# DCON for SPEC

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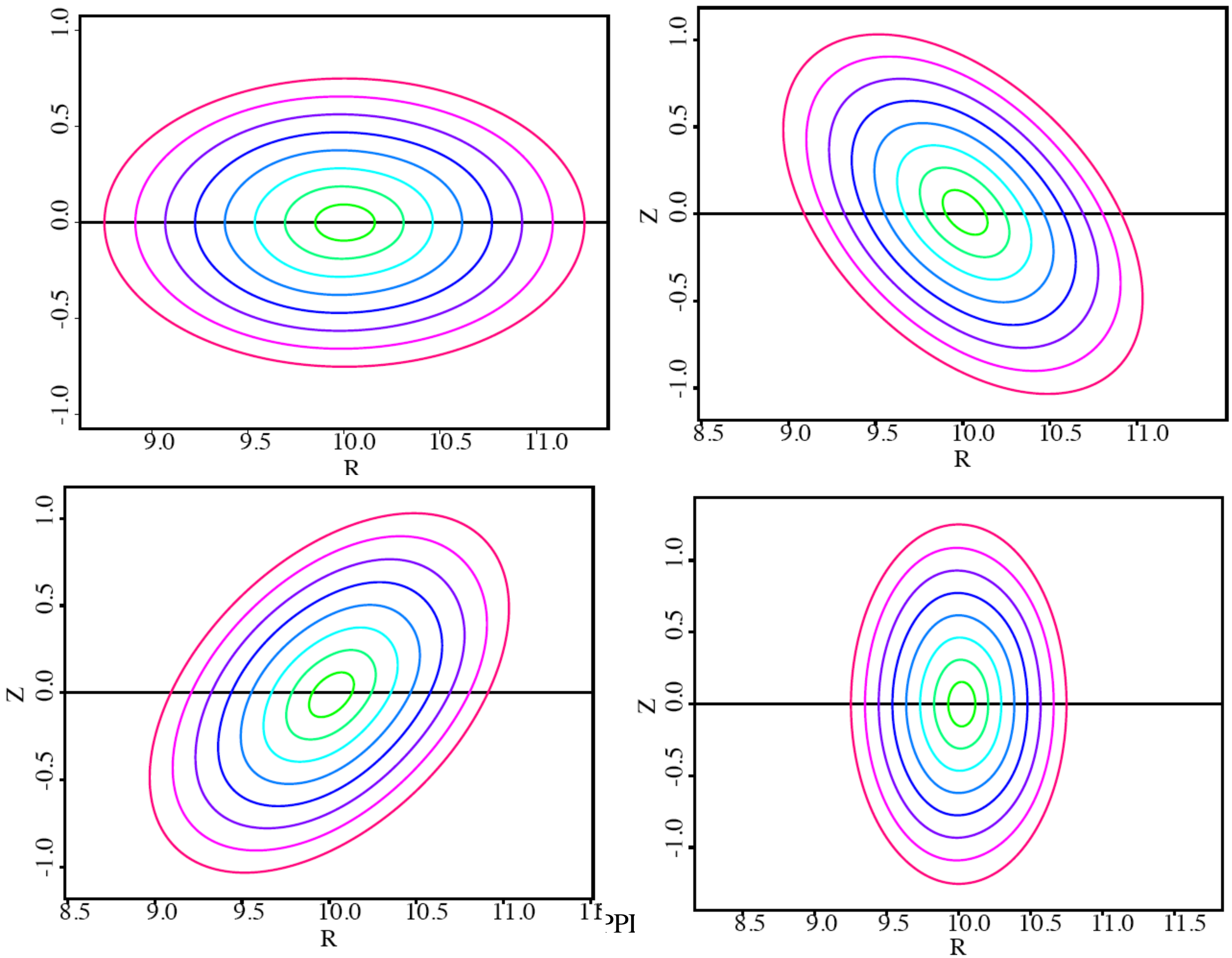
# Ideal MHD Stability Criteria

- Bernstein, Frieman, Kruskal & Kulsrud (1958) derived an ideal MHD potential energy functional $\delta W[\boldsymbol{\xi}(\mathbf{x})]$. They showed that a static equilibrium is unstable iff there is a perturbation $\boldsymbol{\xi}(\mathbf{x})$ that makes $\delta W < 0$.
- Newcomb (1960) applied this to a cylindrical plasma. He derived a 2nd-order Euler-Lagrange equation, $(f\,\xi')' - g\,\xi = 0$, and showed that there is a $\xi(r)$ that makes $\delta W < 0$ iff the solution $\xi(r)$ changes sign between singular points, where $f = 0$. This criterion is limited to fixed-boundary modes.
- PEST, ERATO, GATO, CAS3D, TERPSICHORE (1975++) expansion in basis functions, large matrix eigenvalue problem.
- Glasser (2016) generalized Newcomb's criterion to axisymmetric toroidal plasmas, with $\xi(r)$ replaced by a vector $\boldsymbol{\Xi}(\psi)$ of complex Fourier coefficients of coupled poloidal harmonics $m$ and a $2M^{th}$ order ODE. Implemented in the DCON code.
- A simple procedure extends this to free-boundary modes, with a vacuum region surrounding the plasma.
- The present work further generalizes this to a nonaxisymmetric plasma with stellarator symmetry, coupling multiple toroidal harmonics $n = n_0 + k*l$, with l the number of stellarator field periods and k any integer. Formulated for SPEC equilibrium.

# Stellarator Equilibrium Codes

- **VMEC**
  - Developed by Steve Hirshman et al, ORNL.
  - Imposes nested toroidal flux surfaces on equilibrium, not true energy minimum.
  - Radial finite differences equally space in toroidal magnetic flux.
  - Problems: very coarse near magnetic axis, fails to ensure div **B** = 0.
- **SPEC**
  - S.R. Hudson, R.L. Dewar *et al.*, "Computation of multi-region relaxed magnetohydrodynamic equilibria," Phys. Plasmas 19, 112502 (2012).
  - Infinitesimal interfaces with finite pressure discontinuities, [[p+B^2/2]] = 0, $B_{normal} = 0$.
  - Volumes between interfaces where **B** may be stochastic.
  - All plasma confinement is localized to infinitesimal interfaces.
  - Radial discretization uses Chebyshev polynomials.
  - Regularization near coordinate axis.
  - Solves for vector potential **A** rather than magnetic field **B**, ensures div **B** = 0

# SPEC Interfaces at Four Toroidal Locations



# Outline of Stability Calculation, 1

- Read data from SPEC equilibrium output file, using HDF5 and `read_spec.f90`, written by Jonathan Schilling of IPP Greifswald.

- Compute $R, Z(\theta, \zeta)$ for each interface and $A_\theta, A_\zeta(s, \theta, \zeta)$ for each volume, using Fourier series, Chebyshev polynomials, and regularization.

- Construct Fourier components of metric tensor in each volume.

- For each volume, use covariant components of $\boldsymbol{\xi} \times \mathbf{B}$ and contravariant components of perturbed magnetic field $\mathbf{Q} = \nabla \times (\xi \times \mathbf{B})$ to derive Euler-Lagrange equation minimizing ideal MHD energy principle $\delta W$, using metric tensor.

