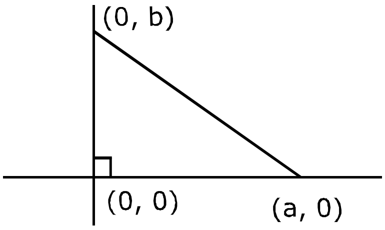
[MATHEMATICS] 09-01-2019_Evening

1. Let S be the set of all triangles in the xy-plane, each having one vertex at the origin and the other two vertices lie on coordinate axes with integral coordinates. If each triangle in S has area 50 sq. units, then the number of elements in the set S is :

(A) 32 (B) 9 (C) 18 (D) 36

Sol. **D**



$a, b \in I$

$|a.b| = 100.$

$ab = \pm 100.$

(i) $ab = 100 = 2^2\, 5^2$

$^{++}_{--}$ tatal factors = 9

18 cases possible for a and b.

(ii)

ab = -100
+ −
− +
} Same possible cases as above

$\therefore$ tatal Ans = 36

2. If the lines x = ay + b, z = cy + d and x = a′z + b′, y = c′z + d′ are perpendicular, then:

(A) bb′ + cc′ + 1 = 0 (B) cc′ + a + a′ = 0 (C) aa′ + c + c′ = 0 (D) ab′ + bc′ + 1 = 0

Sol. **C**

$$\frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}, \frac{x-b'}{a'} = \frac{y-d'}{c'} = \frac{z}{1}$$

For perpandicular lines

a a' + c' + c = 0

3. $f(x) = \int \frac{5x^8 + 7x^6}{\left(x^2 + 1 + 2x^7\right)^2} dx, (x \geq 0)$, and f(0) = 0, then the value of f(1) is:

(A) $-\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) $-\frac{1}{2}$

Sol. **C**

$$f(x) = \int \frac{5x^8 + 7x^6}{\left(x^2 + 1 + 2x^7\right)^2} dx,\ x > 0$$

take x^7 common from denominator $\dfrac{\frac{5}{x^6} + \frac{7}{x^8}}{\left(\frac{1}{x^5} + \frac{1}{x^7} + 2\right)^2} = \int \dfrac{\frac{5}{x^6} + \frac{7}{x^8}}{\left(\frac{1}{x^5} + \frac{1}{x^7} + 2\right)^2}$

$= \int \frac{-dt}{t^2}$ Let $\left(\frac{1}{x^5} + \frac{1}{x^7} + 2\right) = t \Rightarrow \left(\frac{-5}{x^6} - \frac{7}{x^8}\right) dx = dt$

$= \frac{1}{t} + C$

$= \frac{1}{\frac{1}{x^5} + \frac{1}{x^7} + 2} + C$

$f(x) = \frac{x^7}{x^2 + 1 + 2x^7} + C$

C = 0

$f(1) = \frac{1}{4} = \frac{1}{4}$

4. A data consists of n observations :

$x_1, x_2, \ldots\ldots, x_n$. If $\sum_{i=1}^{n}(x_i + 1)^2 = 9n$ and $\sum_{i=1}^{n}(x_i - 1)^2 = 5n$, then the standard deviation of this data is:

(A) $\sqrt{7}$ (B) 2 (C) 5 (D) $\sqrt{5}$

Sol. **D**

$\sum_{i=1}^{n}(x_i + 1)^2 = \sum x_i^2 + 2\sum x_i + n = 9n$

$\sum x_i^2 + 2\sum x_i = 8n$ —(1)

and $\sum x_i^2 + 2\sum x_i = 4n$ —(2)

$\therefore$ $\sum x_i^2 = 6n$ and $\sum x_i = n$

$\therefore$ $s.d = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2} = \sqrt{\frac{6n}{n} - \left(\frac{n}{n}\right)^2} = \sqrt{5}$

5. The number of natural numbers less than 7,000 which can be formed by using the digits 0, 1, 3, 7, 9 (repitition of digits allowed) is equal to :

(A) 372 (B) 375 (C) 374 (D) 250

Sol. **C**

0, 1, 3, 7, 9

□ + □□ + □□□ + □□□□

4 + 4×5 + 4×5×5 + 4×5×5×5

4 + 20 + 100 + 250

= 374

6. The equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lins $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is :

