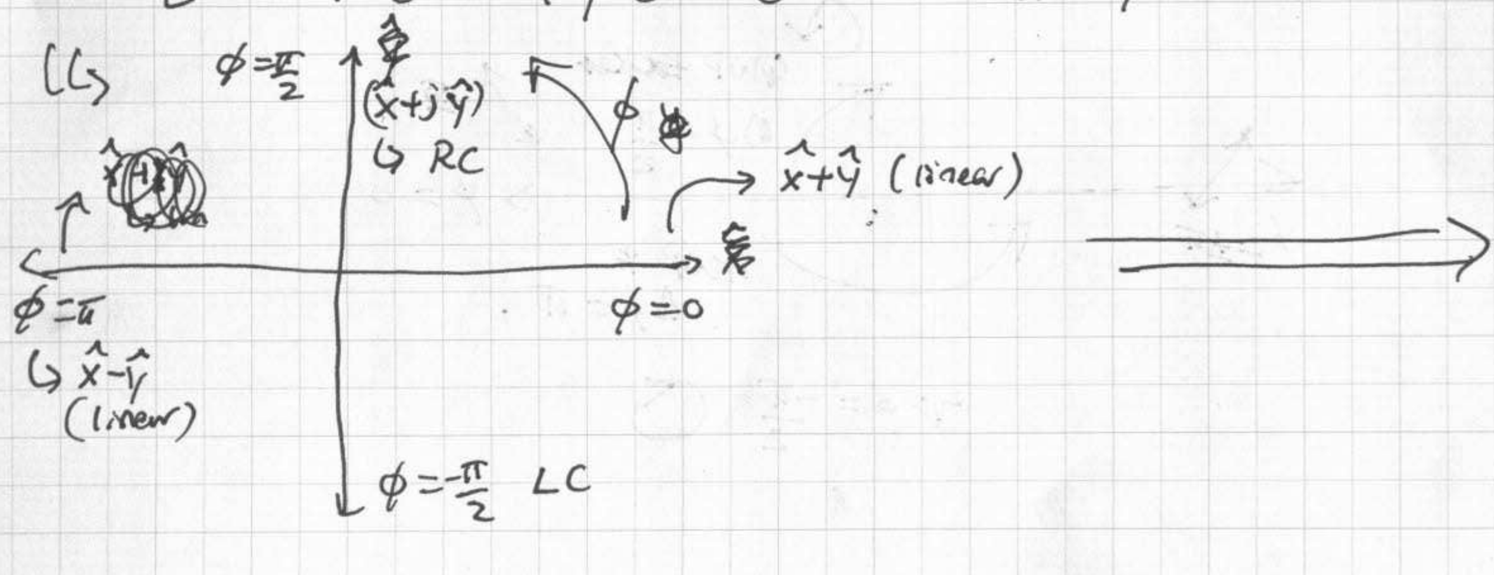
$\bar{E}(t) = \hat{x}\cos\theta - \hat{y}\sin(\theta) \Rightarrow$ @ 2 times $\omega t = 0$ $\hat{x}$

$\omega t = \frac{\pi}{2}$ $-\hat{y}$



We can also look @ polarization in terms of relative phase for $\hat{x}/\hat{y}$ components. Consider,

$$\tilde{E} = \hat{x}\, e^{-jkz} + \hat{y}\, e^{-jkz} e^{j\phi} \xrightarrow{z=0} \hat{x} + \hat{y}\, e^{j\phi}.$$



Birefringence

Now consider, propagation in medium over length L w/ n_x, n_y.

Then $\cancel{E = \hat{x} e^{-jk_x z} + \hat{y} e^{-jk_y Y}}$

$$E = \hat{x} e^{-jk_x z} + y e^{-jk_y z} e^{j\phi}$$

each of ~~t~~ $\hat{x}$ & $\hat{y}$ see a different k in their directions of polarization.

So $k_x = k n_x$, $k_y = k n_y$, where $k = \frac{2\pi}{\lambda}$.

Over a distance L, the term $e^{j\phi}$ changes as the waves in $\hat{x}/\hat{y}$ ~~point in~~ ~~differ~~ have different speeds.

The resulting phase @ $z = L$ is

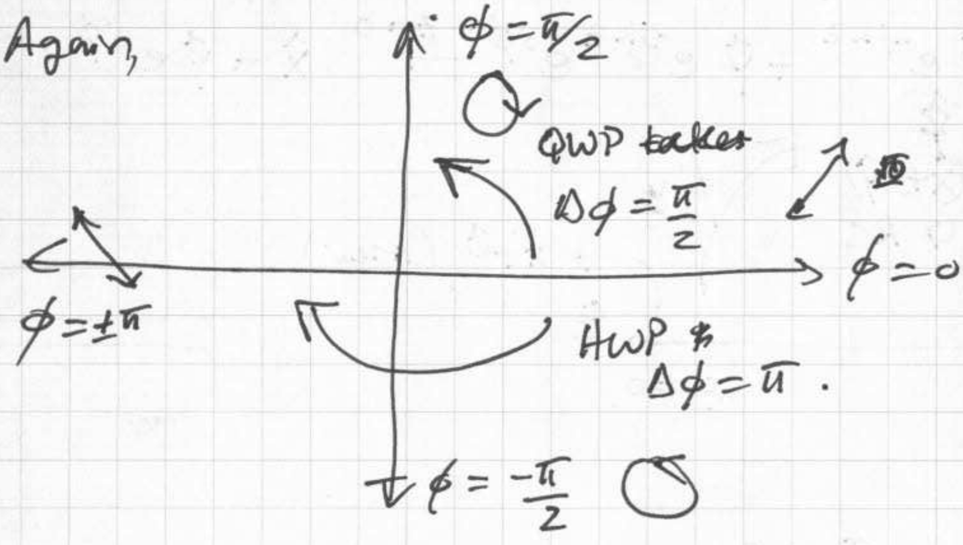
$$\phi(L) = \phi(0) + L(k_x - k_y)$$

$$= \phi(0) + kL(n_x - n_y) = \phi(0) + \frac{2\pi}{\lambda} L(n_x - n_y)$$

$$\Rightarrow \boxed{\Delta\phi = \frac{2\pi}{\lambda} L \Delta n}, \text{ where } n_x > n_y.$$

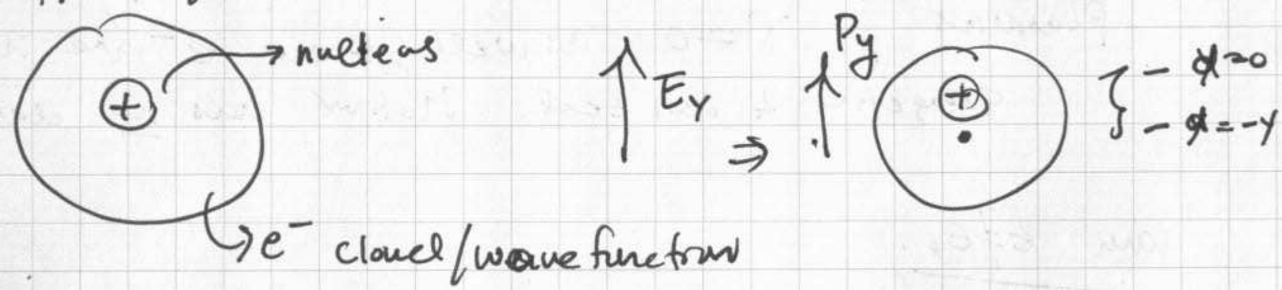
How do we construct HWP/QWPs ?

Again,



Waves in Polarizable Media – Lorentz Oscillator

- Driving electric field induces a frequency dependent polarization in a material.
- Typically the $\bar{E}$ field here is light/radiation.



We use a spring – dashpot – mass oscillator, (2nd-order)

$$m\frac{d^2y}{dt^2} + c\frac{dx}{dt} + ky = qE_Y$$

Terms:

① eqn of motion for free e^-.