- Use LSODE to integrate Euler-Lagrange equations, monitor critical determinant $D_C(s)$ whose vanishing is the generalized Newcomb criterion, indicating fixed-boundary instability. Proof involves integrated terms that couple to adjacent interfaces.

# Outline of Stability Calculation, 2

- Treat inifinitesimal interfaces as finite width $\delta$, then take $\delta \to 0$. Produces discontinuity in $\Xi'(s)$.

- Construct straight-fieldline coordinates, greatly simplify interface equations. Transformation between straight-fieldline and native coordinates.

- For each interface, use contravariant components of $\boldsymbol{\xi}$ and perturbed magnetic field $\mathbf{Q} = \nabla \times (\xi \times \mathbf{B})$ to derive Euler-Lagrange equation minimizing ideal MHD energy principle $\delta W$, using metric tensor.

- Use LSODE to integrate Euler-Lagrange equations, monitor critical determinant $D_C(S)$, $D_C = 0$ is generalized Newcomb criterion, indicates instability. Proof involves integrated terms that couple to adjacent volumes.

- In each region, initialize Euler-Lagrange equation from previous region. Causes integrated terms to cancel.

- If there are no Newcomb crossings, then construct plasma and vacuum response matrices, add them together to obtain total response matrix. Negative eigenvalues imply free-boundary instabilities. Requires vacuum Green's function solution, Peter Merkel of IPP Garching.

# SPEC Equilibrium Description

## Interface Coordinates

$$\mathbf{x}(\theta,\zeta) = x(\theta,\zeta)\hat{\mathbf{e}}_x + y(\theta,\zeta)\hat{\mathbf{e}}_y + z(\theta,\zeta)\hat{\mathbf{e}}_z$$
$$= R(\theta,\zeta)\hat{\mathbf{e}}_R + Z(\theta,\zeta)\hat{\mathbf{e}}_Z$$

$$x = R\cos\phi, \quad y = R\sin\phi, \quad \zeta = \phi$$

$$R = \sum_{m,n} R_{m,n}\cos(m\theta - n\zeta)$$
$$Z = \sum_{m,n} Z_{m,n}\sin(m\theta - n\zeta)$$

## Equilibrium Vector Potential Between Interfaces

$$s \in (-1,1), \quad \mathcal{J} \equiv (\nabla s \times \nabla\theta \cdot \nabla\zeta)^{-1}, \quad \mathcal{F}' \equiv \partial_s \mathcal{F}$$
$$\mathbf{A} = A_\theta(s,\theta,\zeta)\nabla\theta + A_\zeta(s,\theta,\zeta)\nabla\zeta$$

$$A_\theta(s,\theta,\zeta) = \sum_k \sum_j A_{\theta,k,j} T_k(s)\cos(m_j\theta - n_j\zeta)$$
$$A_\zeta(s,\theta,\zeta) = \sum_k \sum_j A_{\zeta,k,j} T_k(s)\cos(m_j\theta - n_j\zeta)$$

## Equilibrium Magnetic Field Between Interfaces

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla A_\theta \times \nabla\theta + \nabla A_\zeta \times \nabla\zeta$$
$$= \mathcal{J}\left(B_s \nabla\theta \times \nabla\zeta + B_\theta \nabla\zeta \times \nabla s + B_\zeta \nabla s \times \nabla\theta\right)$$

$$\mathcal{J}B_s = \partial_\theta A_\zeta - \partial_\zeta A_\theta$$
$$\mathcal{J}B_\theta = -\partial_s A_\zeta$$
$$\mathcal{J}B_\zeta = \partial_s A_\theta$$

# Perturbations

$$\boldsymbol{\xi} = \mathcal{J}\left(\xi^s \nabla\theta \times \nabla\zeta + \xi^\theta \nabla\zeta \times \nabla s + \xi^\zeta \nabla s \times \nabla\theta\right)$$

$$\boldsymbol{\alpha} \equiv \boldsymbol{\xi} \times \mathbf{B} = \alpha_s \nabla s + \alpha_\theta \nabla\theta + \alpha_\zeta \nabla\zeta$$

$$\mathbf{B} \cdot \boldsymbol{\alpha} = B^s \alpha_s + B^\theta \alpha_\theta + B^\zeta \alpha_\zeta = 0$$

$$\boldsymbol{\xi} = \frac{\mathbf{B}}{B^2} \times \left(\alpha_s \nabla s + \alpha_\theta \nabla\theta + \alpha_\zeta \nabla\zeta\right) + \xi_\parallel \mathbf{B}$$

$$\mathcal{J}\nabla \cdot \boldsymbol{\xi} = \partial_s\left(\mathcal{J}\xi^s\right) + \partial_\theta\left(\mathcal{J}\xi^\theta\right) + \partial_\zeta\left(\mathcal{J}\xi^\zeta\right)$$

$$\begin{aligned}
\alpha_s &= \mathcal{J}\left(\xi^\theta B^\zeta - \xi^\zeta B^\theta\right) & \xi^s &= \frac{\mathcal{J}}{B^2}\left(B_\theta \alpha_\zeta - B_\zeta \alpha_\theta\right) + B^s \xi_\parallel \\
\alpha_\theta &= \mathcal{J}\left(\xi^\zeta B^s - \xi^s B^\zeta\right) & \xi^\theta &= \frac{\mathcal{J}}{B^2}\left(B_\zeta \alpha_s - B_s \alpha_\zeta\right) + B^\theta \xi_\parallel \\
\alpha_\zeta &= \mathcal{J}\left(\xi^s B^\theta - \xi^\theta B^s\right) & \xi^\zeta &= \frac{\mathcal{J}}{B^2}\left(B_s \alpha_\theta - B_\theta \alpha_s\right) + B^\zeta \xi_\parallel
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q} = &\nabla \times \boldsymbol{\alpha} = \left(\partial_\theta \alpha_\zeta - \partial_\zeta \alpha_\theta\right) \nabla\theta \times \nabla\zeta \\
&+ \left(\partial_\zeta \alpha_s - \partial_s \alpha_\zeta\right) \nabla\zeta \times \nabla s \quad + \left(\partial_s \alpha_\theta - \partial_\theta \alpha_s\right) \nabla s \times \nabla\theta
\end{aligned}$$

# Fourier Representation

## Fourier Series

$$\alpha_s(s,\theta,\zeta) = \sum_k u_k(s) \exp[i(m_k\theta - n_k\zeta)]$$

$$\alpha_\theta(s,\theta,\zeta) = \sum_k v_k(s) \exp[i(m_k\theta - n_k\zeta)]$$

$$\alpha_\zeta(s,\theta,\zeta) = \sum_k w_k(s) \exp[i(m_k\theta - n_k\zeta)]$$

$n = n_0 + kl$, $\quad n_0 =$ fundamental, $\quad k =$ harmonic, any integer
$l =$ toroidal periodicity, *e.g.* 5 for Wendelstein 7-X