(A) x – 2y + z = 0 (B) 3x + 2y – 3z = 0 (C) 5x + 2y – 4z = 0 (D) x + 2y – 2z = 0

Sol. **A**

Direction ratios of plane : $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} \times \left(2i + 3\hat{j} + 4\hat{k}\right)$

$= \hat{i}(8) - -\hat{j}(1) + \hat{k}(-10)$

$= (8, -1, -2) \times (2, 3, 4)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & -1 & -10 \\ 2 & 3 & 4 \end{vmatrix} = \hat{i}(26) - \hat{j}(52) + \hat{k}(26)$$

$= \left(\hat{i} - 2\hat{j} + \hat{k}\right)$

7. The sum of the following series $1 + 6 + \frac{9\left(1^2 + 2^2 + 3^2\right)}{7} + \frac{12\left(1^2 + 2^2 + 4^2\right)}{9} + \frac{15\left(1^2 + 2^2 + \ldots.5^2\right)}{11} + \ldots.$ up to 15 terms, is:

(A) 7830 (B) 7510 (C) 7820 (D) 7520

Sol. **C**

$$1 + 3.2\ \frac{\left(1^2 + 2^2\right)}{5} + \frac{3.3\left(1^2 + 2^2 + 3^2\right)}{7} + \frac{3.4\left(1^2 + 2^2 + 3^2 + 4^2\right)}{9}$$

T_2 T_3 T_4

$$T_n = \frac{3.n\left(1^2 + 2^2 + \ldots. + n^2\right)}{(2n+1)} = \frac{3n(n)(n+1)(2n+1)}{6(2n+1)}$$

$$= \frac{n(n)(n+1)}{2}$$

$$\Rightarrow \frac{n^3 + n^2}{2}$$

$$\therefore Sn = \frac{1}{2}\left\{\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right\}$$

$$= \frac{1}{2}\left\{(15 \times 8)^2 + \frac{15 \times 16 \times 31}{6}\right\}$$

$$\frac{1}{2}\{14400 + 1240\} = 7820$$

8. The logical statement $[\sim(\sim p \vee q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$ is equivalent to:

(A) $(p \wedge \sim q) \vee r$ (B) $\sim p \vee r$ (C) $(\sim p \wedge \sim q) \wedge r$ (D) $(p \wedge r) \wedge \sim q$

Sol. **D**

$[\sim(\sim p \vee q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$

$[(p \wedge \sim q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$

$[p \wedge (\sim q \vee r)] \wedge (\sim q \wedge r)$

$p \wedge (\sim q \wedge r)$

$(p \wedge r) \wedge \sim q$

9. Let f be a differentiable function from R to R such that $|f(x)-f(y)|\le 2|x-y|^{3/2}$, for all x, y, ∈ R. If f(0) = 1 then $\int_0^1 f^2(x)dx$ is equal to:

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) 2

Sol. B

$$\frac{|f(x)-f(y)|}{|x-y|}\le 2|x-y|^{1/2}$$

$$\left|\frac{f(x)-f(y)}{x-y}\right|\le 2|x-y|^{1/2}$$

$$\lim_{y\to x}|f'(x)|\le 0$$

∴ $f'(x) = 0$

∴ $f(x) = $ Constant

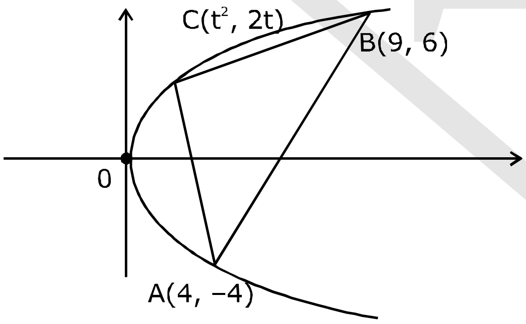
Given $f(0) = 1$ ∴ $f(x) = 1$

∴ $\int_0^1 dx = 1$

10. Let A(4, –4) and B(9, 6) be points on the parabola, $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of △ACB is maximum. Then, the area (in sq.units) of △ACB, is:

