S2 Text. Basic reproduction number

Here we compute the basic reproduction number $R_0$ of the model (3)-(5). First let us set for $i \geq 0$ and $a \in [0,a_{\max}]$ the following functions

$$\pi_s(a,i) = \exp \left( -i \mu_{nat}(a) - \int_0^i [\gamma_{dtr}(a)t]_{\text{sympt,pmax}}(\sigma) + h_s(a,\sigma)] d\sigma \right),$$

$$\pi_m(a,i) = \exp \left( -i \mu_{nat}(a) - \int_0^i h_m(a,\sigma)d\sigma \right),$$

$$\pi_p(a,i) = \exp \left( -i \mu_{nat}(a) - \int_0^i h_p(a,\sigma)d\sigma \right),$$

that describe the survival probability of infected individuals (in the respective compartment), with age $a$, from their infection until the time since infection $i$, in case of no hospitalisation (i.e. $H \equiv 0$). We get the following Volterra formulation of the linearized system of (3)-(5):

$$I_s(t,a) = \begin{cases} I_{s,0}(a,i-t) \frac{\pi_s(a,i)}{\pi_s(a,i-t)}, & \text{for } t \in [0,i), \\ (1-p)q(a)\lambda_0(t-i,a)S_0(a)\pi_s(a,i), & \text{for } t \geq i, \end{cases} \quad (B.1)$$

$$I_m(t,a) = \begin{cases} I_{m,0}(a,i-t) \frac{\pi_m(a,i)}{\pi_m(a,i-t)}, & \text{for } t \in [0,i), \\ (1-p)(1-q(a))\lambda_0(t-i,a)S_0(a)\pi_m(a,i), & \text{for } t \geq i \end{cases} \quad (B.2)$$

and

$$I_p(t,a) = \begin{cases} I_{p,0}(a,i-t) \frac{\pi_p(a,i)}{\pi_p(a,i-t)}, & \text{for } t \in [0,i), \\ p\lambda_0(t-i,a)S_0(a)\pi_p(a,i), & \text{for } t \geq i \end{cases} \quad (B.3)$$

where $\lambda_0 = \lambda(\cdot,\cdot,0)$ is defined by

$$\lambda_0(t,a) = \int_0^{a_{\max}} K(a,a') \int_0^t (\beta_s(a',i)I_s(t,a',i) + \beta_m(a',i)I_m(t,a',i) + \beta_p(a',i)I_p(t,a',i)) \text{ d}a' \quad (B.4)$$

where $\beta_k$, $k \in \{s,m,p\}$ are defined transmission probabilities. Let $I_N(t,a) = \lambda_0(t,a)S_0(a)$ be the density of newly infected of age $a$ at time $t$, with $c \equiv 0$. Then (B.1)-(B.2)-(B.3) can be rewritten as the following Volterra formulation:

$$I_N(t,a) = S_0(a) \int_0^t \int_0^{a_{\max}} K(a,a')\omega(a',i)I_N(t-i,a') \text{ d}a' \text{ d}i + f(t,a),$$

where

$$\omega(a',i) = \beta_s(a',i)(1-p)q(a')\pi_s(a',i) + \beta_m(a',i)(1-p)(1-q(a'))\pi_m(a',i) + \beta_p(a',i)p\pi_p(a',i)$$

and $f(t,a)$ is the density of new infections produced by the initial population. Therefore, the basic reproduction number $R_0$ is the spectral radius, denoted by $r(U)$, of the next generation operator $U$ defined on $L^1(0,a_{\max})$ by

$$U : L^1(0,a_{\max}) \ni v \mapsto S_0(\cdot) \int_0^\infty \int_0^{a_{\max}} K(\cdot,a')\omega(a',i)v(a') \text{ d}a' \text{ d}i \in L^1(0,a_{\max})$$
As explained before, it is estimated in [1] that each average infectiousness \( \beta_k \) \( (k \in \{s, m, p\}) \) takes the form of a Weibull distribution \( W(3, 5.65) \) so that the mean and median are equal to 5.0 days while the standard deviation is 1.9 days. Based on this estimation, we assume that \( \beta_k(a, i) = \alpha \overline{B}(i) \xi_k(i) \) where \( \overline{B} \sim W(3, 5.65) \) and \( \alpha \) is a positive parameter to be determined. Consequently, it follows that \( \alpha \) is given by

\[
\alpha = \frac{R_0}{r(\overline{U})},
\]

where \( \overline{U} \) is the operator defined by

\[
\overline{U} : L^1(0, a_{\text{max}}) \ni v \mapsto S_0(\cdot) \int_0^\infty \int_0^{d_{\text{max}}} K(\cdot, a') \overline{\Omega}(a') v(a') \, da' \, di \in L^1(0, a_{\text{max}})
\]

with

\[
\overline{\Omega}(a') = \int_0^\infty \overline{\omega}(a', i) \, di.
\]