## Fourier Vectors

$$\boldsymbol{\alpha}_s(s) \equiv \{u_{m,n}(s)\}$$

$$\boldsymbol{\alpha}_\theta(s) \equiv \{v_{m,n}(s)\}$$

$$\boldsymbol{\alpha}_\zeta(s) \equiv \{w_{m,n}(s)\}$$

$$\boldsymbol{\alpha} \equiv \begin{pmatrix} \boldsymbol{\alpha}_s \\ \boldsymbol{\alpha}_\theta \\ \boldsymbol{\alpha}_\zeta \end{pmatrix}$$

# Ideal MHD Energy Principle
## Volumes Between Interfaces

$$\delta W = \frac{1}{2}\int d\mathbf{x}\left[\mathbf{Q}^2 + \mathbf{J}\cdot\boldsymbol{\xi}\times\mathbf{Q} + (\boldsymbol{\xi}\cdot\nabla P)(\nabla\cdot\boldsymbol{\xi}) + \gamma P(\nabla\cdot\boldsymbol{\xi})^2\right]$$

$$= \frac{1}{2}\int d\mathbf{x}\Big\{\left[\left(\partial_\theta\alpha_\zeta - \partial_\zeta\alpha_\theta\right)\nabla\theta\times\nabla\zeta + \left(\partial_\zeta\alpha_s - \partial_s\alpha_\zeta\right)\nabla\zeta\times\nabla s + \left(\partial_s\alpha_\theta - \partial_\theta\alpha_s\right)\nabla s\times\nabla\theta\right]^2 - \frac{\mu}{\mathcal{J}}\left[\alpha_s\left(\partial_\theta\alpha_\zeta - \partial_\zeta\alpha_\theta\right) + \alpha_\theta\left(\partial_\zeta\alpha_s - \partial_s\alpha_\zeta\right) + \alpha_\zeta\left(\partial_s\alpha_\theta - \partial_\theta\alpha_s\right)\right]\Big\}$$

$$= \frac{V'}{2}\int_{-1}^{1} ds\left(\boldsymbol{\alpha}'^\dagger\mathsf{A}\boldsymbol{\alpha}' + \boldsymbol{\alpha}'^\dagger\mathsf{B}\boldsymbol{\alpha} + \boldsymbol{\alpha}^\dagger\mathsf{B}^\dagger\boldsymbol{\alpha}' + \boldsymbol{\alpha}^\dagger\mathsf{C}\boldsymbol{\alpha}\right)$$

# Metric Tensor

$$d\xi^i = (ds, d\theta, d\zeta), \quad dx^j = (dR, dZ, Rd\phi)$$

$$J_i^j \equiv \frac{\partial x^j}{\partial \xi^i}, \quad \mathcal{J} \equiv \det \mathbf{J}, \quad \mathbf{g} \equiv \frac{\mathbf{J}\mathbf{J}^\dagger}{\mathcal{J}^2}$$

$$\begin{aligned} g_{11} &\equiv |\nabla\theta \times \nabla\zeta|^2 & g_{23} &\equiv (\nabla\zeta \times \nabla s) \cdot (\nabla s \times \nabla\theta) \\ g_{22} &\equiv |\nabla\zeta \times \nabla s|^2 & g_{31} &\equiv (\nabla s \times \nabla\theta) \cdot (\nabla\theta \times \nabla\zeta) \\ g_{33} &\equiv |\nabla s \times \nabla\theta|^2 & g_{12} &\equiv (\nabla\theta \times \nabla\zeta) \cdot (\nabla\zeta \times \nabla s) \end{aligned}$$

$$\langle \mathcal{F} \rangle_{m,n;m',n'} \equiv \frac{\oint d\theta \oint d\zeta \mathcal{J} \mathcal{F} \exp\left[i(m-m')\theta - i(n-n')\zeta\right]}{\oint d\theta \oint d\zeta \mathcal{J}}$$

$$(G_{ij})_{m,n;m',n'} \equiv \langle g_{ij} \rangle_{m,n;m',n'}, \quad \mathbf{G}_{ij}^\dagger = \mathbf{G}_{ij}$$

$$M_{m,m'} \equiv m\delta_{m,m'}, \quad N_{n,n'} \equiv n\delta_{n,n'}$$

$$Q_{m,n;m',n'} \equiv (m - nq)\delta_{m,m'}\delta_{n,n'}$$

# Coefficient Matrices

## Coefficient Matrix $\mathbf{A}$

$$\mathbf{A} = \mathbf{A}^\dagger = \begin{pmatrix} \mathbf{A}_{ss} & \mathbf{A}_{s\theta} & \mathbf{A}_{s\zeta} \\ \mathbf{A}_{\theta s} & \mathbf{A}_{\theta\theta} & \mathbf{A}_{\theta\zeta} \\ \mathbf{A}_{\zeta s} & \mathbf{A}_{\zeta\theta} & \mathbf{A}_{\zeta\zeta} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{33} & -\mathbf{G}_{23} \\ \mathbf{0} & -\mathbf{G}_{23} & \mathbf{G}_{22} \end{pmatrix}$$

## Coefficient Matrix $\mathbf{B}$

$$\begin{aligned} \mathbf{B} &= \begin{pmatrix} \mathbf{B}_{ss} & \mathbf{B}_{s\theta} & \mathbf{B}_{s\zeta} \\ \mathbf{B}_{\theta s} & \mathbf{B}_{\theta\theta} & \mathbf{B}_{\theta\zeta} \\ \mathbf{B}_{\zeta s} & \mathbf{B}_{\zeta\theta} & \mathbf{B}_{\zeta\zeta} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -i\left(\mathbf{G}_{33}\mathbf{M} + \mathbf{G}_{23}\mathbf{N}\right) & i\mathbf{G}_{31}\mathbf{N} & i\mathbf{G}_{31}\mathbf{M} - (\mu/V')\mathbf{I} \\ i\left(\mathbf{G}_{22}\mathbf{N} + \mathbf{G}_{23}\mathbf{M}\right) & -i\mathbf{G}_{12}\mathbf{N} + (\mu/V')\mathbf{I} & -i\mathbf{G}_{12}\mathbf{M} \end{pmatrix} \neq \mathbf{B}^\dagger \end{aligned}$$

## Coefficient Matrix $\mathbf{C}$

$$\mathbf{C} = \mathbf{C}^\dagger = \begin{pmatrix} \mathbf{C}_{ss} & \mathbf{C}_{s\theta} & \mathbf{C}_{s\zeta} \\ \mathbf{C}_{\theta s} & \mathbf{C}_{\theta\theta} & \mathbf{C}_{\theta\zeta} \\ \mathbf{C}_{\zeta s} & \mathbf{C}_{\zeta\theta} & \mathbf{C}_{\zeta\zeta} \end{pmatrix}$$

$$\begin{aligned} \mathbf{C}_{ss} =& \mathbf{M}\mathbf{G}_{33}\mathbf{M} + \mathbf{M}\mathbf{G}_{23}\mathbf{N} + \mathbf{N}\mathbf{G}_{23}\mathbf{M} + \mathbf{N}\mathbf{G}_{22}\mathbf{N} \\ \mathbf{C}_{\theta\theta} =& \mathbf{N}\mathbf{G}_{11}\mathbf{N} \\ \mathbf{C}_{\zeta\zeta} =& \mathbf{M}\mathbf{G}_{11}\mathbf{M} \\ \mathbf{C}_{s\theta} =& \mathbf{C}_{\theta s}^\dagger = -\mathbf{N}\mathbf{G}_{12}\mathbf{N} - \mathbf{M}\mathbf{G}_{31}\mathbf{N} - (i\mu/V')\mathbf{N} \\ \mathbf{C}_{\zeta s} =& \mathbf{C}_{s\zeta}^\dagger = -\mathbf{M}\mathbf{G}_{31}\mathbf{M} - \mathbf{M}\mathbf{G}_{12}\mathbf{N} + (i\mu/V')\mathbf{M} \\ \mathbf{C}_{\theta\zeta} =& \mathbf{C}_{\zeta\theta}^\dagger = \mathbf{N}\mathbf{G}_{11}\mathbf{M} \end{aligned}$$