② $\gamma\frac{dP}{dt} \Rightarrow$ damping

③ $\omega_0^2 P \Rightarrow$ bond spring

$$① \frac{d^2y}{dt^2} + ② \frac{c}{m}\frac{dx}{dt} + ③ \frac{k}{m}y = \frac{qE}{m}y \qquad P_Y = qN_y$$

$$\frac{1}{qN}\left[\frac{d^2P_Y}{dt^2} + \gamma\frac{dP_Y}{dt} + \omega_0^2 P_Y\right] = \frac{qE_Y}{m}$$

generally $\Rightarrow$

$$\frac{d^2\bar{P}}{dt^2} + \gamma\frac{d\bar{P}}{dt} + \omega_0^2\bar{P} = \frac{Nq^2}{M}\bar{E} = \frac{Nq^2}{\epsilon_0 m}\bar{E} = \epsilon_0\omega_p^2\bar{E} = \epsilon_0$$

Assuming $\bar{P}_y(t) = \bar{P}(\omega)e^{j\omega t}$

yields complex dielectric constant:

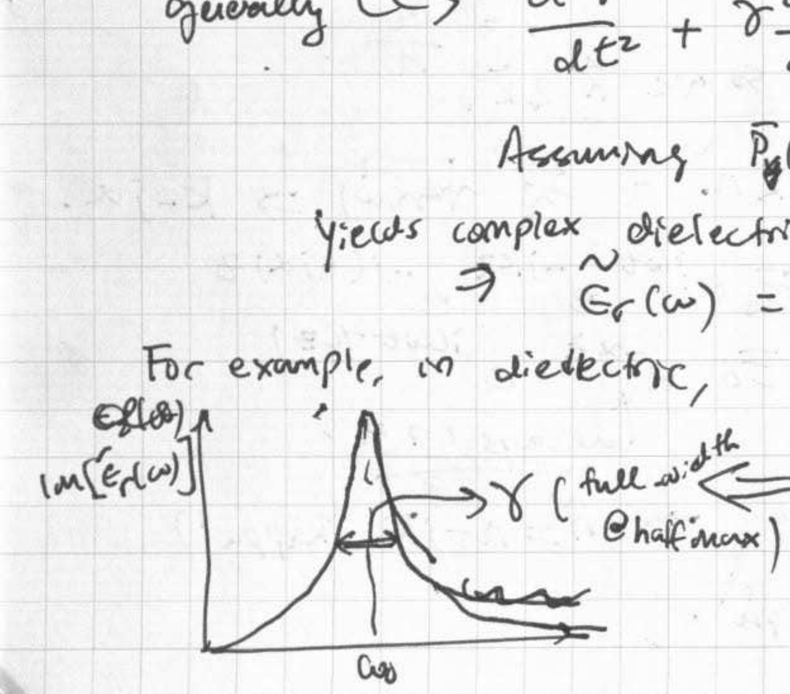
$$\Rightarrow \tilde{\epsilon}_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\gamma\omega}$$

$\omega_p^2 \leftarrow$ plasma frequency

$\omega_0^2 \uparrow$ resonant frequency

$j\gamma\omega \uparrow$ linewidth / damping

For example, in dielectric,



- Review TART & qualitative behavior for $\epsilon_r(\omega)$ in handout.
- For example, consider what ω_p^2, γ, ω_0 would be for various media.

In metals/plasmas $\omega_0 = 0$, as electrons are <u>unbound</u> to nuclei (think of Fermi sea from 5.11x / 3.091)

Plasmas, $\gamma = 0$ as well, since e^- are <u>highly</u> energetic & ionized. Motion has <u>no</u> damping.

<u>when $\epsilon \neq \epsilon_0$.</u>

$$n = \frac{c}{v_p} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\mu_r\epsilon_r} \overset{\mu_r \approx 1}{=} \sqrt{\epsilon_r}$$

(1) temporal frequency does not change (ω).

(2) Spatial frequency <u>does</u> change (k or β)

$$k = \frac{\omega}{v_p} = \frac{\omega}{c} n = n k_0 = n \frac{2\pi}{\lambda_0} \leftarrow \text{free-space wavelength}$$

$$\lambda = \frac{\lambda_0}{n}.$$

<u>If $\tilde{\epsilon}_r(\omega)$ is complex, so are $\tilde{n}$ & $\tilde{k}$.</u>

$$n = \sqrt{\tilde{\epsilon}_r(\omega)} \Rightarrow \tilde{k} = k\tilde{n} = \frac{2\pi}{\lambda}\sqrt{\tilde{\epsilon}_r(\omega)} \Rightarrow k - j\alpha.$$

$$E = E_0 e^{j(\omega t - \tilde{k} z)} = E_0 e^{j\omega t} e^{-jkz} e^{-j(-j\alpha)z}$$

$$= E_0 \underset{\uparrow \text{decays in } z}{e^{-\alpha z}} e^{j(\omega t - kz)}$$

$$\alpha = \frac{2\pi}{\lambda_0} * \operatorname{Im}\{\tilde{n}\} \Rightarrow \tilde{n} = n - j\kappa \text{ (kappa')}$$

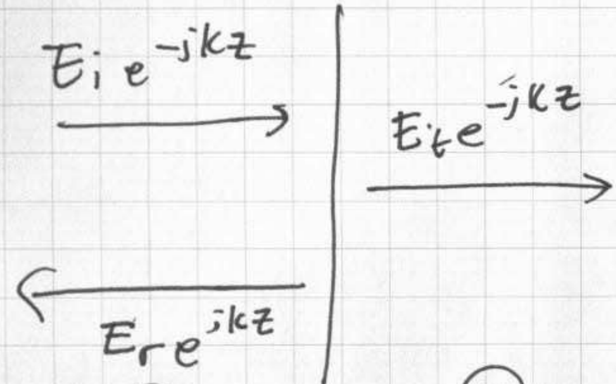
$$= \frac{2\pi}{\lambda_0} \kappa \text{ (kappa)}.$$

Reflection / Transmission / Diffraction Boundaries

$\Rightarrow$ Tangential $\bar{E}$ continuous @ surface
Normal $\mu_0 \bar{H}$ continuous @ surface.

Boundary coefficients

$$= \frac{E_r}{E_i}$$



Media: 1 $z=0$ 2

Impedances: η_1 η_2
or
Indices: n_1 n_2

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{n_1 - n_2}{n_1 + n_2}$$

$$t = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{2n_1}{n_1 + n_2}$$

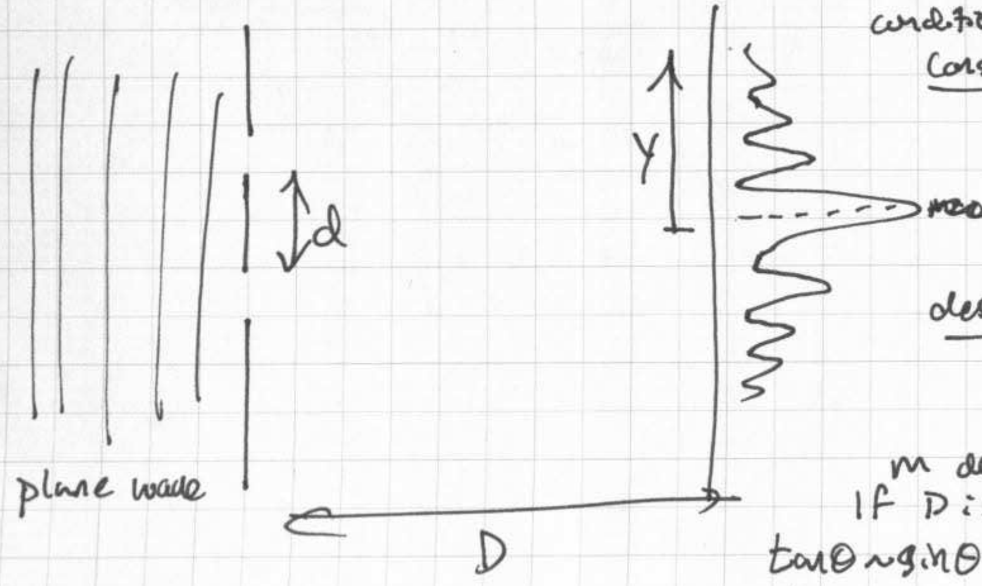
Lossless Media

$-1 \le r \le 1$ $\quad 0 \le t \le 2$

Lossy Media, $\tilde{r}, \tilde{t}, \tilde{\eta}, \tilde{n}$ are complex

$|r| \le 1$, $|t| \le 2$

Diffraction



Define $\tan\theta = \frac{y}{D}$.

condition for Constructive (@ maxima)

m^{th}

$$d \sin\theta = m\lambda$$

destructive

$$d \sin\theta = \left(m + \tfrac{1}{2}\right)\lambda.$$

m denotes node from center.