We see that \( \overline{U} \) can be rewritten as

\[
\overline{U} v(a) = S_0(a) \int_0^{d_{\text{max}}} K(a, a') \overline{\Omega}(a') v(a') \, da', \quad \forall v \in L^1(0, a_{\text{max}}),
\]

where \( \overline{\Omega}(a') = \int_0^\infty \overline{\omega}(a', i) \, di \). Now, in order to compute the spectral radius \( r(\overline{U}) \), we first make the following assumptions:

**Assumption 1** We suppose that functions \( S_0, K, \overline{\Omega} \) are bounded and positive almost everywhere.

Then, we can show that \( r(\overline{U}) \) is given by the spectral radius of the following linear operator:

\[
L^1(0, a_{\text{max}}) \ni v \mapsto \int_0^{d_{\text{max}}} K(\cdot, a') \overline{\Omega}(a') S_0(a') v(a') \, da' \in L^1(0, a_{\text{max}})
\]

which can be easily computed since the age \( a \) is numerically divided into \( N \) classes, so that the term inside the integral of the latter equation is a \( N \times N \) matrix. Finally, we obtain \( \alpha \) from (B.5).

In addition to Assumption 1, if the function \( K \) is symmetric, we can define the positive self-adjoint operator \( S \) by

\[
S : L^2(0, a_{\text{max}}) \ni v \mapsto \sqrt{S_0(\cdot) \overline{\Omega}(\cdot)} \int_0^{d_{\text{max}}} K(\cdot, a') \sqrt{S_0(a') \overline{\Omega}(a')} v(a') \, da' \in L^2(0, a_{\text{max}}).
\]

We then deduce the following

**Proposition 2** Let \( K \) be symmetric and Assumption 1 be satisfied. Then, operators \( \overline{U} \) and \( S \) are positive and compact, their spectra \( \sigma(\overline{U}) \setminus \{0\} \) and \( \sigma(S) \setminus \{0\} \) are composed of isolated eigenvalues with finite algebraic multiplicity. Moreover, we have \( \sigma(\overline{U}) = \sigma(S) \subset \mathbb{R}_+ \) and the following Rayleigh formula holds:

\[
r(\overline{U}) = r(S) = \sup_{\|v\|_{L^2(0, a_{\text{max}})} = 1} \int_0^{d_{\text{max}}} \int_0^{d_{\text{max}}} K(a, a') \sqrt{S_0(a') \overline{\Omega}(a')} \sqrt{S_0(a) \overline{\Omega}(a)} v(a') v(a) \, da' \, da.
\]
**Proof.** The compactness of both integral operators follows from the fact that \( a_{\text{max}} < \infty \) by assumption (see Table 1), hence their spectra are punctual. Now we prove that \( \sigma(\mathcal{U}) = \sigma(S) \). Let \( \nu \in \sigma(\mathcal{U}) \) be an eigenvalue of \( \mathcal{U} \) and \( \phi \in L^1(0,a_{\text{max}}) \) be the associated eigenvector, i.e.

\[
\mathcal{U}\phi(a) = S_0(a) \int_0^{a_{\text{max}}} K(a,a') \sqrt{\Omega(a')} \phi(a') da' = \nu \phi(a), \quad \forall a \in [0,a_{\text{max}}]
\]

so that \( \phi \in L^\infty(0,a_{\text{max}}) \subset L^2(0,a_{\text{max}}) \). Defining the function

\[
\psi = \frac{\phi \sqrt{\Omega}}{\sqrt{S_0}} \in L^2(0,a_{\text{max}})
\]

leads to

\[
\nu \psi(a) = \sqrt{S_0(a) \Omega(a)} \int_0^{a_{\text{max}}} K(a,a') \sqrt{\Omega(a')} S_0(a') \psi(a') da' = S \psi(a), \quad \forall a \in [0,a_{\text{max}}]
\]

i.e. \( \nu \in \sigma(S) \) is an eigenvalue of \( S \) associated to the eigenvector \( \psi \), so that \( \sigma(\mathcal{U}) \subset \sigma(S) \). For the reverse inclusion, let \( \nu \in \sigma(S) \) and \( \psi \in L^2(0,a_{\text{max}}) \subset L^1(0,a_{\text{max}}) \) be the associated eigenvector for \( S \). It follows that the function

\[
\phi = \frac{\psi \sqrt{S_0}}{\sqrt{\Omega}} \in L^1(0,a_{\text{max}})
\]

is an eigenvector of \( \mathcal{U} \) related to the eigenvalue \( \nu \in \sigma(\mathcal{U}) \), whence \( \sigma(\mathcal{U}) = \sigma(S) \). In particular, both spectral radius are equal. Finally, the Rayleigh formula is classical for positive and symmetric operators.

**References**