# Appendix

We start with the equations described in the main text, *i.e.*

$$\begin{aligned}
\frac{dS_0}{dt} &= \mu - (\mu+\nu)S_0 - \beta S_0 I, \\
\frac{dS_1}{dt} &= m\omega S_2 - (\mu+\nu)S_1 - \beta\varepsilon(1)S_1 I, \\
\frac{dS_i}{dt} &= m\omega S_{i+1} - (\mu+\nu+m\omega)S_i - \beta\varepsilon(i)S_i I, \qquad (1<i<n) \\
\frac{dS_n}{dt} &= \gamma I + \nu\sum_{j=0}^{m} S_j - (\mu+m\omega)S_n, \\
\frac{dI}{dt} &= \beta S_0 I + \beta\sum_{j=1}^{m}\varepsilon(j)S_j I - (\mu+\gamma)I.
\end{aligned}$$

At the disease free equilibrium, $I = 0$. Since the other differential equations are also set to 0, solving these gives $S_0^*$ immediately, *i.e.*,

$$S_0^* = \frac{\mu}{\mu+\nu}.$$

To obtain $S_n^*$, we start with the $\frac{dS_n}{dt}$ equation where it is set equal to 0 and $I = 0$, *i.e.*,

$$0 = \nu\sum_{j=0}^{m} S_j - (\mu+m\omega)S_n.$$

If we add and subtract $\nu S_n$, and use the fact that because $I = 0$ the remaining classes $S_0$ to $S_n$ must sum to 1, it follows that

$$\begin{aligned}
0 &= \nu\sum_{j=0}^{m} S_j - (\mu+m\omega)S_n, \\
0 &= \nu\sum_{j=0}^{m} S_j - (\mu+m\omega)S_n + \nu S_n - \nu S_n, \\
0 &= \nu\sum_{j=0}^{n} S_j - (\mu+\nu+m\omega)S_n, \\
0 &= \nu - (\mu+\nu+m\omega)S_n, \\
S_n^* &= \frac{\nu}{\mu+\nu+m\omega}.
\end{aligned}$$

Thus, there is a recurrence relation for the susceptible classes between 1 and $n$, *i.e.*,

$$S_i^* = \frac{m\omega}{\mu+\nu+m\omega} S_{i+1}^*.$$

We can then use this recurrence relation to obtain an expression for $S_i^*$ as a function of $S_n$, *i.e.*,

$$\begin{aligned} S_i^* &= (\frac{m\omega}{\mu+\nu+m\omega})^{n-i} S_n^*, \\ S_i^* &= (\frac{m\omega}{\mu+\nu+m\omega})^{n-i} \frac{\nu}{\mu+\nu+m\omega}. \end{aligned}$$

Since $S_0$ to $S_n$ sum to 1 at the disease free equilibrium, it follows that

$$S_1^* = 1 - S_0 - \sum_{i=2}^{n} S_i^*.$$

This sum is a geometric sum, so we can use a substitution trick, giving

$$\begin{aligned} S_1^* &= 1 - \frac{\mu}{\mu+\nu} - \frac{\nu}{\mu+\nu+m\omega}(\frac{1-(\frac{m\omega}{\mu+\nu+m\omega})^m}{1-\frac{m\omega}{\mu+\nu+m\omega}}) \\ &= 1 - \frac{\mu}{\mu+\nu} - \frac{\nu}{\mu+\nu+m\omega}(\frac{1-(\frac{m\omega}{\mu+\nu+m\omega})^m}{\frac{\mu+\nu}{\mu+\nu+m\omega}}) \\ &= 1 - \frac{\mu}{\mu+\nu} - \frac{\nu}{\mu+\nu+m\omega}\frac{\mu+\nu+m\omega}{\mu+\nu}(1-(\frac{m\omega}{\mu+\nu+m\omega})^m) \\ &= 1 - \frac{\mu}{\mu+\nu} - \frac{\nu}{\mu+\nu}(1-(\frac{m\omega}{\mu+\nu+m\omega})^m) \\ &= 1 - \frac{\mu}{\mu+\nu} - \frac{\nu}{\mu+\nu} + \frac{\nu}{\mu+\nu}(\frac{m\omega}{\mu+\nu+m\omega})^m \\ S_1^* &= \frac{\nu}{\mu+\nu}(\frac{m\omega}{\mu+\nu+m\omega})^m. \end{aligned}$$

This therefore gives us the complete set of equilibrium values

$$\begin{aligned} S_0^* &= \frac{\mu}{\mu+\nu}, \\ S_1^* &= \frac{\nu}{\mu+\nu}(\frac{m\omega}{\mu+\nu+m\omega})^m, \\ S_i^* &= (\frac{m\omega}{\mu+\nu+m\omega})^{n-i}\frac{\nu}{\mu+\nu+m\omega}, \qquad (1<i<n) \\ S_n^* &= \frac{\nu}{\mu+\nu+m\omega}. \end{aligned}$$