If D is really big.

$\tan\theta \sim \sin\theta \sim \theta \sim \frac{y}{D}$.

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Tutorial 10: Midterm 2 Questions and Oblique Transmission/Reflection

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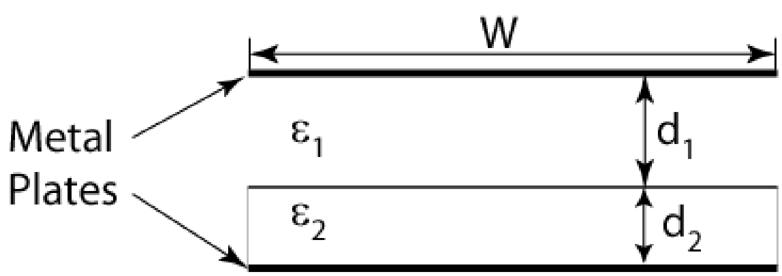
Notes:

1. Optics lab writeup due tomorrow (November 17); problem Set 8 due Thursday (November 19).
2. Recommended reading in Shen and Kong, Chapter 4 (pg. 85—112). This chapter does a great job of explaining TE and TM wave reflection and transmission.

1 Quiz 2 material

1.1 Partially Filled Capacitor



Partially Filled Capacitor, similar to the quiz problem.

There are two methods to solve for the capacitance of this structure. You can either apply boundary conditions at the interface to get the fields in both regions or you can recognize that this is two capacitors in series (NOT two capacitors in parallel as in previous problem sets).

Boundary Conditions Solution:

Which of the following equations describes the fields we are interested in at the boundary between material 1 and 2?

$$\hat{n} \times (H_1 - H_2) = J_s \qquad \hat{n} \times (E_1 - E_2) = 0 \tag{1}$$
$$\hat{n} \cdot (B_1 - B_2) = 0 \qquad \hat{n} \cdot (D_1 - D_2) = \rho_s \tag{2}$$

What is the E-field in Region 1 and how do we find it?

What is the E-field in Region 2 and how do we find it?

What is the capacitance of the structure?

Series Capacitors Solution:

What is the capacitance of the top half of the structure?

What is the capacitance of the bottom half of the structure?

What is the capacitance of the total structure?

1.2 Forces using the energy method revisited

In previous tutorials/lectures, we developed a method for computing forces using the stored energy in the system.

Recall that for magnetic systems,

$$dW_m = id\lambda - F_x dx. \tag{3}$$

Differentiating by dt relates the power flow in the system, and we can integrate over a constant contour of λ to solve for the force

$$F_x = -\frac{dW_m}{dx}\Big|_\lambda \qquad W_m = \frac{\lambda^2}{2L}, \tag{4}$$

Similarly, for electric systems,

$$dW_e = VdQ - F_x dx \tag{5}$$

which yields the force,

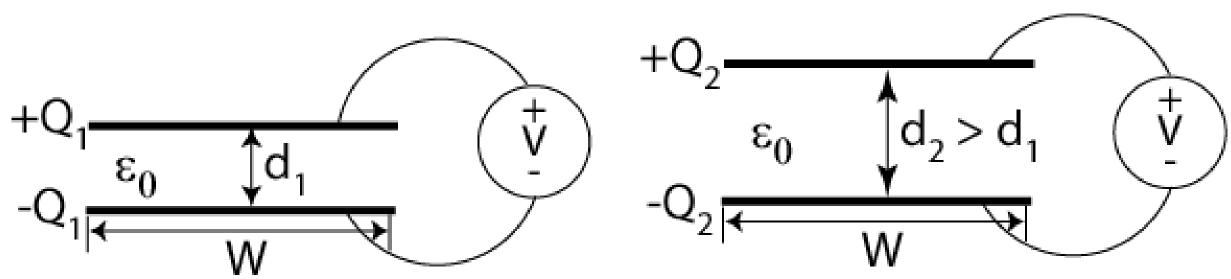
$$F_x = -\frac{dW_e}{dx}\Big|_Q \qquad W_e = \frac{Q^2}{2C}. \tag{6}$$

How do we apply the formulas above?

First, understand that we need to define the variable x as increasing in the direction we expect the force to be acting. Then, write an expression for the stored energy (or inductance/capacitance) in terms of the problem dimensions and the variable x.

I would advise you on problem sets and quizzes to define the magnitude of the force and then describe how it's acting on the system (closing the gap, opening the gap, drawing the plates together, forcing the plates apart, etc.).

1.3 What happens when you leave the battery connected? Or why we keep Q and λ constant.



Partially Filled Capacitor, similar to the quiz problem.

In applying the energy method we need to keep Q constant for electrostatic problems or λ constant for magnetostatic problems.

Consider the diagram above, what happens to Q as we change the distance between plates? Is $Q_2 > Q_1$ or is $Q_2 < Q_1$?

We know that the charge changes, so in addition to moving the plates we are also moving a charge either along or against the voltage drop in the source.

2 Oblique Incidence Reflection and Transmission

In questions of oblique incidence, we consider a plane wave from a medium (ϵ_0, μ_0) incident on another medium (ϵ_t, μ_t). Such a case is shown in Figure 1, where the incident plane wave is linear polarized in $\hat{y}$. The $\hat{x} - \hat{y}$ plane is called the *plane of incidence*, which is the plane formed by the normal to the boundary surface and the incident incident **k**. The incident wave in Figure 1 is called transverse electric (or 'TE') because **E** is perpendicular (or transverse) to the plane of incidence. If **E** is in the plane of incidence and **H** is transverse, the incident wave is a TM (transverse magnetic) wave. Remember that a TE (TM) wave's incident, reflected, and transmitted **E**'s (**H**'s) will *always* be orthogonal to the plane of incidence.

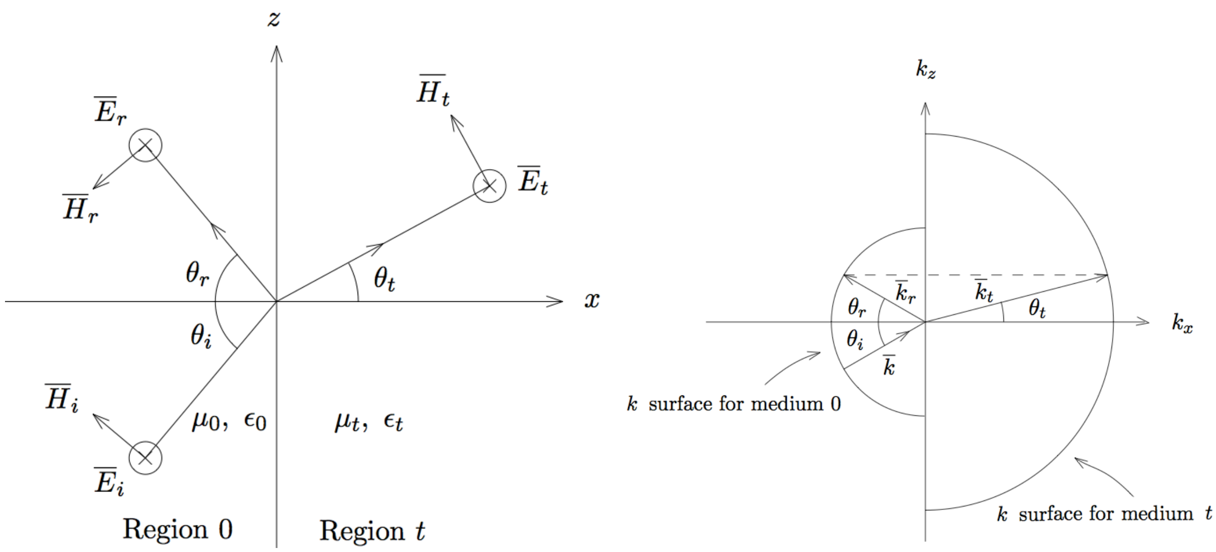


Figure 1: Reflection and transmission of TE waves at a plane boundary separating Regions 0 and t. Each arrow has a magnitude k and direction θ given by $\mathbf{k}$.