(A) $30\frac{1}{2}$ (B) 32 (C) $31\frac{3}{4}$ (D) $31\frac{1}{4}$

Sol. D



$$\text{Area} = \frac{1}{2}\begin{vmatrix} t^2 & 2t & 1 \\ 9 & 6 & 1 \\ 4 & -4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left\{t^2(10)-2t(5)-1(60)\right\}$$

$A = 5\,|t^2 - t - 6|$

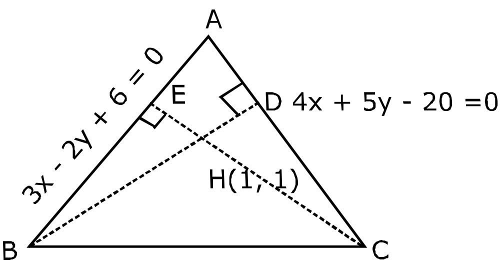
$$\frac{dA}{dt}=0, t=\frac{1}{2}$$

$$\text{Aera}_{max} = 5\left|\frac{1}{4}-\frac{1}{2}-6\right| = 5\left|\frac{1-2-24}{4}\right| = \frac{125}{4} = 31\frac{1}{4}$$

11. Let the equations of two sides of a triangle be 3x – 2y + 6 = 0 and 4x + 5y – 20. If the orthocentre of this triangle it at (1, 1) then the equation of its third side is:

(A) 122y – 26x – 1675 = 0 (B) 122y + 26x + 1675 = 0
(C) 26x – 122y – 1675 = 0 (D) 26x + 61y + 1675 = 0

Sol. **C**



Equation of BD: 5x – 4y = 1 Solve with 3x - 2y + 6 = 0
Equation of CE : 2x + 3y = 5 Solve with 4x + 5y - 20 = 20

Co-ordinates of B = $\left(-13, \frac{33}{2}\right)$

Co-ordinates of C = $\left(\frac{35}{2}, -10\right)$

$\therefore$ Equation of BC

y + 10 = $\frac{+13}{61}\left(x - \frac{35}{2}\right)$

61y + 610 = + 13x + $\frac{445}{2}$

– 26x + 122y + 1675 = 0

12. The area of the region A = $\{(x, y) : 0 \le y \le x|x| + 1\}$ and $-1 \le x \le 1$ in sq. units, is:

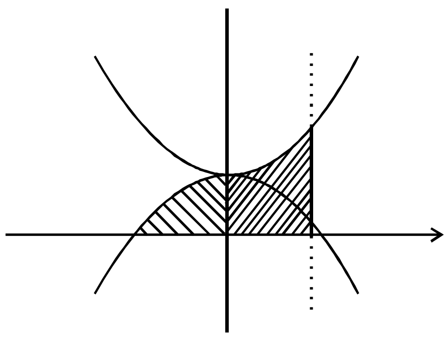
(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) 2 (D) $\frac{4}{3}$

Sol. **C**

$o \le y \le x\,|\,x\,| + 1$, $x \in [-1, 1]$

Case-I $x \in [0, 1]$

$y \le x^2 + 1$



Case-II

$x \in [-1, 0]$

A = $\int_{-1}^{0}\left(-x^2 + 1\right)dx + \int_{0}^{1}\left(x^2 + 1\right)dx$

$$= \left(\frac{-x^3}{3}+x\right)_{-1}^{0} + \left(\frac{x^3}{3}+x\right)_{0}^{1}$$

$$= -\left(\frac{1}{3}-1\right)+\left(\frac{1}{3}+1\right)$$

$$\frac{2}{3}+\frac{4}{3}=2$$

13. Let f:[0, 1] → R be such that f(xy) = f(x).f(y), for all x, y ∈[0, 1], and f(0) ≠ 0. If y = y(x) satisfies the differential equation $\frac{dy}{dx} = f(x)$ with y(0) = 1, then $y\left(\frac{1}{4}\right)+y\left(\frac{3}{4}\right)$ is equal to:

(A) 4 (B) 3 (C) 5 (D) 2

Sol. **B**

f(x. y) = f(x) . f(y), x,y∈ [0, 1] f(0) ≠ 0

x = y = 0 $\frac{dy}{dx} = f(x)$

f(0) = $f^2(0)$ y(0) = 1

∴ f(0) = 1

y = 0

f(0) = f(x) = 1

∴ $\frac{dy}{dx} = 1$

y = x + c

x = 0, y= 1 ∴ c = 1

y = x + 1

$y\left(\frac{1}{4}\right)+y\left(\frac{3}{4}\right) = \frac{1}{4}+1+\frac{3}{4}+1$ = 3.