# Elimination of $\alpha_s$

## Minimization with Respect to $\alpha_s^\dagger$

$$\mathsf{B}_{s\theta}\alpha_\theta' + \mathsf{B}_{s\zeta}\alpha_\zeta' + \mathsf{C}_{ss}\alpha_s + \mathsf{C}_{s\theta}\alpha_\theta + \mathsf{C}_{s\zeta}\alpha_\zeta = 0$$

$$\alpha_s = -\mathsf{C}_{ss}^{-1}\left(\mathsf{B}_{s\theta}\alpha_\theta' + \mathsf{B}_{s\zeta}\alpha_\zeta' + \mathsf{C}_{s\theta}\alpha_\theta + \mathsf{C}_{s\zeta}\alpha_\zeta\right)$$

## Schur Complements

$$\begin{array}{lll}
\overline{\mathsf{A}}_{\theta\theta} \equiv \mathsf{A}_{\theta\theta} - \mathsf{B}_{\theta s}\mathsf{C}_{ss}^{-1}\mathsf{B}_{s\theta} & \overline{\mathsf{B}}_{\theta\theta} \equiv \mathsf{B}_{\theta\theta} - \mathsf{B}_{\theta s}\mathsf{C}_{ss}^{-1}\mathsf{C}_{s\theta} & \overline{\mathsf{C}}_{\theta\theta} \equiv \mathsf{C}_{\theta\theta} - \mathsf{C}_{\theta s}\mathsf{C}_{ss}^{-1}\mathsf{C}_{s\theta} \\
\overline{\mathsf{A}}_{\theta\zeta} \equiv \mathsf{A}_{\theta\zeta} - \mathsf{B}_{\theta s}\mathsf{C}_{ss}^{-1}\mathsf{B}_{s\zeta} & \overline{\mathsf{B}}_{\theta\zeta} \equiv \mathsf{B}_{\theta\zeta} - \mathsf{B}_{\theta s}\mathsf{C}_{ss}^{-1}\mathsf{C}_{s\zeta} & \overline{\mathsf{C}}_{\theta\zeta} \equiv \mathsf{C}_{\theta\zeta} - \mathsf{C}_{\theta s}\mathsf{C}_{ss}^{-1}\mathsf{C}_{s\zeta} \\
\overline{\mathsf{A}}_{\zeta\theta} \equiv \mathsf{A}_{\zeta\theta} - \mathsf{B}_{\zeta s}\mathsf{C}_{ss}^{-1}\mathsf{B}_{s\theta} & \overline{\mathsf{B}}_{\zeta\theta} \equiv \mathsf{B}_{\zeta\theta} - \mathsf{B}_{\zeta s}\mathsf{C}_{ss}^{-1}\mathsf{C}_{s\theta} & \overline{\mathsf{C}}_{\zeta\theta} \equiv \mathsf{C}_{\zeta\theta} - \mathsf{C}_{\zeta s}\mathsf{C}_{ss}^{-1}\mathsf{C}_{s\theta} \\
\overline{\mathsf{A}}_{\zeta\zeta} \equiv \mathsf{A}_{\zeta\zeta} - \mathsf{B}_{\zeta s}\mathsf{C}_{ss}^{-1}\mathsf{B}_{s\zeta} & \overline{\mathsf{B}}_{\zeta\zeta} \equiv \mathsf{B}_{\zeta\zeta} - \mathsf{B}_{\zeta s}\mathsf{C}_{ss}^{-1}\mathsf{C}_{s\zeta} & \overline{\mathsf{C}}_{\zeta\zeta} \equiv \mathsf{C}_{\zeta\zeta} - \mathsf{C}_{\zeta s}\mathsf{C}_{ss}^{-1}\mathsf{C}_{s\zeta}
\end{array}$$

## Reduced Matrices

$$\overline{\alpha} \equiv \begin{pmatrix} \alpha_\theta \\ \alpha_\zeta \end{pmatrix} \qquad \overline{\mathsf{A}} \equiv \begin{pmatrix} \overline{\mathsf{A}}_{\theta\theta} & \overline{\mathsf{A}}_{\theta\zeta} \\ \overline{\mathsf{A}}_{\zeta\theta} & \overline{\mathsf{A}}_{\zeta\zeta} \end{pmatrix}$$

$$\overline{\mathsf{B}} \equiv \begin{pmatrix} \overline{\mathsf{B}}_{\theta\theta} & \overline{\mathsf{B}}_{\theta\zeta} \\ \overline{\mathsf{B}}_{\zeta\theta} & \overline{\mathsf{B}}_{\zeta\zeta} \end{pmatrix} \qquad \overline{\mathsf{C}} \equiv \begin{pmatrix} \overline{\mathsf{C}}_{\theta\theta} & \overline{\mathsf{C}}_{\theta\zeta} \\ \overline{\mathsf{C}}_{\zeta\theta} & \overline{\mathsf{C}}_{\zeta\zeta} \end{pmatrix}$$

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# Euler-Lagrange Equation
## Volumes Between Interfaces

### Euler-Lagrange Equation

$$\delta W = \frac{V'}{2}\int_{-1}^{1} ds\left(\overline{\alpha}^{\dagger\prime}\overline{\mathbf{A}}\overline{\alpha}' + \overline{\alpha}^{\dagger\prime}\overline{\mathbf{B}}\overline{\alpha} + \overline{\alpha}^{\dagger}\overline{\mathbf{B}}^{\dagger}\overline{\alpha}' + \overline{\alpha}^{\dagger}\overline{\mathbf{C}}\overline{\alpha}\right)$$

$$\overline{\mathbf{A}}^{\dagger} = \overline{\mathbf{A}}, \quad \overline{\mathbf{B}}^{\dagger} \neq \overline{\mathbf{B}}, \quad \overline{\mathbf{C}}^{\dagger} = \overline{\mathbf{C}}$$

$$\left(\overline{\mathbf{A}}\overline{\alpha}' + \overline{\mathbf{B}}\overline{\alpha}\right)' - \left(\overline{\mathbf{B}}^{\dagger}\overline{\alpha}' + \overline{\mathbf{C}}\overline{\alpha}\right) = 0$$

The equation has singular points wherever $\det\overline{\mathbf{A}} = 0$.
Not yet understood.