2.1 Boundary Conditions and Phase Matching

Solving an oblique incidence problem boils down to knowing two things: first, reflected and transmitted wave amplitudes (given dutifully by the Fresnel coefficients), and second, the reflected and transmitted wave vectors $\mathbf{k}$, whose angles are determined by Snell's law (how are the magnitudes determined?). Let's examine the TE case in Figure 1.

The wave solutions in each region are written in Equation 7. Recall that the $\mathbf{E}$'s will be pointing in $\hat{y}$ for a TE wave, so we can write down each $\mathbf{E}$ and relate their amplitudes using the Fresnel coefficients (which we use assumed values):

$$\begin{array}{ccc} \text{Incident} & \text{Reflected} & \text{Transmitted} \\ \mathbf{k} = \hat{x}k_x + \hat{z}k_z & \mathbf{k} = -\hat{x}k_{rx} + \hat{z}k_{rz} & \mathbf{k} = \hat{x}k_{tx} + \hat{z}k_{tz} \\ \mathbf{E}_i = \hat{y}E_i e^{-j\mathbf{k}\cdot\mathbf{r}} & \mathbf{E}_r = \hat{y}r_{\text{TE}}E_i e^{-j\mathbf{k}_r\cdot\mathbf{r}} & \mathbf{E}_t = \hat{y}t_{\text{TE}}E_i e^{-j\mathbf{k}_t\cdot\mathbf{r}} \\ \mathbf{H}_i = \dfrac{\mathbf{k}\times\mathbf{E_i}}{\omega\mu_0} & \mathbf{H}_r = \dfrac{\mathbf{k}_r\times\mathbf{E_r}}{\omega\mu_0} & \mathbf{H}_t = \dfrac{\mathbf{k}_r\times\mathbf{E_t}}{\omega\mu_t} \end{array} \tag{7}$$

The Fresnel coefficients can be derived by applying boundary conditions to the incident, transmitted, and reflected waves at the $x = 0$ boundary. If the incident wave medium has index n_0 and the transmitted wave medium has index n_1, the Fresnel coefficients to relate the amplitudes of E_r^{TE} and E_t^{TE} are[1]

$$t_{\text{TE}} = \frac{(2\cos\theta_i)\, n_1}{(\cos\theta_i)\, n_1 + (\cos\theta_t)\, n_2} \qquad r_{\text{TE}} = \frac{(\cos\theta_i)\, n_1 - (\cos\theta_t)\, n_2}{(\cos\theta_i)\, n_1 + (\cos\theta_t)\, n_2}. \tag{8}$$

The $\mathbf{H}$ fields in these regions are then determined using Ampere's law, assuming we figure out the $\mathbf{k}$'s (which we'll get to in a minute).

[1] There are several equivalent ways of writing the Fresnel coefficients, this one is the easiest to calculate using n's and θ's.

We now know the amplitudes for the TE wave case, so we need to determine $\mathbf{k}_r$ and $\mathbf{k}_t$. A second consequence of applying boundary conditions is called phase matching. These conditions imply that the tangential components of three wave vectors $\mathbf{k}$ (incident), $\mathbf{k}_t$, and $\mathbf{k}_r$ are equal,

$$k_z = k_{rz} = k_{tz}. \tag{9}$$

In terms of angles defined in Figure 1,

$$k \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t. \tag{10}$$

This equality has a few practical consequences,

1. The first equality relates the incident and reflective angles. Because the $\mathbf{k}$ vector magnitudes in the incident medium are equal, $\theta_i = \theta_r$ as $\sin\theta_i = \sin\theta_r$.

2. The second equality yields Snell's law, which relates the $\mathbf{k}$ vectors in the incident (index n_0) and transmitted (index $n_1 = n_t$) media,

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k}{k_t} = \frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\mu_t\epsilon_t}} = \frac{n_0}{n_t}, \tag{11}$$

   or,

$$n_0 \sin\theta_i = n_1 \sin\theta_t. \tag{12}$$

   Recall that the critical angle $\theta_c = \theta_i$ for total internal reflection is given when $\theta_t = \pi/2$.

3. It can be shown by setting $r_{\mathrm{TM}} = 0$ (Equation 16) that

$$\theta_r + \theta_t = \theta_i + \theta_t = \frac{\pi}{2}, \tag{13}$$

   which implies that

$$n_0 \sin\theta_i = n_1 \sin\theta_t = n_1 \sin\left(\frac{\pi}{2} - \theta_i\right) = n_1 \cos\theta_i, \tag{14}$$

   or

$$\tan\theta_i = \frac{n_1}{n_0}. \tag{15}$$

   The initial angle for which the TM wave is totally transmitted is called Brewster's angle. Physically we can explain this by visualizing the dielectric media as consisting of dipoles that are excited by the transmitted wave and radiating at the same frequency. Each individual dipole has a radiation pattern that is maximum in a direction perpendicular to the dipole axis and null along the dipole axis. For a TM wave excitation, all dipoles oscillate parallel to the plane of incidence along the **E**-field lines. At the Brewster angle of incidence, the reflected $\mathbf{k}_r$ vector is in the same direction as the dipole oscillation in the transmitted medium. Thus, no TM wave is reflected.

The TM analysis is a straightforward permutation of the analysis in Equation 7 (where TE $\to$ TM, $\mathbf{E} \to \mathbf{H}$, and $\mu \to -\epsilon$), and Fresnel coefficients are,

$$t_{\mathrm{TM}} = \frac{(2\cos\theta_i)/n_1}{(\cos\theta_i)/n_1 + (\cos\theta_t)/n_2} \qquad r_{\mathrm{TM}} = \frac{(\cos\theta_i)/n_1 - (\cos\theta_t)/n_2}{(\cos\theta_i)/n_1 + (\cos\theta_t)/n_2}. \tag{16}$$

Example 1: Optical Fiber

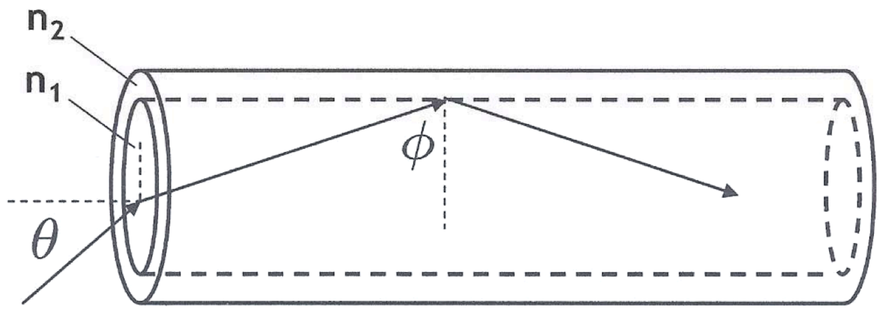


Figure 2: An optical fiber consists of a glass core of index n_1 surrounded by a coating of index $n_2 < n_1$. Suppose the beam of light enters the fiber from air at an index of θ

1. Using Snell's law, write down the expression for the smallest angle ϕ for which total internal reflection will occur.

2. Show that the greatest possible value for ϕ for which a ray can be propagated down a fiber is given by $\sin\theta_{\max} = (n_1^2 - n_2^2)^{1/2}$.

Example 2: Incidence Waves

A uniform plane wave in air impinges on a lossless dielectric medium at $\pi/4$. The transmitted wave propagates at $\pi/6$ with respect to the normal. It's frequency is 300MHz.

1. Find ϵ_2 in terms of ϵ_0.

2. Find the reflection coefficient r.

3. Obtain the mathematical expressions for the incident $\mathbf{E}$, reflected $\mathbf{E}$, and transmitted $\mathbf{E}$.

