Supporting information

S1 Appendix.

Theorem 1. Let $X_{n \times p}$ be a matrix of full column rank. Assume $y \in \text{colspace}(X)$, $\tilde{y} = y + \delta$ where $\delta \in \mathbb{R}^n$, and $\hat{y} = P \tilde{y}$, where $P = X(X^T X)^{-1} X^T$. Then,

$||\tilde{y} - y||_2 \geq ||\hat{y} - y||_2.$

Proof. Properties for the proof:

Property 1: Let $X$ be an $n \times p$ matrix such that $\text{rank}(X) = p$.

Property 2: Define $P = X(X^T X)^{-1} X^T$. Then

Property 2a: $P$ is idempotent (i.e., $P = PP = P^2$).

Property 2b: $P$ is symmetric (i.e., $P = P^T$).

Property 2c: $I - P$ is idempotent.

Property 2d: $I - P$ is symmetric.

Property 3: If $A$ is a symmetric matrix, then $A = Q \Lambda Q^T$ where $Q$ is a matrix whose columns are equal to the eigenvectors of $A$ and $\Lambda$ is a diagonal matrix whose diagonal elements $\lambda$ are the eigenvalues of $A$.

Property 4: If $A$ is idempotent, then its eigenvalues are either 0 or 1.

Property 5: Assume $y \in \text{colspace}(X)$.

Property 6: $\tilde{y} = y + \delta$, where $\delta \in \mathbb{R}^n$.

Property 7: $\hat{y} = P \tilde{y}$

The proof proceeds as follows:

$$||\tilde{y} - y||_2 - ||\hat{y} - y||_2 = ||y + \delta - y||_2 - ||P \tilde{y} - y||_2$$

$= ||\delta||_2 - ||Py + P\delta - y||_2$ (Properties 6 and 7)

$= ||\delta||_2 - ||y + P\delta - y||_2$ (Property 5)

$= ||\delta||_2 - ||P\delta||_2$

$= \delta^T \delta - \delta^T PP \delta$

$= \delta^T \delta - \delta^T P^T P \delta$

$= \delta^T (I - P) \delta$ (Property 2b)

$= \delta^T \Lambda \Lambda^T \delta$ (Properties 2d and 3)

$= a^T \Lambda a$

$= \sum_{i=1}^{n} \lambda_i a_i^2$

$\geq 0.$ (Property 3)

Thus, $||\tilde{y} - y||_2 - ||\hat{y} - y||_2 \geq 0$ implies $||\tilde{y} - y||_2 \geq ||\hat{y} - y||_2$. $\square$