S1 Eigenvalues of the homogeneous contact matrix

Here we will demonstrate a general case for the eigenvalues of a homogeneous contact matrix, for which every column accounts for the fraction age-groups represent respect to the total population.

**Theorem S1.1.** Let $C$ be a square $n \times n$ matrix, such that all columns are identical, i.e., $C_{i, \cdot} = f_i$, $f \in \mathbb{R}^n$, and $\sum_i f_i \neq 0$. Then $C$ is diagonalizable and has a single non-zero eigenvalue $\lambda = \sum_i f_i$.

**Proof.** First, we note that the dimension of the kernel of $T : \mathbb{R}^n \to \mathbb{R}^n$, $T(u) = Cu$, i.e, the vector space $\ker (T) = \{u|Cu = 0\}$ is $n - 1$. Thus, there are $n - 1$ linearly independent vectors associated to the eigenvalue $\lambda = 0$, which algebraic multiplicity has therefore to be equal or larger than $n - 1$. Then, we study the nature of the characteristic polynomial:

$$p(\lambda) = \det (C - \lambda I)$$

(1)

$$= \begin{vmatrix} f_1 - \lambda & f_1 & \ldots & f_1 \\ f_2 & f_2 - \lambda & \ldots & f_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_n & \ldots & f_n - \lambda \end{vmatrix}$$

(2)

$$= \begin{vmatrix} f_1 - \lambda & \lambda & \lambda & \ldots & \lambda \\ f_2 & -\lambda & 0 & \ldots & 0 \\ f_3 & 0 & -\lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & 0 & 0 & \ldots & -\lambda \end{vmatrix}$$

(3)

$$= \begin{vmatrix} \sum_i f_i - \lambda & 0 & 0 & 0 & 0 \\ -\lambda & 0 & \ldots & 0 \\ f_3 & 0 & -\lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & 0 & 0 & \ldots & -\lambda \end{vmatrix}$$

(4)

The highlighted zone in the determinant corresponds to $-\lambda I_{n-1}$, with $I_{n-1}$ the identity matrix in $\mathbb{R}^{n-1 \times n-1}$. Let $I_{n-1}$ be $I_{n-1}$, but with the $j$’th row replaced by a row of zeros. Using that $|aA| = a^n |A|$ for an $n \times n$ arbitrary matrix, and that $|D| = \prod d_i$ for a diagonal matrix, we calculate $p(\lambda)$ by minor determinants:

$$p(\lambda) = \left( \sum_i f_i - \lambda \right) \left| -\lambda I_{n-1} \right| + \sum_{i=2}^n (-1)^{i-1} f_i \left| \tilde{I}_{n-1} \right|$$

(5)

$$= \left( \sum_i f_i - \lambda \right) (-1)^{n-1} \lambda^{n-1}.$$  

(6)

As we found the last eigenvalue, and, by definition, it has at least one eigenvector, we completed the required set of $n$ eigenvectors and concluded the demonstration.

**Corollary S1.1.1.** When $C$ is a contact matrix as defined in theorem S1.1 and $f$ accounts for the fraction age-groups represent respect to the total population, the largest eigenvalue of matrix $C$ is 1.

**Proof.** Direct from theorem S1.1 knowing that $\sum_i f_i = 1$. 

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