4. In both media, sketch the standing-wave pattern of $|E_{x,total}|$ as a function of z.

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Tutorial 11: Quantum Mechanics in 1-D Potentials

Notes:

1. Problem set 9 due Thursday *after* Thanksgiving.
2. Recommended reading in French and Taylor: Chapters 3 and 4. Chapters 1 and 2 in David Griffiths' *Introduction to Quantum Mechanics* are also useful. Both should be in Hayden, Barker, and/or the Physics Department Reading Room.

Quantum mechanics is an amazing model of the universe, allowing us describe atomic scale behavior with great accuracy—but in a way completely divorced from our perception of reality. Are such small things particles? Waves? Wave-particles? Not really. It's equally well-said that they may behavior like neither, and in this way we must treat them with mathematical abstraction.

1 Rules for 1-D Quantum Mechanics

Our mathematical abstraction of choice is the wave function, sometimes denoted as ψ, and it allows us to predict the statistical outcomes of experiments (i.e., the outcomes of our measurements) according to a few rules

1. At any given time, the state of a physical system is represented by a wave function $\psi(x)$, which—for our purposes—is a complex, scalar function dependent on position. The quantity $\psi(x) = \psi^*(x)\,\psi(x)$ is a probability density function. Furthermore, ψ is complete, and tells us everything there is to know about the particle.
2. Every measurable attribute of a physical system is represented by an operator that acts on the wave function. In 6.007, we're largely interested in position ($\hat{x}$) and momentum ($\hat{p}$) which have operator representations in the x-dimension

$$\hat{x} \rightarrow x \qquad \hat{p} \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}. \tag{1}$$

Outcomes of measurements are described by the expectation values of the operator

$$\langle \hat{x} \rangle = \int dx \psi^* x \psi \qquad \langle \hat{p} \rangle = \int dx \psi^* \frac{\hbar}{i}\frac{\partial \psi}{\partial x}. \tag{2}$$

In general, any dynamical variable Q can be expressed as a function of x and p, and we can find the expectation value of

$$\langle Q(x,p)\rangle = \int dx \psi^* Q\left(x, \frac{h}{i}\frac{\partial}{\partial x}\right)\psi. \tag{3}$$

3. The time evolution of the wave function is described by the Schrodinger wave equation, a partial differential equation that is fundamentally a statement of energy conservation:

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)\psi \\ &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(\hat{x})\psi. \end{aligned} \tag{4}$$

The operator acting on ψ on the right is called the Hamiltonian. Fixed energy solutions—eigenstates—of this equation are called stationary states (because they don't evolve in time), and are found by solving

$$E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(\hat{x})\psi. \tag{5}$$

2 Heisenberg Uncertainty and Photons

In lecture, our characterization of measurement uncertainty dealt with the observation of electrons with photons of definite energy and momentum

$$E = \hbar\omega \qquad p = \hbar k = \frac{2\pi\hbar}{\lambda}. \tag{6}$$

The Heisenberg microscope shows that it's difficult to *simultaneously* know the position and momentum of observed electrons with great precision:

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{7}$$

From basic probability theory, it can be shown that the uncertainty in x is given by,

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2. \tag{8}$$

2.1 Infinite Square-Well

There are painfully few exactly-solvable potentials in quantum mechanics. The energy eigenstates of the infinite square well problem has several important features:

1. Solutions to 1-D Schrodinger equation are eigenstates given by
$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin(k_n x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \tag{9}$$
with associated energies
$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{2mL^2}. \tag{10}$$
Note that these are called energy eigenstates because they were states of definite energy, meaning that $\Delta E = 0$.
2. The eigenstates are alternately even and odd with respect to the center of the well.
3. The eigenstates are mutually orthogonal, meaning that
$$\int \psi_m^*(x)\, \psi_n(x) = \delta_{mn}, \tag{11}$$
where δ_{mn} is 0 if $m \neq n$, and 1 if $m = 1$. For calculating expectation values for the infinite square well where $m = n$, the following identity will be useful:
$$\sin^2(\theta) = \frac{1}{2}\left[1 - \cos(2\theta)\right]. \tag{12}$$

3 Qualitative Properties of Wave Functions

When we are only interested in qualitative behavior of the wave function then we can sketch expected wave functions for different energy levels following some simple rules:

1. Outside of the potential well we expect the wave function to decay smoothly to zero. The larger the difference between the potential V and the particle energy E, the faster we expect the wave function to decay. If the potential V at the boundary is infinite, then the wave function will go to zero right at the boundary.
2. Inside the potential well, we expect the wave function to behave roughly like a sine or cosine function.
3. We expect any symmetry in the potential well to be reflected in the wave function. If we identify a point of symmetry in the potential, then the wave function should be either an even or odd function about that point.
4. The number of nodes in the wave function for a state should be the state number (n) minus 1 ($n = 1$ for ground state, 2 for first excited state, and so on).
5. The curvature of the wave function is related to the kinetic energy of the state. If the well has a potential that varies with position, then in regions with higher kinetic energy the wave function should have a shorter "wavelength".
6. If the well has a potential that varies with position, the particle will spend less time in regions where it has higher potential energy so the wave function will be (relatively) smaller in those regions.

Example 1: Mystery Wave Function

Consider the wave function

$$\psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}, \tag{13}$$

where A, λ, and ω are positive, real constants.

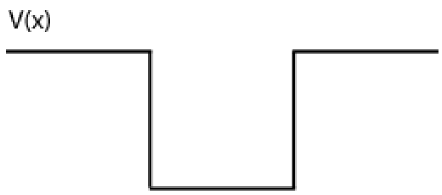
1. Normalize ψ.

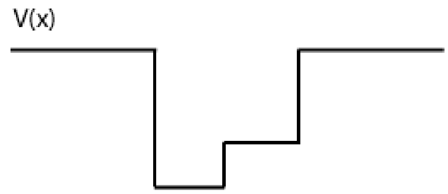
2. Determine the expectation values of x and x^2.

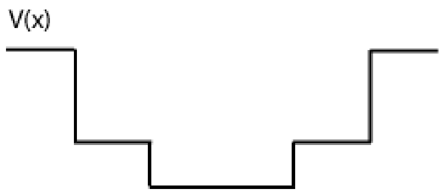
3. Find the standard deviation of x. Sketch the graph of $|\psi|^2$, as a function of x, and mark the points $\langle x\rangle + \sigma$ and $\langle x\rangle - \sigma$ to illustrate the sense in which σ represents the 'spread' in x. What is the probability that the particle would be found outside this range?

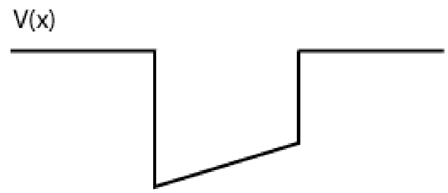
Example 2: **Qualitative Wave Functions**

Using the rules from "Qualitative Properties of Wave Functions", sketch the wave function for the first several energy states for the following potential wells.









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Tutorial 12: Tunneling and Flash Memory

Notes:

1. Quantum tunneling lab tonight and tomorrow. Final problem set due Thursday.

The coming two weeks is devoted to understanding the implications of quantum mechanics to modern electronics (semiconductors and metals, in particular) and optoelectronics (lasers). Wave function transmission/reflection is analogous to examples of evanescent wave reflection/transmission for EM waves incident at the critical angle or sound waves leaking through walls, so it may help to make these connections as we finish this material.

1 1D Wave Functions, Potentials, and Scattering

Evanescence occurs in a wave propagation problem when a thin medium (labeled '2') is situated between two identical media (labeled '1') and the properties of these media are such that the wave equation has propagating solutions in medium 1, but decaying solutions in medium 2. In optics, medium 1 might be vacuum and medium 2 might be a conductor (metal) or plasma. In quantum mechanics, medium 1 is a region of space where the particle's total energy is greater than its potential energy, and medium 2 is a region of space (or 'barrier') where the particle total energy is less than its potential energy.