### Matrix Form; Hamiltonian and Symplectic Symmetries

$$\mathbf{u} \equiv \begin{pmatrix} \overline{\alpha} \\ \overline{\mathbf{A}}\overline{\alpha}' + \overline{\mathbf{B}}\overline{\alpha} \end{pmatrix}, \quad \mathbf{L} \equiv \begin{pmatrix} -\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}} & \overline{\mathbf{A}}^{-1} \\ \overline{\mathbf{C}} - \overline{\mathbf{B}}^{\dagger}\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}} & \overline{\mathbf{B}}^{\dagger}\overline{\mathbf{A}}^{-1} \end{pmatrix}, \quad \mathbf{u}' = \mathbf{L}\mathbf{u}$$

$$\mathbf{L}_{22}^{\dagger} = -\mathbf{L}_{11}, \quad \mathbf{L}_{12}^{\dagger} = \mathbf{L}_{12}, \quad \mathbf{L}_{21}^{\dagger} = \mathbf{L}_{21}$$

$$\mathbf{J} \equiv \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{J}\mathbf{L}\mathbf{J} = \mathbf{L}^{\dagger}$$

$$\mathbf{U}' = \mathbf{L}\mathbf{U}, \quad \mathbf{U}(0) = \mathbf{I}, \quad \left(\mathbf{U}^{\dagger}\mathbf{J}\mathbf{U}\right)' = 0, \quad \mathbf{U}^{\dagger}\mathbf{J}\mathbf{U} = \mathbf{J}$$

# Generalized Newcomb Criterion
## Volumes Between Interfaces

### Energy Principle

$$\delta W = \frac{V'}{2}\int_{-1}^{1} ds \left(\overline{\boldsymbol{\alpha}}^{\dagger\prime}\overline{\mathsf{A}}\overline{\boldsymbol{\alpha}}' + \overline{\boldsymbol{\alpha}}^{\dagger\prime}\overline{\mathsf{B}}\overline{\boldsymbol{\alpha}} + \overline{\boldsymbol{\alpha}}^{\dagger}\overline{\mathsf{B}}^{\dagger}\overline{\boldsymbol{\alpha}}' + \overline{\boldsymbol{\alpha}}^{\dagger}\overline{\mathsf{C}}\overline{\boldsymbol{\alpha}}\right)$$

### Perfect Derivative

For any $\mathsf{P} = \mathsf{P}^{\dagger}$,

$$\frac{V'}{2}\int_{-1}^{1} ds \left(\boldsymbol{\alpha}^{\dagger}\mathsf{P}\boldsymbol{\alpha}\right)' = \frac{V'}{2}\int_{-1}^{1} ds \left(\boldsymbol{\alpha}'^{\dagger}\mathsf{P}\boldsymbol{\alpha} + \boldsymbol{\alpha}^{\dagger}\mathsf{P}\boldsymbol{\alpha}' + \boldsymbol{\alpha}^{\dagger}\mathsf{P}'\boldsymbol{\alpha}\right) = \frac{V'}{2}\,\boldsymbol{\alpha}^{\dagger}\mathsf{P}\boldsymbol{\alpha}\Big|_{-1}^{1}$$

Neglect boundary terms, subtract from $\delta W$ without changing its value.

$$\delta W = \frac{V'}{2}\int_{-1}^{1} ds \left[\overline{\boldsymbol{\alpha}}^{\dagger\prime}\overline{\mathsf{A}}\overline{\boldsymbol{\alpha}}' + \overline{\boldsymbol{\alpha}}^{\dagger\prime}\left(\overline{\mathsf{B}} - \mathsf{P}\right)\overline{\boldsymbol{\alpha}} + \overline{\boldsymbol{\alpha}}^{\dagger}\left(\overline{\mathsf{B}}^{\dagger} - \mathsf{P}\right)\overline{\boldsymbol{\alpha}}' + \overline{\boldsymbol{\alpha}}^{\dagger}\left(\overline{\mathsf{C}} - \mathsf{P}'\right)\overline{\boldsymbol{\alpha}}\right]$$

### Completing the Square

Choose $\mathsf{P} \equiv \mathsf{U}_{22}\mathsf{U}_{12}^{-1}, \quad D_C \equiv \det \mathsf{P}^{-1}$

$\mathsf{P}$ is self-adjoint by the symplectic symmetry of $\mathsf{U}$
It exists if and only if $D_C \neq 0$ for $s \in (0, a)$.

$$\mathsf{P}' = \overline{\mathsf{C}} - \left(\mathsf{P} - \overline{\mathsf{B}}\right)\overline{\mathsf{A}}^{-1}\left(\mathsf{P} - \overline{\mathsf{B}}^{\dagger}\right)$$

$$\delta W = \frac{V'}{2}\int_{-1}^{1}\left[\overline{\boldsymbol{\alpha}}'^{\dagger} + \overline{\boldsymbol{\alpha}}^{\dagger}\left(\overline{\mathsf{B}}^{\dagger} - \mathsf{P}\right)\overline{\mathsf{A}}^{-1}\right]\overline{\mathsf{A}}\left[\overline{\boldsymbol{\alpha}}' + \overline{\mathsf{A}}^{-1}\left(\overline{\mathsf{B}} - \mathsf{P}\right)\overline{\boldsymbol{\alpha}}\right] ds$$

$\delta W \geq 0$. $D_C \neq 0$ is generalized newcomb criterion.
Boundary terms couple each region to neighboring regions.

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# Straight Fieldline Coordinates
## Layers Inside Interfaces

$$\mathbf{B} = \mathcal{J}\left(B^\theta \nabla\zeta \times \nabla s + B^\zeta \nabla s \times \nabla\theta\right)$$

$$\chi'(s) \equiv \int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \mathcal{J} B^\theta, \quad \Phi'(s) \equiv \int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \mathcal{J} B^\zeta$$

$$\bar{\mathcal{J}}\mathbf{B} \cdot \nabla\bar{\theta} = \chi', \quad \bar{\mathcal{J}}\mathbf{B} \cdot \nabla\bar{\zeta} = \Phi', \quad \bar{\mathcal{J}} \equiv \left(\nabla s \cdot \nabla\bar{\theta} \times \nabla\bar{\zeta}\right)^{-1}$$

$$\mathbf{B} = \chi' \nabla\bar{\zeta} \times \nabla s + \Phi' \nabla s \times \nabla\bar{\theta} = \chi'\left(\nabla\bar{\zeta} - q\nabla\bar{\theta}\right) \times \nabla s, \quad q \equiv \Phi'/\chi'$$

$$\mathbf{B} \cdot \nabla\bar{\theta} = B^\theta \partial_\theta \bar{\theta} + B^\zeta \partial_\zeta \bar{\theta} = \chi'/\bar{\mathcal{J}}, \quad \mathbf{B} \cdot \nabla\bar{\zeta} = B^\theta \partial_\theta \bar{\zeta} + B^\zeta \partial_\zeta \bar{\zeta} = \Phi'/\bar{\mathcal{J}}$$

$$\bar{\theta}(\theta, \zeta) = \theta + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u_{m,n} e^{i(m\theta - n\zeta)}$$
$$\bar{\zeta}(\theta, \zeta) = \zeta + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} v_{m,n} e^{i(m\theta - n\zeta)}$$

$$i(mB^\theta - nB^\zeta) u_{m,n} = \chi'(\nabla s \cdot \nabla\bar{\theta} \times \nabla\bar{\zeta})_{m,n}$$
$$i(mB^\theta - nB^\zeta) v_{m,n} = \Phi'(\nabla s \cdot \nabla\bar{\theta} \times \nabla\bar{\zeta})_{m,n}$$

Solution procedure: Let $\bar{\mathcal{J}} = \mathcal{J}$. Since $u_{m,n}/v_{m,n}$ is independent of $\theta$ and $\zeta$, these terms don't contribution to $\bar{\mathcal{J}}$. Only the nonzero terms in $(\nabla s \cdot \nabla\theta \times \nabla\zeta)_{m,n}$ contribute nonzero terms to $u_{m,n}$ and $v_{m,n}$.

