S4 Appendix: The $R_0$ expression of the SI model of $T. cruzi$ transmission in its host community.

We used the ‘Next Generation’ approach (Diekmann et al., 2010) to identify the expression of $R_0$ in the SI model of transmission of $T. cruzi$ in its host community that is described by Equ. 1 to 4 in the main text.

To derive the expression of $R_0$, we focus on the equations that in our system of ordinary differential equations describe the variations in the number of vector and host infected individuals, i.e. Equ 2 and 4 described in the main text;

$$\frac{dI}{dt} = M - d_i I + \beta (N_v, N) \sum_{i \in C} \phi_i (N) \frac{I_i}{N_i} S_v p_v,$$

$$\frac{dI_i}{dt} = -d_i I_i + I_v \beta (N_v, N) \phi_i (N) \frac{S_i}{N_i} p_v,$$

for all $i \in C$

At the disease-free equilibrium, $N_i = S_i$, $N_v = S_v$ and further considering $M = 0$, this subsystem (also known as the linearized infection subsystem) becomes:

$$\frac{dI}{dt} = -d_i I + \beta (N_v, N) \sum_{i \in C} \phi_i (N) \frac{I_i}{N_i} N_v p_v,$$

$$\frac{dI_i}{dt} = -d_i I_i + I_v \beta (N_v, N) \phi_i (N) p_v,$$

for all $i \in C$

As explained in Diekmann et al. (2010), this system of ordinary differential equations can be represented by a matrix $T$ made of transmission terms and a matrix $\Sigma$ containing transition terms;

$$T = \beta (N_v, N) \begin{pmatrix}
0 & p_v \phi_1 (N) \frac{N_v}{N_i} & \cdots & p_v \phi_{n_c} (N) \frac{N_v}{N_{n_c}} \\
\phi_1 (N) p_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n_c} (N) p_{n_c} & 0 & \cdots & 0
\end{pmatrix}$$

and
\[ \Sigma = - \begin{pmatrix} d_v & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n_c} \end{pmatrix}. \]

The expression of \( R_0 \) can then be found by identifying the spectral radius (the dominant eigenvalue) of the matrix \( K_L = -T\Sigma^{-1} \) where \( K_L \) is called the next generation matrix with large domain.

Since \( \Sigma \) is a diagonal non-singular square matrix, its inverse is:

\[
\Sigma^{-1} = - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{d_v} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_{n_c}} \end{pmatrix}
\]

and basic algebraic manipulations allow to show that the next generation matrix with large domain writes:

\[
K_L = -T\Sigma^{-1} = \beta (N_V, N) \begin{pmatrix} 0 & p_V \phi_1 (N) \frac{N_V}{N_1} & \cdots & p_V \phi_{n_c} (N) \frac{N_V}{N_{n_c}} \\ \frac{1}{d_v} \phi_1 (N)p_{1V} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{d_v} \phi_{n_c} (N)p_{n_cV} & 0 & \cdots & 0 \end{pmatrix}.
\]

In order to compute the eigenvalues of \( K_L \), we need to find the solutions of \( |K_L - \lambda I| = 0 \).

It can be shown that every \( n \) dimensional square matrix of the form:

\[
A = \begin{pmatrix} -\lambda & a_n \\ b_1 & -\lambda & 0 \\ \vdots & \vdots & \vdots \\ b_n & 0 & -\lambda \end{pmatrix}
\]
has as determinant $|A| = (-1)^n \lambda^n + (-1)^{n+1} \lambda^{n-2} \sum_{i=1}^n a_i b_i$.

Since $K_L - \lambda$ corresponds to such a square matrix of dimension $(n_c + 1)$ and because $|kA - \lambda I| = k^n \left| A - \frac{\lambda}{k} I \right|$ for any non-null real value of $k$, one can show that

$$|K_L - \lambda I| = \beta^{n_c+1} (N_V, N) \left[ (-1)^{n_c+1} \left( \frac{\lambda}{\beta(N_V, N)} \right)^{n_c+1} + (-1)^{n_c} \left( \frac{\lambda}{\beta(N_V, N)} \right)^{n_c+1} \frac{N_V}{d_V} \sum_{i \in C} \left( \phi_i^2 (N) \frac{p_{IV}}{d_i N_i} \right) \right].$$

As we look at the solutions of $|K_L - \lambda I| = 0$, it comes:

$$(-1)^{n_c+1} \left( \frac{\lambda}{\beta(N_V, N)} \right)^{n_c+1} + (-1)^{n_c} \left( \frac{\lambda}{\beta(N_V, N)} \right)^{n_c+1} \frac{N_V}{d_V} \sum_{i \in C} \left( \phi_i^2 (N) \frac{p_{IV}}{d_i N_i} \right) = 0.$$

Reducing and factoring the above equality, one find it equivalent to:

$$- \left( \frac{\lambda}{\beta(N_V, N)} \right)^{n_c-1} \left( \frac{\lambda}{\beta(N_V, N)} \right)^{2} \frac{N_V}{d_V} \sum_{i \in C} \left( \phi_i^2 (N) \frac{p_{IV}}{d_i N_i} \right) = 0,$$

whose following non-trivial solution corresponds to the expression of $\Re_0$:

$$\Re_0 = \sqrt{\sum_{i \in C} \beta^2 (N_V, N) \phi_i^2 (N) \frac{p_{IV} p_{IV N_V}}{d_i d_V N_i}}.$$