Lesson

SPEAKING PRIME NUMBERS

CREATED BY

Richard Born
Associate Professor Emeritus
Northern Illinois University
rborn2@niu.edu

TOPICS

Mathematics, Programming

GRADES

7-12

METHOD

OzoBlockly

DURATION

30-45 minutes

Speaking Prime Numbers
The Sieve of Eratosthenes

By Richard Born
Associate Professor Emeritus
Northern Illinois University
rborn2@niu.edu

Introduction

A ***prime number*** is any natural (counting) number greater than 1 that is evenly divisible only by itself and 1. For example, 5 is a prime number, as it is divisible only by 5 and 1. Mankind has been fascinated with prime numbers for centuries, dating back as far as the ancient Greeks and possibly even much further back to the ancient Egyptians. The Greek mathematician Euclid is credited as the first person to prove that the number of prime numbers in infinite. Beginning with the 1600's and continuing to this day, interest in studying prime numbers grew significantly. Mathematicians including Fermat, Mersenne, Legendre, and Gauss, just to mention a few, each played their part in prime number studies. Since the 1950's, many *very large* prime numbers have been found by the use of computers. By the 1970's the use of large prime numbers extended beyond the realm of purely mathematical number theory when it was found that prime numbers are useful in ***public key encryption (PKE)***. PKE is based on the difficulty of factoring the product of two very large prime numbers.

With this historical background in mind, it seems reasonable that students would benefit from learning about prime numbers, if for no reason other than curiosity. In this lesson, students learn a version of one of the oldest and probably the most well-known method for generating prime numbers—the ***Sieve of Eratosthenes***. A sieve (pronounced *siv*) is a sifter, in this case sifting out numbers that are not prime (i.e., are ***composite***) and keeping only those that are prime. Eratosthenes was an ancient Greek scholar who studied many scientific disciplines. After learning how Eratosthenes' algorithm for generating primes works, students then load Evo with an OzoBlockly program for which the algorithm allows Evo to speak the prime numbers as they are generated, starting with 2 and proceeding through all primes that are less than 127. Evo moves on a circular wheel and stops on numbered spokes, blinking green if the number is prime and red otherwise.

The Sieve of Eratosthenes

The algorithm discussed here is not quite as efficient as Eratosthenes' original, but is probably easier for students to understand. Figure 1 shows step-by-step, with steps labeled A through J on the far left, how the algorithm works:

A: Write down the natural numbers in order starting with 2 and ending with the highest number for which you want to search for prime numbers. For our discussion, we'll assume that 25 is the highest number.
B: Circle 2, which is prime since it is evenly divisible only by itself and 1. Then cross out every second number after 2. These crossed-out numbers cannot be prime since they are multiples of 2.

C: Grayed numbers were crossed out in step B. Circle 3, which is prime since it is evenly divisible only by itself and 1. Then cross out every third number after 3. These crossed-out numbers cannot be prime since they are multiples of 3. Note that crossing out a number more than once is permitted, as seen by 6, 12, 18, and 24.

D: Circle the next number that has not already been crossed out. This number is 5, which is prime since it is evenly divisible only by itself and 1. Then cross out every fifty number after 5. These crossed-out numbers cannot be prime since they are multiples of 5.

E: Circle the next number that has not already been crossed out. This number is 7, which is prime since it is evenly divisible only by itself and 1. Then cross out every seventh number after 7. These crossed-out numbers cannot be prime since they are multiples of 7.

F: Circle the next number that has not already been crossed out. This number is 11, which is prime since it is evenly divisible only by itself and 1. Then cross out every eleventh number after 11. These crossed-out numbers cannot be prime since they are multiples of 11.

G: Circle the next number that has not already been crossed out. This number is 13, which is prime since it is evenly divisible only by itself and 1. There is no need to worry about crossing out numbers here, as the thirteenth number after 13 is 26, and 26 is beyond the highest number in our list.

In a similar manner, we cross out 17, 19, and 23, as shown in steps H, I, and J, respectively. The circled numbers 2, 3, 5, 7, 11, 13, 17, 19, and 23 are the prime numbers in the range 2 through 25.