In each region, we are looking for stationary (fixed in time) solutions to Schrodinger's equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi. \tag{1}$$

As in normal incidence reflection/transmission for EM plane waves, we can follow several steps for relating the amplitudes of the reflected and transmitted waves—

1. Write down wave functions in each region of constant potential. Solutions will either oscillate (e^{ikx}) or decay (e^{-kx}) depending on the sign of k^2 when solving the wave equation—

$$\frac{d^2\psi}{dx^2} = k^2\psi \qquad k^2 = \frac{2m}{\hbar^2}\left(V - E\right). \tag{2}$$

—which, in turn, is dependent on weather $V > E$ or $V < E$.

2. Match boundary conditions for wave function and its derivative at the boundary (here $x = 0$) of *finite* potential.

$$\psi_1\left(0^+\right) = \psi_2\left(0^-\right). \tag{3}$$

$$\frac{d\psi_1}{dx}|_{x=0^+} = \frac{d\psi_2}{dx}_{x=0^-}. \tag{4}$$

3. Normalize wave functions or use initial conditions.

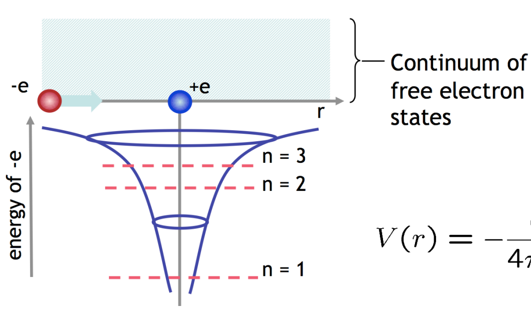
In class, we discussed a few examples using constant potentials. Potentials with spatial variation can be handled by the WKB approximation technique (see 6.728).

1. **Finite potential step**. For this scattering problem, an propagating wave is incidence at a barrier V ($x = 0 \to \infty$) at $x = 0$. If $E > V$, the reflection and transmission solution,
$$R = |\frac{1 - k_2/k_1}{1 + k_2/k_1}|^2 \qquad T = \frac{4k_2/k_1}{|1 + k_2/k_1|^2}, \tag{5}$$
was exactly similar to the case of EM normal incidence. If $E < V$, $R = 1$ and $T = 0$, and the incident wave is totally reflected.

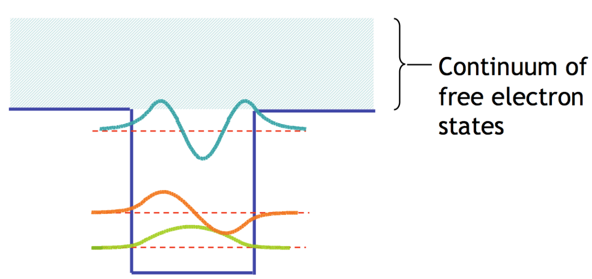
2. **Finite potential barrier.** An incident wave here is can be transmitted partially through finite barrier of potential V spanning from $x = -a$ to $x = a$. For the $E < V$ case,
$$\begin{aligned} T &= \frac{1}{1 + \frac{V^2}{4E_0(V-E_0)\sinh^2(2\kappa a)}} \\ &\approx \frac{1}{1 + \frac{V^2}{16E_0(V-E_0)}} e^{-4\kappa a} = \frac{16E_0\,(V - E_0)}{V^2} e^{-4\kappa a}, \end{aligned} \tag{6}$$
where the last step is an approximation taken for wide barriers. This problem and its applications to flash memory are examples later in the handout.

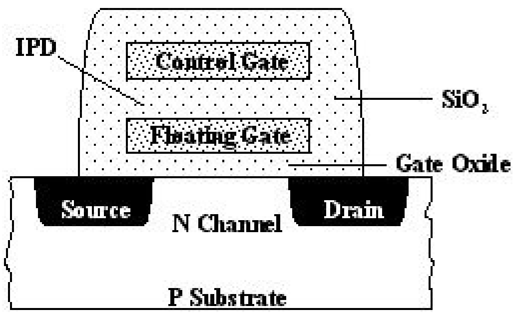
2 Simple Approximations of Molecular Structure

Our discussion on solids this week will be mostly qualitative, starting with the square-well approximation of the Coulomb potential. More on band structure of solids next week.



$$V(r) = -\frac{e^2}{4\pi\epsilon_o r}$$





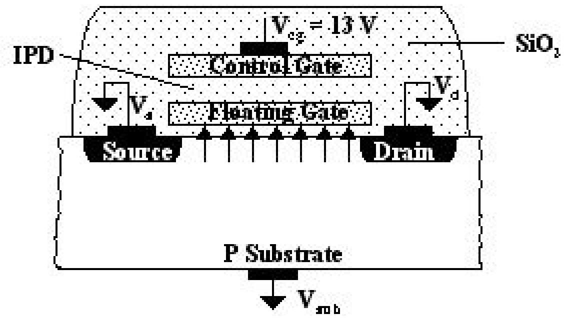


Figure 1: Floating Gate Flash Memory Structure. The dielectric constant of SiO_2 is $3\epsilon_o$. The thickness of the oxide between the substrate and the floating gate is 10 nm and the thickness of the oxide between the control gate and floating gate is 45 nm. The floating gate is 10 nm thick.

3 Flash Memory Structure

Flash memory stores information by changing the charge on a floating gate (a piece of metal that is not contacted to other conductors). We interpret a floating gate with a positive charge as a `1` and a floating gate with a negative charge as a `0`. The details by which the flash memory is read (e.g. how we detect the presence of a positive or negative charge) is a topic best left to 6.012. A one sentence explanation is: when there is a positive charge on the floating gate, current can flow from the source to the drain and we read a `1`; when there is negative charge on the gate, current cannot flow and we read a `0`.

The interesting part (that you won't learn in 6.012) is how we *change* the charge on the floating gate. To write a bit, we must somehow put electrons on or take electrons off of the isolated bit of metal. The write operation is accomplished through electron tunneling and is explored in a following problem.

Example 1: **Finite Potential Barrier Reflection/Transmission**

Consider a potential barrier $V = V_0$ from $x = 0$ to $x = a$ and 0 everywhere else.

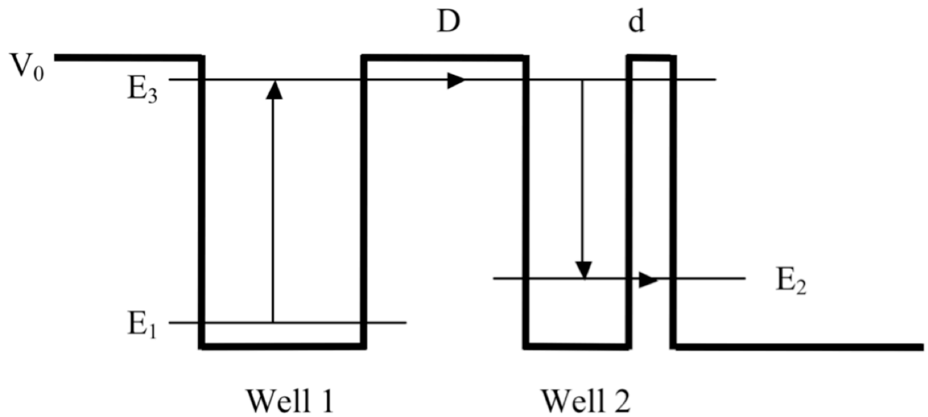
1. Write the Schrodinger wave solutions in each region? What is the wave vector k for each solution?

2. What are the boundary condition equations for each reflecting interface?

3. What ratio of amplitude coefficients determine T? Using the boundary equations from the previous part, relate the incoming and outgoing coefficients for the wave function amplitudes as a matrix equation.

Example 2: Tunneling of Electrons out of Quantum Wells

George Gamow is interested in designing a delay circuit using a quantum well structure. The basic idea is to capture an electron in a quantum well temporarily, and then detect after it has leaked out. The potential energy diagram of the structure is drawn in the figure below. The electron is initially in the ground state of Well 1. The electron is promoted to an excited



state near the top of the well by absorbing light from a laser pulse (the laser pulse lasts only a few femtoseconds). When the electron reaches the excited state, there is a possibility that it will tunnel into an excited state of Well 2, where it can lose energy to lattice vibrations and decay to the ground state (this decay takes about 1 psec). If designed properly, the electron will make it to the ground state of Well 2 where it will remain for a time τ (about 1 nsec) before tunneling out to the right side of Well 2 where it will be collected. In the figure all potential barriers are at V_0 above the bottom of the wells, which is at $E = 0$. Assume that Well 1 has a width L_1, and that well 2 has a width L_2.