# Transformation of Fourier Series

$$y(x+2\pi) = y(x) + 2\pi, \quad g(y) = g(y+2\pi) = g(y(x)) = f(x)$$

$$f(x) = c_0 + \sum_{m=1}^{\infty} (c_m \cos mx + s_m \sin mx)$$
$$g(y) = \bar{c}_0 + \sum_{m=1}^{\infty} (\bar{c}_m \cos my + \bar{s}_m \sin my)$$

## Transformation Matrices

$$T^{00}_{0,0} = 1, \quad T^{01}_{0,m'} = \frac{1}{2\pi}\int_0^{2\pi} dy \cos m'x, \quad T^{02}_{0,m'} = \frac{1}{2\pi}\int_0^{2\pi} dy \sin m'x$$

$$T^{11}_{m,m'} = \frac{1}{\pi}\int_0^{2\pi} dy \cos my \cos m'x, \quad T^{12}_{m,m'} = \frac{1}{\pi}\int_0^{2\pi} dy \cos my \sin m'x$$
$$T^{21}_{m,m'} = \frac{1}{\pi}\int_0^{2\pi} dy \sin my \cos m'x, \quad T^{22}_{m,m'} = \frac{1}{\pi}\int_0^{2\pi} dy \sin my \sin m'x$$

$$\mathbf{a} \equiv \begin{pmatrix} c_0 \\ \mathbf{c} \\ \mathbf{s} \end{pmatrix}, \quad \bar{\mathbf{a}} \equiv \begin{pmatrix} \bar{c}_0 \\ \bar{\mathbf{c}} \\ \bar{\mathbf{s}} \end{pmatrix}, \quad \mathsf{T} \equiv \begin{pmatrix} 1 & \mathsf{T}^{01} & \mathsf{T}^{02} \\ 0 & \mathsf{T}^{11} & \mathsf{T}^{12} \\ 0 & \mathsf{T}^{21} & \mathsf{T}^{22} \end{pmatrix}$$

$$\bar{\mathbf{a}} = \mathsf{T}\mathbf{a}, \quad \mathbf{a} = \mathsf{T}^{-1}\bar{\mathbf{a}}$$

# Equilibrium Magnetic Field
## Layers Inside Interfaces

$$s \in (-1,1), \quad \mathcal{J} = \left(\nabla s \cdot \nabla\theta \times \nabla\zeta\right)^{-1}, \quad \mathbf{B} \cdot \nabla s = 0$$

$$\begin{aligned} \mathbf{B} =& \mathcal{J}\left(B^\theta \nabla\zeta \times \nabla s + B^\zeta \nabla s \times \nabla\theta\right) \\ =& \mathcal{J}\left(B_s \nabla s + B_\theta \nabla\theta + B_\zeta \nabla\zeta\right) \end{aligned}$$

$$\mathcal{J} B^\theta = \chi'(s), \quad \mathcal{J} B^\zeta = q(s)\chi'(s)$$

$$\begin{aligned} B_s =& \mathbf{B} \cdot (\nabla\theta \times \nabla\zeta) = \mathcal{J}\left(g_{s\theta} B^\theta + g_{s\zeta} B^\zeta\right) \\ B_\theta =& \mathbf{B} \cdot (\nabla\zeta \times \nabla s) = \mathcal{J}\left(g_{\theta\theta} B^\theta + g_{\theta\zeta} B^\zeta\right) \\ B_\zeta =& \mathbf{B} \cdot (\nabla s \times \nabla\theta) = \mathcal{J}\left(g_{\zeta\theta} B^\theta + g_{\zeta\zeta} B^\zeta\right) \end{aligned}$$

$$\mathcal{J} \nabla \cdot \mathbf{B} = \partial_\theta\left(\mathcal{J} B^\theta\right) + \partial_\zeta\left(\mathcal{J} B^\zeta\right) = 0$$

$$\begin{aligned} |\mathbf{B}|^2 =& \mathbf{B} \cdot \mathbf{B} = \mathcal{J}\left(B_\theta B^\theta + B_\zeta B^\zeta\right) \\ =& \mathcal{J}\left[(B^\theta)^2 g_{\theta\theta} + 2 B^\theta B^\zeta g_{\theta\zeta} + (B^\zeta)^2 g_{\zeta\zeta}\right] \\ =& \chi'^2\left(g_{\theta\theta} + 2 q g_{\theta\zeta} + q^2 g_{\zeta\zeta}\right) \end{aligned}$$

# Equilibrium Current and Pressure
## Layers Inside Interfaces

$$\mathbf{J} = \mathcal{J}\left(J^\theta \nabla\zeta \times \nabla s + J^\zeta \nabla s \times \nabla\theta\right)$$

$$\begin{aligned}
\mathcal{J} J^s &= 0 \\
\mathcal{J} J^\theta &= -\partial_s B_\zeta \\
\mathcal{J} J^\zeta &= \partial_s B_\theta
\end{aligned}$$

$$\begin{aligned}
&\mathcal{J}\left(\mathbf{J} \times \mathbf{B}\right) \cdot \left(\nabla\theta \times \nabla\zeta\right) = \mathcal{J}\left(J^\theta B^\zeta - J^\zeta B^\theta\right) \\
&\quad = -\left(\partial_s B_\zeta\right) B^\zeta - \left(\partial_s B_\theta\right) B^\theta = -\frac{1}{2}\partial_s\left(B_\theta B^\theta + B_\zeta B^\zeta\right) = \partial_s P
\end{aligned}$$

$$\partial_s\left(2P + B_\theta B^\theta + B_\zeta B^\zeta\right) = 0$$

# Perturbations
## Layers Inside Interfaces

### Normal Displacement

$$\xi^s \equiv \boldsymbol{\xi} \cdot \nabla s$$

### Parallel Displacement

$$\xi_\parallel \equiv \frac{\boldsymbol{\xi} \cdot \mathbf{B}}{B^2}$$

### Normal Vector Potential

$$\alpha_s \equiv \mathcal{J} \left( \xi \times \mathbf{B} \right) \cdot \left( \nabla\theta \times \nabla\zeta \right) = \mathcal{J} \left( \xi^\theta B^\zeta - \xi^\zeta B^\theta \right)$$

Proportional to displacement in the magnetic surface
and across the equilibrium magnetic field.