14. If both the roots of the quadratic equation $x^2 - mx + 4 = 0$ area real and distinct and they lie in the intervasl [1, 5], then m lie in the interval :

(A) (5, 6) (B) (−5, −4) (C) (4, 5) (D) (3, 4)

Sol. **C / Bonus**

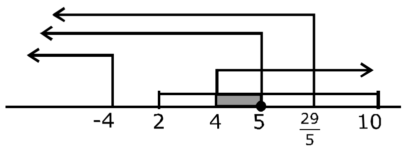
$x^2 - mx + 4 = 0$

(1) D> 0 (2) f(1) ≤ > 0 (3) f(5) ≥ 0

(4) $1 < -\frac{b}{2a} < 5$

$m^2 - 16 > 0$

$m \in (-\infty, -4) \cup (4, \infty)$ Solving : m∈(4, 5)



5-m ≥ 0 25-5m + 4 ≥ 0

m ≤ 5 m ≤ 29/5

$1 < \frac{m^2}{2} < 5$

2<M<10

m ∈ (4, 5]

15. An urn contains 5 red and 2 green balls. A ball is drawn at random from the urn. If the drawn ball is green, then a red ball is added to the urn and if the drawn ball is red, then a green ball is added to the urn; the original ball is not returned to the urn. Now, a second ball is drawn at random from it. The probability that the second ball is red, is :

(A) $\frac{27}{49}$ (B) $\frac{32}{49}$ (C) $\frac{21}{49}$ (D) $\frac{26}{49}$

Sol. **B**

5R
2G

P(G). P(R) + P(R). P(R)

$\frac{2}{7}\times\frac{6}{7}+\frac{5}{7}\times\frac{4}{7}=\frac{12+20}{49}=\frac{32}{49}$.

16. If the system of linear equations x – 4y + 7z = g, 3y – 5z = h, –2x + 5y – 9z = k is consistent, then:

(A) g + 2h + k = 0 (B) g + h + k = 0 (C) 2g + h + k = 0 (D) g + h + 2k = 0

Sol. **C**

$$\begin{cases} x-4y+7z=g \\ 3y-5z=h \\ -2x+5y-9z=k \end{cases} \qquad D=\begin{vmatrix} 1 & -4 & 7 \\ 0 & 3 & -5 \\ -2 & 5 & -9 \end{vmatrix}$$

= 1(–27 + 25)+ 4(–10)+ 7(6)
= – 2 – 40 + 42 = 0

$$D_1=\begin{vmatrix} g & -4 & 7 \\ h & 3 & -5 \\ k & 5 & -9 \end{vmatrix}$$

= g(–27 + 25) +4(-9h + 5k) + 7 (5h – 3k) = 0
= – 2g 36h + 20k + 35h - 21k = 0
– 2g – h – k = 0
2g + h + k = 0

17. If $\int_0^{\pi/3}\frac{\tan\theta}{\sqrt{2k\sec\theta}}d\theta=1-\frac{1}{\sqrt{2}},(k>0)$, then the value of k is:

(A) 2 (B) 4 (C) $\frac{1}{2}$ (D) 1

Sol. **A**

$$\int_0^{\pi/3}\frac{\tan\theta}{\sqrt{2k\sec\theta}}d\theta=1-\frac{1}{\sqrt{2}},k>0$$

$$\frac{1}{\sqrt{2k}}\int_0^{\pi/3}\frac{\sin\theta}{\sqrt{\cos\theta}}d\theta$$

Let $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$
One Solving K = 2

18. Let A ={$x\in R$: x is not a positive interger}.

Define a fucntion f: A $\rightarrow$ R as f(x) = $\frac{2x}{x-1}$, then f is :

(A) neither injective nor surjective (B) surjective but not injectivbe
(C) injective but not surjective (D) not injective