1. If a classical electron, of mass m, is in Well 2 at energy E_2, what is the classical velocity of the electron?

2. How many times per second would the classical electron inside Well 2 with energy E_2 hit the thin barrier (of width d)?

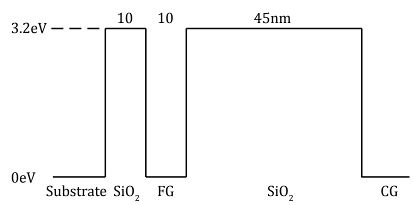
3. What is the probability that quantum mechanical electron tunnels through the thin barrier (of width d) during a bounce?

4. If the electron is at energy E_2 inside Well 2, find an expression for the tunneling rate out of Well 2. (Note $1/\tau$ is not an acceptable answer.)

Example 3: Reading and Writing Flash Memory

To begin, let's assume that we have already written information to our memory and charge is stored on the floating gate. Let's further assume that we have an excess of electrons on the floating gate (thus the bit represented is a logical `0`). We are interested in how long the bit will remain a logical `0`. We know that the electrons will eventually tunnel through the oxide, depleting the charge on the floating gate. When the charge on the floating gate is nearly depleted, we will no longer be sure of the logical value of the bit.

If a single free electron is present in the floating gate, it will see the 10 nm thick potential barrier, shown in the figure below. The 3.2 eV potential barrier represents the energy difference between the electron levels of the conductor that forms the floating gate and the electron levels of SiO_2 (an insulator). As the floating gate is only 10 nm wide, it is a quantum well within which the energy of the electron is quantized.

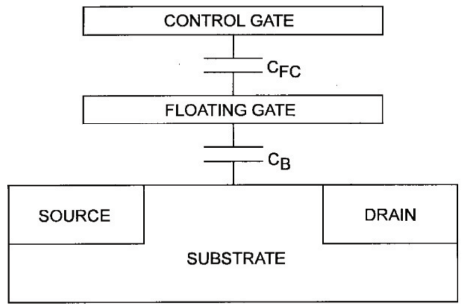


1. Estimate the ground state energy level ($n = 1$) of the electron on the floating gate by using the energy level spacings of the infinite potential well.

2. What is the transmission probability T that the electron will tunnel through thick dielectric? What is T for the thin dielectric?

To write a flash memory bit, we apply a 13 V potential across the floating gate as shown in the right figure. (The source, drain, and substrate electrodes are grounded, while the control gate is biased at 13 V.)

1. Sketch the magnitude of the electric field across the structure. Assume that the gates and substrate are metals (hence the electric field inside of them is zero). It may help to visualize the structure as capacitors in the following manner:



2. What is the potential drop (in volts) across the thick SiO_2 dielectric? What is the potential drop across the thin SiO_2 dielectric?

3. Use your answer to **(b)** to sketch the potential well that will be experienced by an electron incident from the substrate. Sketch the ground state wave function that would be a solution to the Schrodinger equation for this potential for an electron incident from the substrate. Finally sketch the probability distribution associated with the wave function.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.007 – Applied Electromagnetics
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Tutorial 13: Band Structure and Semiconductors

Notes:

1. Optional extra-credit labs tonight and tomorrow.
2. Review session and extra office hours for the final exam TBA. Let your TA know if there are any topics you might need help with before the week is over.

1 Band Structure in Solids

Band structure in solids is an emergent behavior of the periodicity of the electrostatic potential that electrons experience. Bonding in chains of atoms is representationed by superpositions of Coulomb potentials (centered at the proton):

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}. \tag{1}$$

You can think of it this way, that a solid is a crystalline material with a basic unit cell (i.e., the atoms that are repeated through the lattice) repeats itself everywhere in space in three dimensions. Under the periodic boundary conditions throughout the entire crystal, a fundamental result is that allowed electron states cluster in sharply defined energy bands, leaving bandgaps of forbidden energies in-between. Electrons fill the allowed bands following the Pauli Exclusion Principle, and the location of the last filled electron state (either at the edge of the valence band or in the conduction band) determines if a material is a metal, insulator, or semiconductor.

2 Semiconductors

Semiconductors have small bandgaps. Diamond (an insulator), for example, has a 6eV bandgap. For electronic devices that operate near room temperature, the most widely used semiconductors have bandgaps on the order of 0.7eV to 2eV. Si and GaAs have room temperature bandgaps of 1.12eV and 1.42eV, and graphene (a 1-D sheet of carbon atoms) is a 0eV semiconductor.

1. **Intrinsic semiconductors.** An intrinsic semiconductor fulfills our notion of an 'ideal' semiconductor, a perfectly crystalline material that is sufficiently pure (i.e., low concentrations of foreign atoms or impurities). At $T = 0$, Si is strictly an insulator. We

can make an intrinsic semiconductor conduct through either introducing foreign ions (extrinsic semiconductors), or through the following:

(a) *Increasing the temperature.* You might remember from the previous problem set on tunneling that the mean kinetic energy of an electron is $3k_BT/2$, where k_B is the Boltzmann constant ($k_B = 1.38 \times 10^{-23}$J/K) and T is the temperature in Kelvin. At room temperature, $3k_BT/2 \approx 26$meV. While this isn't quite comparable to a semiconductor bandgap, the probability that a rogue electron is liberated from the valence band and jumps into the conduction band grows with increasing temperature[1]. An electron excited to the conduction band leaves behind an empty state in the valence band called a hole, with an effective charge e^+. These holes can be filled by other electrons in the valence band, causing the hole to effectively migrate like an electron.

(b) *Interactions with light.* An electron-hole pair results from the break up of a covalent bond in the valence band. Energy conservation says that a photon with energy $\hbar\omega$ liberates an electron-hole pair with kinetic energy $\hbar\omega - E_{\text{gap}}$.

2. **Extrinsic semiconductors.** We can also add foreign atoms to change the number of electrons or holes in conduction and valence bands in two ways.

   (a) *N-type.* Electrons are the majority. Column V elements (P, As, Sb) in the periodic table use four electrons to bond to Si (column IV), and spare a fifth electron that is 'donated' to the conduction band.

   (b) *P-type.* Holes are the majority. Column III elements (B, Al) are short one electron and can only bond with three Si neighbors. A hole in the covalent bonding structure can 'migrate' away and be replaced by an 'accepted' bonding electron.

2.1 Electron Mobility and Charge Transport

In 6.007's short discussion of charge transport, we defined related conductivity and current density to a new property of charge carriers, m^*, which we called the *effective mass*:

$$\mathbf{J} = \sigma \mathbf{E}_{\text{DC}} \qquad \sigma = \frac{ne^2\tau}{m^*}. \tag{2}$$

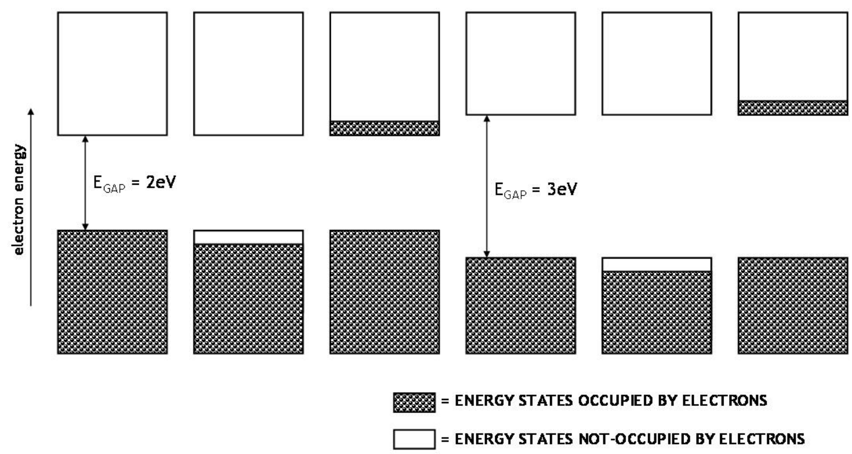
Here, n is the concentration of dopant atoms and τ is the time between scattering events of electrons. Note that this mass is used to describe the movement of the electron according to $F = m^*a$. In conductors and doped semiconductors, m^* is a fraction of m_e.