### Perturbed Magnetic Field

$$\mathbf{Q} = \nabla \times \boldsymbol{\alpha} = \mathcal{J} \left( Q^s \nabla\theta \times \nabla\zeta + Q^\theta \nabla\zeta \times \nabla s + Q^\zeta \nabla s \times \nabla\theta \right)$$

$$\begin{aligned}
\mathcal{J} Q^s &= \mathcal{J} \left( B^\theta \partial_\theta + B^\zeta \partial_\zeta \right) \xi^s \\
\mathcal{J} Q^\theta &= \partial_\zeta \alpha_s - \partial_s \left( \mathcal{J} B^\theta \xi^s \right) \\
\mathcal{J} Q^\zeta &= -\partial_s \left( \mathcal{J} B^\zeta \xi^s \right) - \partial_\theta \alpha_s
\end{aligned}$$

# Fourier Representation and Ordering Assumptions

## Fourier Representation

$$\xi^s(s,\theta,\zeta) = \sum_k u_k(s) \exp[2\pi i(m_k\theta - n_k\zeta)]$$
$$\alpha_s(s,\theta,\zeta) = \sum_k v_k(s) \exp[2\pi i(m_k\theta - n_k\zeta)]$$

$$\Xi^s(s) \equiv \{u_k(s)\}, \quad \boldsymbol{\alpha}_s(s) \equiv \{v_k(s)\}$$

## Ordering Assumptions

$$x \equiv s - s_0 \sim \delta, \quad \partial_x \sim \delta^{-1}, \quad \partial_\theta \sim \partial_\zeta \sim 1$$
$$\xi^s \sim Q^s \sim \delta, \quad \partial_s \xi^s \sim \xi^\theta \sim \xi^\zeta \sim Q^\theta \sim Q^\zeta \sim 1, \quad \nabla \cdot \boldsymbol{\xi} \sim \delta$$

# Current and Pressure Terms

$$
\begin{aligned}
\mathcal{I}_1 \equiv& \mathbf{J} \cdot \xi \times \mathbf{Q} \\
=& \mathcal{J} \left[ (J^\zeta Q^\theta - J^\theta Q^\zeta) \xi^s + (J^\theta \xi^\zeta - J^\zeta \xi^\theta) Q^s \right] \\
=& \left\{ J^\zeta \left[ \partial_\zeta \xi_\wedge - \partial_s \left( \mathcal{J} B^\theta \xi^s \right) \right] + J^\theta \left[ \partial_\theta \xi_\wedge + \partial_s \left( \mathcal{J} B^\zeta \xi^s \right) \right] \right\} \xi^s \\
&+ \left( J^\theta \xi^\zeta - J^\zeta \xi^\theta \right) \chi' \left( \partial_\theta + q \partial_\zeta \right) \xi^s \\
=& \xi^s \left( J^\theta \partial_\theta + J^\zeta \partial_\zeta \right) \xi_\wedge - \xi_\wedge \left( J^\theta \partial_\theta + J^\zeta \partial_\zeta \right) \xi^s \\
&+ \left[ J^\theta \partial_s (\mathcal{J} B^\zeta) - J^\zeta \partial_s \left( \mathcal{J} B^\theta \right) \right] \xi^{s2} + P' \left( \xi^s \partial_\psi + \xi^\theta \partial_\theta + \xi^\zeta \partial_\zeta \right) \xi^s
\end{aligned}
$$

$$
\mathcal{I}_2 \equiv (\xi \cdot \nabla P)(\nabla \cdot \xi) = \frac{\xi^s P'}{\mathcal{J}} \left[ \partial_\psi \left( \mathcal{J} \xi^s \right) + \partial_\theta \left( \mathcal{J} \xi^\theta \right) \partial_\zeta \left( \mathcal{J} \xi^\zeta \right) \right]
$$

$$
\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 =& \xi^s \mathbf{J} \cdot \nabla \xi_\wedge - \xi_\wedge \mathbf{J} \cdot \nabla \xi^s + \left[ J^\theta \partial_s \left( \mathcal{J} B^\zeta \right) - J^\zeta \partial_s \left( \mathcal{J} B^\theta \right) \right] (\xi^s)^2 \\
&+ \frac{1}{\mathcal{J}} \left[ P' \partial_s \left( \mathcal{J} \xi^{s2} \right) + \partial_\theta \left( \mathcal{J} P' \xi^s \xi^\theta \right) + \partial_\zeta \left( \mathcal{J} P' \xi^s \xi^\zeta \right) \right]
\end{aligned}
$$

# Ideal MHD Energy Principle
## Layers Inside Interfaces

$$\delta W = \frac{1}{2}\int d\mathbf{x}\left[Q^2 + \mathbf{J}\cdot\boldsymbol{\xi}\times\mathbf{Q} + (\boldsymbol{\xi}\cdot\nabla P)(\nabla\cdot\boldsymbol{\xi}) + \gamma P(\nabla\cdot\boldsymbol{\xi})^2\right]$$

$$\begin{aligned}
=&\frac{1}{2}\int ds d\theta d\zeta \mathcal{J}\Big\{\Big[\mathcal{J}\left[B^\theta\partial_\theta + B^\zeta\partial\zeta\right]\xi^s\nabla\theta\times\nabla\zeta \\
&+\left[\partial_\zeta\alpha_s - \partial_s\left(\mathcal{J}B^\theta\xi^s\right)\right]\nabla\zeta\times\nabla s - \left[\partial_s\left(\mathcal{J}B^\zeta\xi^s\right) + \partial_\theta\alpha_s\right]\nabla s\times\nabla\theta\Big]^2 \\
&+\xi^s\mathbf{J}\cdot\nabla\alpha_s - \alpha_s\mathbf{J}\cdot\nabla\xi^s + \left[J^\theta\partial_s\left(\mathcal{J}B^\zeta\right) - J^\zeta\partial_s\left(\mathcal{J}B^\theta\right)\right](\xi^s)^2 \\
&+\frac{1}{\mathcal{J}}\left[P'\partial_s\left[\mathcal{J}(\xi^s)^2\right] + \partial_\theta\left(\mathcal{J}P'\xi^s\xi^\theta\right) + \partial_\zeta\left(\mathcal{J}P'\xi^s\xi^\zeta\right)\right] + \gamma P(\nabla\cdot\xi)^2\Big\}
\end{aligned}$$

$$\begin{aligned}
=&\frac{V'}{2}\int_{-\delta}^{\delta} ds\left[\boldsymbol{\alpha}_s^\dagger\mathsf{A}\boldsymbol{\alpha}_s + \boldsymbol{\alpha}_s^\dagger(\mathsf{B}\boldsymbol{\alpha}_s' + \mathsf{C}\boldsymbol{\alpha}_s) + (\boldsymbol{\alpha}_s'^\dagger\mathsf{B}^\dagger + \boldsymbol{\alpha}_s^\dagger\mathsf{C}^\dagger)\boldsymbol{\alpha}_s\right. \\
&\left.+\boldsymbol{\alpha}_s'^\dagger\mathsf{D}\boldsymbol{\alpha}_s' + \boldsymbol{\alpha}_s'^\dagger\mathsf{E}\boldsymbol{\alpha}_s + \boldsymbol{\alpha}_s^\dagger\mathsf{E}^\dagger\boldsymbol{\alpha}_s' + \boldsymbol{\alpha}_s^\dagger\mathsf{H}\boldsymbol{\alpha}_s\right]
\end{aligned}$$