Sol. **C**

$f : A \to R$

$f(x) = \dfrac{2x}{x-1}$

$\dfrac{\text{linear}}{\text{linear}}$ is always one-one

19. Let a, b nd c be the 7th, 11th and 13th terms respectively of a non-constant A.P. if these are also the three consecutive terms of a G.P., then $\dfrac{a}{c}$ is equal to :

(A) 2 (B) $\dfrac{7}{13}$ (C) $\dfrac{1}{2}$ (D) 4

Sol. **D**

$t_7 = a = A + 6d$
$b = A + 10d$
$c = A + 12d$

$$r = \frac{b}{a} = \frac{b}{b}$$

$$= \frac{A+10d}{A+6d} = \frac{A+12d}{A+10d}$$

$$r \Rightarrow \frac{2d}{4d} = \frac{1}{2}$$

$$\therefore \frac{1}{r^2} = 4$$

20. If $\begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$ then A is :

(A) not invertible for any $t \in R$. (B) invertible only if $t = \dfrac{\pi}{2}$.

(C) invertible only if $t = \pi$. (D) invertible for all $t \in R$.

Sol. **D**

$$|A| = e^{-t}\begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\cos t - \sin t & -\sin t + \cos t \\ 1 & 2\sin t & -2\cos t \end{vmatrix}$$

$$= e^{-t}\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -2\cos t - \sin t & -2\sin t + \cos t \\ 0 & 2\sin t - \cos t & -2\cos t - \sin t \end{vmatrix}$$

$= e^{-t}\ \{(2c + s)^2 + (2s - c)^2\}$
$= 5\ e^{-t}$

21. Let z_0 be a root of the quadratic equation, $x^2 + x + 1 = 0$, If $z = 3 + 6i\ z_0^{81} - 3iz_0^{93}$, then arg z is equal to;

(A) 0 (B) $\dfrac{\pi}{3}$ (C) $\dfrac{\pi}{6}$ (D) $\dfrac{\pi}{4}$

Sol. $Z_0 \nearrow w$, $Z_0 \searrow w^2$

$Z = 3 + 6iZ_0^{81} - 3iZ_0^{93}$

$3 + 6iw^{81} - 3iw^{93}$
$= 3 + 3i$
$\therefore \arg(z) = \pi/4$

22. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)$ is :

(A) 12 (B) 14 (C) 15 (D) 10

Sol. **C**

$(1-t^6)^3(1-t)^{-3}$

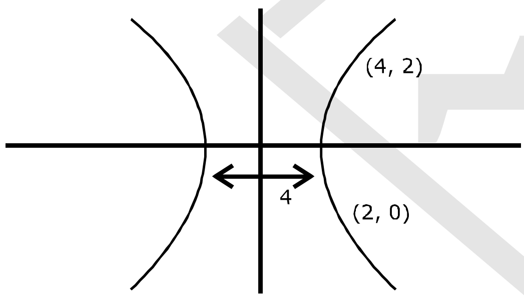
$({}^3C_0 - {}^3C_1t^6 + {}^3C_2t^{12} - {}^3C_3t^{18})(1-t)^{-3}$

${}^{3+4-1}C_4 = {}^6C_4 = 15$

23. A hyperbola has its centre at the origin, passes through the point (4, 2) and has transverse axis of length 4 along the x-axis. Then the eccentricity of the hyperbola is:

(A) $\frac{3}{2}$ (B) $\sqrt{3}$ (C) 2 (D) $\frac{2}{\sqrt{3}}$

Sol. **D**



a = 2

$\frac{x^2}{y} - \frac{4^2}{b^2} = 1$

$4 - \frac{4}{b^2} = 1$

$\frac{3}{4} = \frac{1}{b^2}$ $\Rightarrow$ $b^2 = 4/3$

$\therefore$ $e^2 = 1 + \frac{4/3}{4} = \frac{1}{3} + 1$

$e = \frac{2}{\sqrt{3}}$

24. If x = 3 tan t and y = 3 sec t, the the value of $\frac{d^2y}{dx^2}$ att $= \frac{\pi}{4}$, is :

(A) $\frac{1}{6}$ (B) $\frac{3}{2\sqrt{2}}$ (C) $\frac{3}{3\sqrt{2}}$ (D) $\frac{1}{6\sqrt{2}}$