[1]If the mean kinetic energy of an electron is $3k_BT/2 \approx 26$meV, the approximate deBroglie wavelength is about 7.6nm at room temperature. How many atoms are in a sphere of 7.6nm? Let's say 10000. An electron is somewhat spread out over many atoms, so it's useful to think of an electron or hole probabilistically spread out over many atoms.

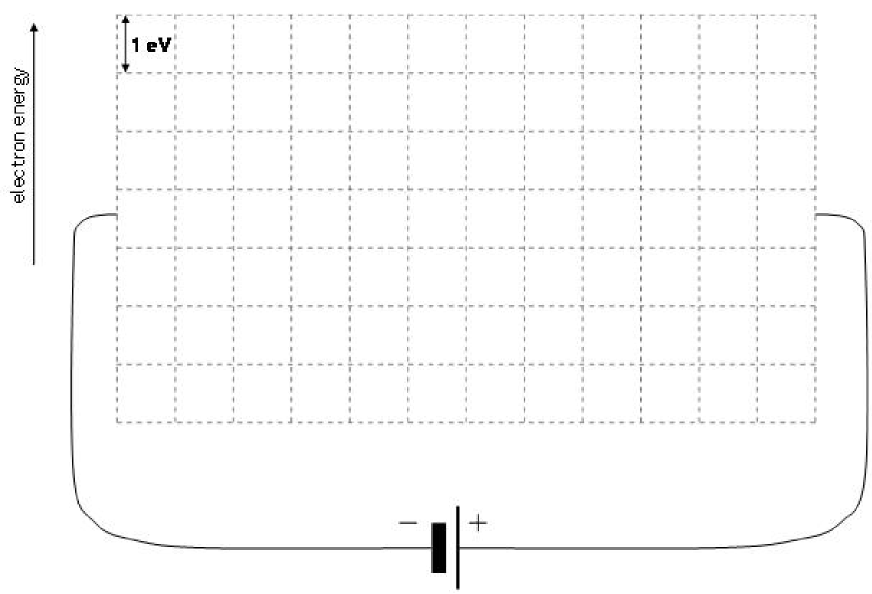
Example 1: **Energy Bands and Optoelectronic Devices**

Energy band diagrams of two different semiconductors are drawn below, each as undoped, p-type doped, or n-type doped semiconductor.

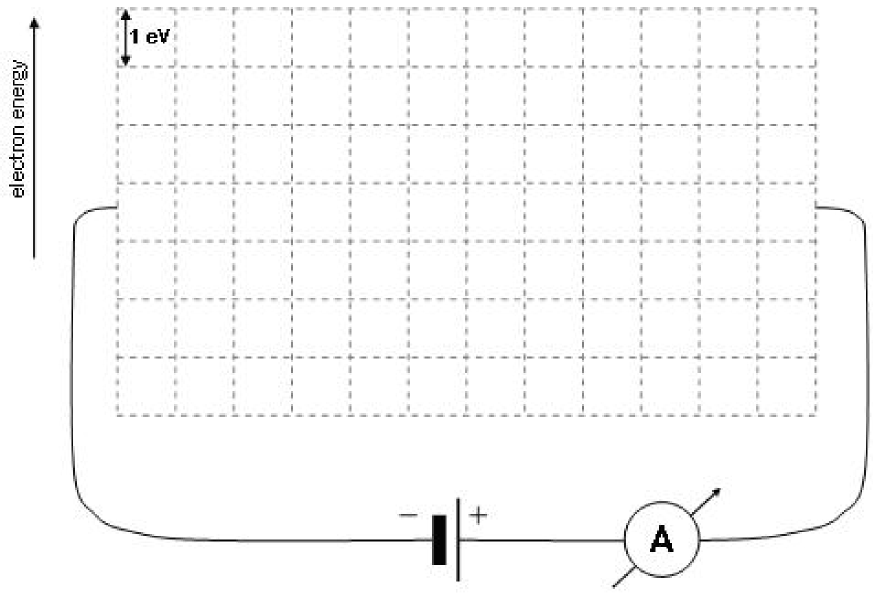
1. Label each semiconductor as undoped, p-type doped, or n-type doped.



2. Using the semiconductors from part (a) you are asked to design a light emitting device that emits red color. Please draw on the energy diagram below the series of semiconducting layers that would form the device. Label the band gap and type of each of the materials you use. Pay attention to the polarity of the battery attached below.

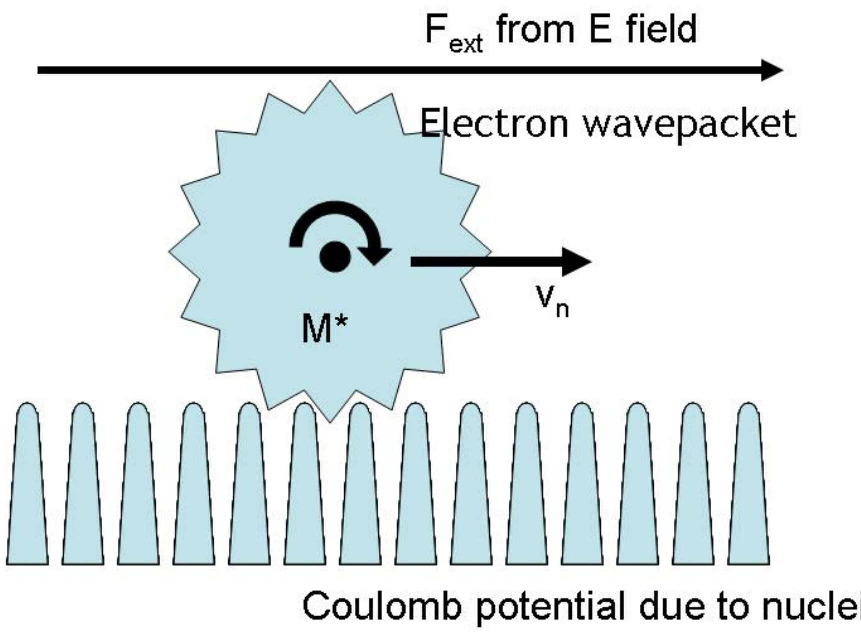


3. Using the semiconductors from part (a) you are asked to design a photodetector that is sensitive to blue color photons and other higher energy photons. Please draw on the energy diagram below the series of semiconducting layers that would form the device. Label the band gap and type of each of the materials you use. Pay attention to the polarity of the battery attached below.



Example 2: **Electron Transport**

1. Electron transport in a solid-state lattice can be modeled in semiclassical form as a mass m^* responding to an external force F_{ext}. If we know that the energy of the lattice



E_n depends on the lattice spatial frequency k, then the velocity expectation value $< v_n >$ and external force F_{ext} are expressed as:

$$< v_n > = \frac{< p >}{m} = \frac{1}{\hbar} \frac{\partial}{\partial k} E_n$$
$$F_{ext} = \hbar \frac{dk}{dt}$$

Derive an expression for the semiclassical effective mass in terms of E_n using $F_{ext} = m^* a$ and equating the two expressions for acceleration. Assume $< v_n >$ is the same as v_n. You may want to use the change of variables $\frac{\partial}{\partial t} = \frac{\partial}{\partial k} \frac{dk}{dt}$.

2. Table 1 below shows different effective masses (as ratio against free-space electron mass m_0)for different semiconductor materials. Which materials makes the faster transistors?

Name	Symbol	Germanium	Silicon	Gallium Arsenide
Effective mass for conductivity calculations				
Electrons	$m_{e,cond}^*/m_0$	0.12	0.26	0.067
Holes	$m_{h,cond}^*/m_0$	0.21	0.36	0.34