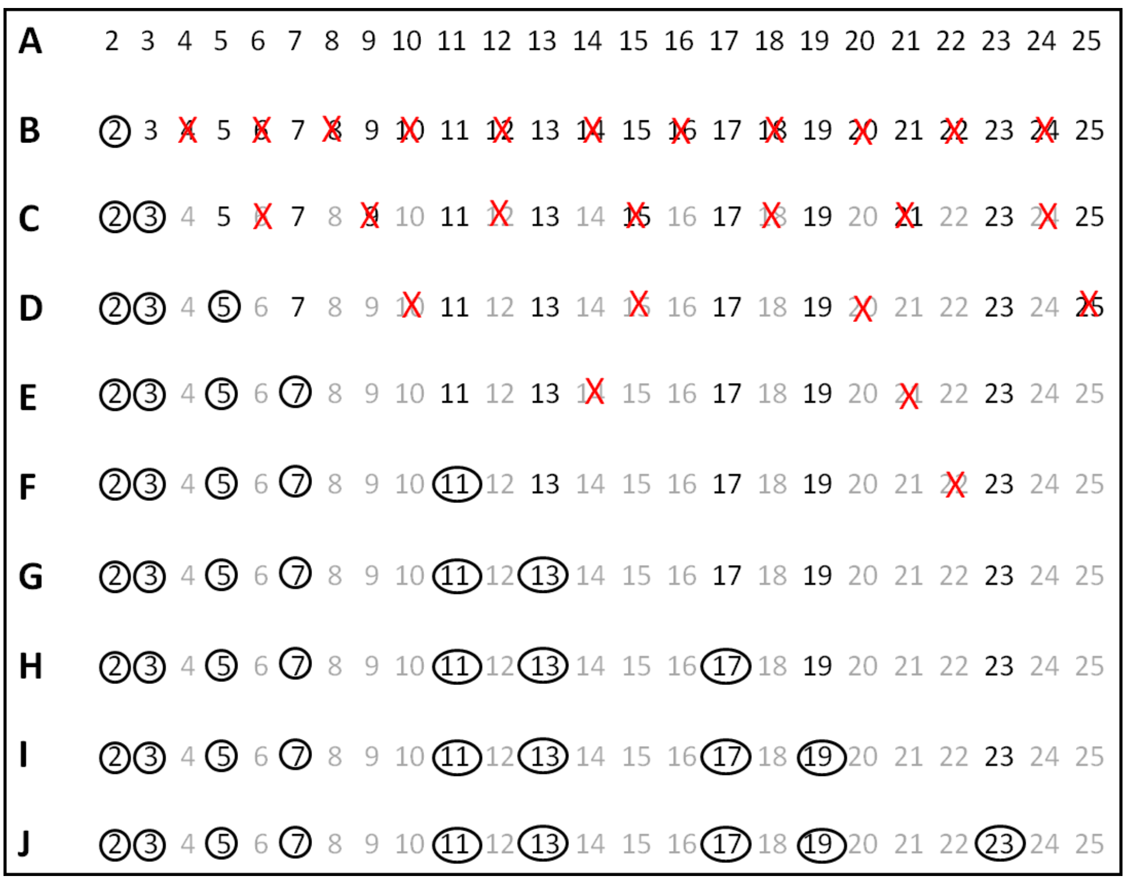


Figure 1

Running the Evo OzoBlockly Prime Numbers Program

1. Load *Prime Number Generator.ozocode* onto Evo and print a copy of the Ozomap on the last page of this document for each student lab group.
2. To run the loaded program on Evo, begin with Evo powered down. Place Evo centered on the zero intersection of the Ozomap and facing to the right. Press Evo's button once to turn Evo on. Then when the front lights show blue, double-press the button. Evo's top light should then be red. The OzoBlockly program has started running.
3. After about three seconds, Evo will begin moving around the circle. Evo will speak the prime numbers as they are generated, starting with 2 and proceeding through all primes that are less than 127. Evo will stop and blink green if the number is prime and show red otherwise. Evo will stop and turn off at the number 126.

Student Activities

1. While Evo is speaking the prime numbers, write them down in a list on a piece of paper in a vertical column. Then investigate the following two things:
    a. How many primes are there between 2 and 25, inclusive? Between 26 and 50? Between 51 and 75? Between 76 and 100?
        i. Do the primes appear to be distributed regularly or irregularly among the natural numbers?
        ii. What seems to happen to the distribution of primes as one moves to higher and higher numbers?
    b. Compute the difference between adjacent primes in your list. How many of these differences are 2? Such pairs of primes are referred to as "twin primes". It has been conjectured, but never proven to this date, that there are an infinite number of twin primes.
2. The ***Fundamental Theorem of Arithmetic*** tells us that every composite number can be factored into primes in only one way, ignoring the order of the primes. For example, $15 = 3 \bullet 5$, $40 = 2^3 \bullet 5$, and $46 = 2 \bullet 23$. Determine the prime factorization of the following composite numbers:
    a. 55
    b. 100
    c. 126
3. The Fundamental Theorem of Arithmetic is important because it indicates that the prime numbers are the building blocks for all numbers. In a sense, the role of primes is similar to the role of the elements in the physical universe. However, what are the differences?
4. The ***Goldbach Conjecture***, unproven to this date, states that every even number greater than or equal to 4 can be written as the sum of two prime numbers. For example, 4 = 2+2, 6 = 3+3, and 20 = 3+17 = 7+13. Note that there can be more than one pair of primes that sum to the desired even number. Demonstrate the Goldbach Conjecture for the following numbers, being sure to include all possible pairs of primes:
    a. 12
    b. 18
    c. 22
    d. 34

Exercise Solutions

1. (a) 9, 6, 6, 4. Irregularly. The "density" of the primes seems to decrease.
   (b) 10 "twin primes" between 2 and 100.