Sol. **D**

x = 3 tant, y = 3 sect

$\frac{dx}{dt} = 3\sec^2 t$

$\frac{dy}{dt} = 3\sec t \tan t$

$\therefore$ $\frac{dy}{dt} = \sin t$

$\frac{d^2y}{dx^2} = \cos t.\frac{dt}{dx} = \frac{\cos^3 t}{3}$

t = $\frac{\pi}{4}$

$\therefore$ $\frac{d^2y}{dx^2} = \frac{1}{6\sqrt{2}}$

25. The number of all possible positive intergral values of α for which the roots of the quadratic equation , $6x^2 - 11x + \alpha = 0$ are rational numbers is :

(A) 3 (B) 2 (C) 4 (D) 5

Sol. **A**

D $\rightarrow$ perfect sq.

D = 121 – 24 $\alpha = \lambda^2$

α = 1, reject

α = 2 reject

α = 3

$\left.\begin{array}{l}\alpha = 3\\ \alpha = 4\\ \alpha = 5\end{array}\right\}$ 3 integration values

26. Let $\vec{a} = \hat{i} + \hat{j} + \sqrt{2}\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + \sqrt{2}\hat{k}$ and $\vec{c} = 5\hat{i} + \hat{j} + \sqrt{2}\hat{k}$ be three vectors such that the projection vector of $\vec{b}$ on $\vec{a}$ is $\vec{a}$ If $\vec{a}+\vec{b}$ is perpendicular to $\vec{c}$, then $|\vec{b}|$ is equal to :

(A) $\sqrt{32}$ (B) 6 (C) 4 (D) $\sqrt{22}$

Sol. **A**

Project of $\vec{b}$ on $\vec{a} = \frac{\vec{b} - \vec{a}}{|\vec{a}|} = |\vec{a}|$

$\frac{b_1 + b_2 + 2}{2} = 2$

$b_1 + b_2 = 2$

$(\vec{a}+\vec{b}) \perp \vec{c} \Rightarrow (\vec{a}+\vec{b}).\vec{c} = 0$

$5b_1 + b_2 = -10$

$b_1 = -3,$ $b_2 = 5$

$\therefore |\vec{b}| = 6$

27. If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to:
(A) 10 (B) 7π (C) 0 (D) π

Sol. D
$x = \sin^{-1}(\sin 10) = -10 + 3\pi$
$y - \cos^{-1}\cos 10 = 4\pi - 10$
$\therefore$ $y - x = 4\pi - 10 + 10 - 3\pi$
$= \pi$

28. For each $x \in R$, let $[x]$ be the greatest integer less than or equal to x. Then $\lim_{x\to 0^-}\frac{x([x]+|x|)\sin[x]}{|x|}$ is equal to:
(A) 1 (B) sin 1 (C) – sin 1 (D) 0

Sol. C

$$\lim_{x\to 0^-}\frac{x\{[x]+|x|\}\sin[x]}{|x|}$$

$$\lim_{x\to 0^-}\frac{x(+x+1)\sin 1}{-x} = -\sin 1$$

29. If the circles $x^2 + y^2 - 16x - 20y + 164 = r^2$ and $(x - 4)^2 + (y - 7)^2 = 36$ intersect at two distinct points, then;
(A) $0 < r < 1$ (B) $r = 11$ (C) $r > 11$ (D) $1 < r < 11$

Sol. D
C_1 (8, 10), $r_1 = r$
C_2 (4, 7) $r_2 = 6$
$|r_1 - r_2| < C_1C_2 < r_1 + r$
$\therefore r \in (1, 11)$

30. If $0 \le x < \frac{\pi}{2}$, then the number of values of x for which $\sin x - \sin 2x + \sin 3x = 0$, is :
(A) 3 (B) 2 (C) 1 (D) 4

Sol. B
$\sin x + \sin 3x - \sin 2x = 0$
$\sin 2x\,(2\cos x - 1) = 0$

$\sin 2x = 0$, $\cos x = \frac{1}{2}$

$x = 0, \frac{\pi}{3}$