2. (a) $55 = 5 \bullet 11$
   (b) $100 = 2^2 \bullet 5^2$
   (c) $126 = 2 \bullet 3^2 \bullet 7$

3. The Periodic Table of the Elements has an orderly pattern, but the primes don't have a regular pattern.

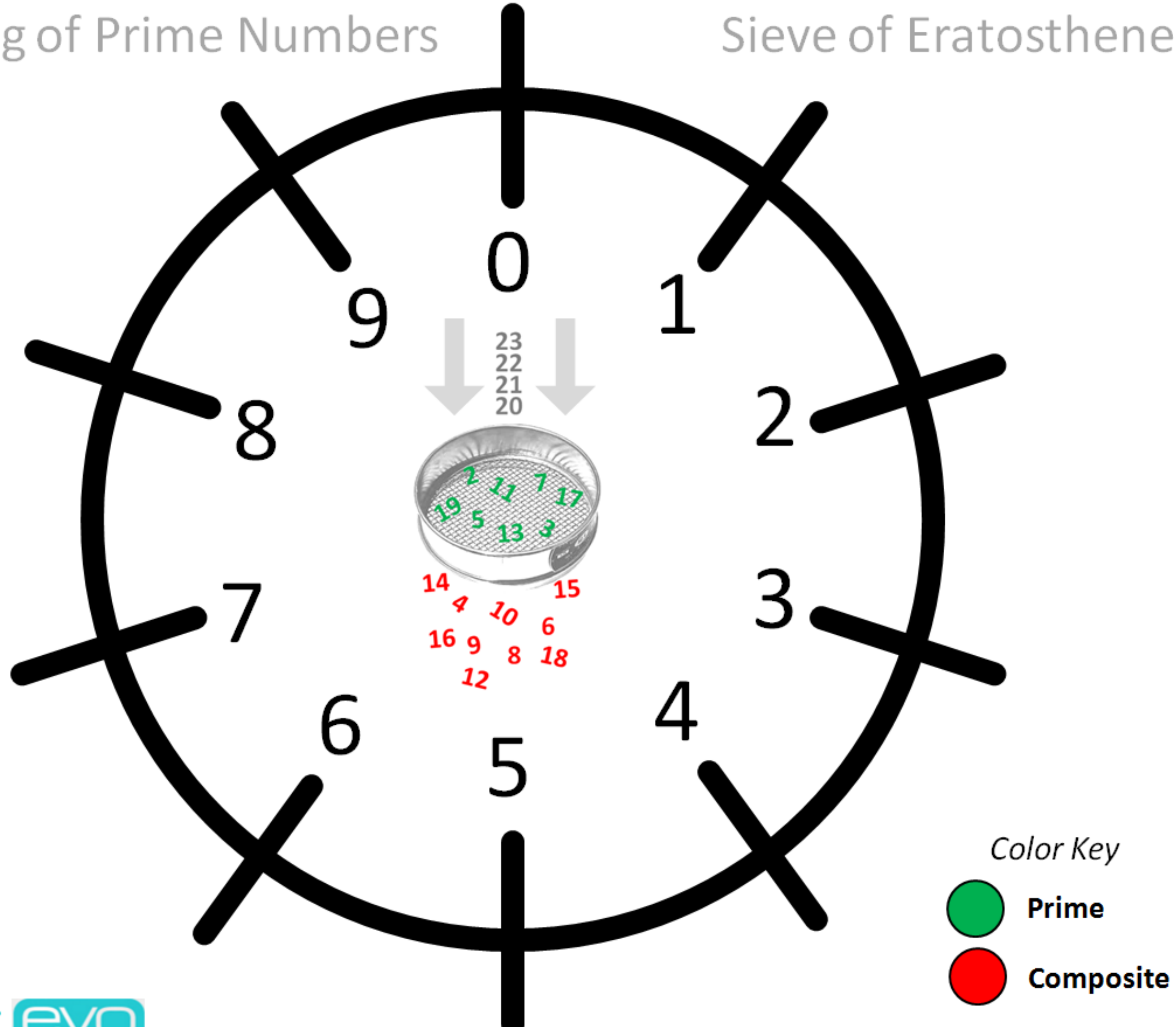
   There are an infinite of primes, but only a finite number of elements (as far as we now know).

4. (a) $12 = 5 + 7$

   (b) $18 = 5 + 13 = 7 + 11$

   (c) $22 = 11 + 11 = 17 + 5 = 19 + 3$

   (d) $34 = 17 + 17 = 11 + 23 = 5 + 29 = 3 + 31$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Color Key
Prime
Composite