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# Euler-Lagrange Equation
## Layers Inside Interfaces

### Minimization with Respect to $\alpha_s$

$$\mathsf{A}\alpha_s + \mathsf{B}\Xi^{s\prime} + \mathsf{C}\Xi^s = 0 \quad \alpha_s = -\mathsf{A}^{-1}(\mathsf{B}\Xi^{s\prime} + \mathsf{C}\Xi^s),$$

$$\mathsf{F} \equiv \mathsf{D} - \mathsf{B}^\dagger\mathsf{A}^{-1}\mathsf{B} = \mathsf{F}^\dagger$$
$$\mathsf{K} \equiv \mathsf{E} - \mathsf{B}^\dagger\mathsf{A}^{-1}\mathsf{C} \neq \mathsf{K}^\dagger$$
$$\mathsf{G} \equiv \mathsf{H} - \mathsf{C}^\dagger\mathsf{A}^{-1}\mathsf{C} = \mathsf{G}^\dagger$$

$$\Xi^s \rightarrow \Xi$$

### Euler-Lagrange Equation

$$\delta W = \frac{1}{2}\int_{-\delta}^{\delta} ds\,\left(\Xi'^\dagger\mathsf{F}\Xi' + \Xi'^\dagger\mathsf{K}\Xi + \Xi'\mathsf{K}'\Xi' + \Xi'\mathsf{G}\Xi\right)$$

$$(\mathsf{F}\Xi' + \mathsf{K}\Xi)' - (\mathsf{K}^\dagger\Xi' + \mathsf{G}\Xi) = 0$$

$$\mathrm{u} \equiv \begin{pmatrix} \Xi^s \\ \mathsf{F}\Xi^{s\prime} + \mathsf{K}\Xi^s \end{pmatrix}, \quad \mathsf{L} \equiv \begin{pmatrix} -\mathsf{F}^{-1}\mathsf{K} & \mathsf{F}^{-1} \\ \mathsf{G} - \mathsf{K}^\dagger\mathsf{F}^{-1}\mathsf{K} & \mathsf{K}^\dagger\mathsf{F}^{-1} \end{pmatrix}, \quad \mathrm{u}' = \mathsf{L}\mathrm{u}$$

# Coefficient Matrices

## Primitive Matrices

$$
\begin{aligned}
\mathbf{A} =& \mathbf{N}\mathbf{G}_{22}\mathbf{N} + \mathbf{N}\mathbf{G}_{23}\mathbf{M} + \mathbf{M}\mathbf{G}_{23}\mathbf{N} + \mathbf{M}\mathbf{G}_{33}\mathbf{M} \\
\mathbf{B} =& -i\chi' \left[\mathbf{N}\mathbf{G}_{22} + (\mathbf{M} + q\mathbf{N})\mathbf{G}_{23} + q\mathbf{M}\mathbf{G}_{33}\right] \\
\mathbf{C} =& -i\left[\chi''(\mathbf{N}\mathbf{G}_{22} + \mathbf{M}\mathbf{G}_{23}) + (q\chi')'(\mathbf{N}\mathbf{G}_{23} + \mathbf{M}\mathbf{G}_{33})\right] \\
& - \chi'(\mathbf{N}\mathbf{G}_{12} + \mathbf{M}\mathbf{G}_{31})\mathbf{Q} + i\left[\mathbf{N}P'/\chi' - \mathbf{Q}\left\langle \mathbf{J}\cdot\nabla u\right\rangle\right] \\
\mathbf{D} =& \chi'^2\left(\mathbf{G}_{22} + 2q\mathbf{G}_{23} + q^2\mathbf{G}_{33}\right) \\
\mathbf{E} =& \chi'\left[\chi''(\mathbf{G}_{22} + q\mathbf{G}_{23}) + (q\chi')'(\mathbf{G}_{23} + q\mathbf{G}_{33})\right] \\
& - i\chi'^2(\mathbf{G}_{12} + q\mathbf{G}_{31})\mathbf{Q} + P'\mathbf{I} \\
\mathbf{H} =& \chi''\left[\chi''\mathbf{G}_{22} + (q\chi')'\mathbf{G}_{23}\right] + (q\chi')'\left[\chi''\mathbf{G}_{23} + (q\chi')'\mathbf{G}_{33}\right] \\
& + i\chi'\left[\chi''(\mathbf{Q}\mathbf{G}_{12} - \mathbf{G}_{12}\mathbf{Q}) + (q\chi')'(\mathbf{Q}\mathbf{G}_{31} - \mathbf{G}_{31}\mathbf{Q})\right] \\
& + \chi'^2\mathbf{Q}\mathbf{G}_{11}\mathbf{Q} + P'\mathbf{V}''
\end{aligned}
$$

## Schur Complements

$$
\begin{aligned}
\mathbf{F} \equiv& \mathbf{D} - \mathbf{B}^\dagger\mathbf{A}^{-1}\mathbf{B} \\
=& \chi'^2\mathbf{N}^{-1}\mathbf{Q}\left[\mathbf{G}_{33} - (\mathbf{G}_{23}\mathbf{N} + \mathbf{G}_{33}\mathbf{M})\,\mathbf{A}^{-1}\,(\mathbf{N}\mathbf{G}_{23} + \mathbf{M}\mathbf{G}_{33})\right]\mathbf{Q}\mathbf{N}^{-1} \\
\mathbf{K} \equiv& \mathbf{E} - \mathbf{B}^\dagger\mathbf{A}^{-1}\mathbf{C} = \chi'\mathbf{N}^{-1}\mathbf{Q}\left\{i(\mathbf{G}_{23}\mathbf{N} + \mathbf{G}_{33}\mathbf{M})\mathbf{A}^{-1}\mathbf{C}\right. \\
& \left. - \left[\chi''\mathbf{G}_{23} + (q\chi')'\mathbf{G}_{33}\right] - i\chi'\mathbf{G}_{33} + \left\langle \mathbf{J}\cdot\nabla u\right\rangle\right\} \\
\mathbf{G} \equiv& \mathbf{H} - \mathbf{C}^\dagger\mathbf{A}^{-1}\mathbf{C}
\end{aligned}
$$

$\mathbf{F}$ vanishes wherever $m = nq$; singular surfaces.

# Generalized Newcomb Criterion
## Layers Inside Interfaces

### Energy Principle

$$\delta W = \frac{V'}{2}\int_{-\delta}^{\delta} ds \left(\Xi^{\dagger\prime}\mathbf{F}\Xi' + \Xi^{\dagger\prime}\mathbf{K}\Xi + \Xi^{\dagger}\mathbf{K}^{\dagger}\Xi' + \Xi^{\dagger}\mathbf{C}\Xi\right)$$

### Perfect Derivative

For any $\mathbf{P} = \mathbf{P}^{\dagger}$,

$$\frac{V'}{2}\int_{-\delta}^{\delta} ds \left(\Xi^{\dagger}\mathbf{P}\Xi\right)' = \frac{V'}{2}\int_{-\delta}^{\delta} ds \left(\Xi'^{\dagger}\mathbf{P}\Xi + \Xi^{\dagger}\mathbf{P}\Xi' + \Xi^{\dagger}\mathbf{P}'\Xi\right) = \frac{V'}{2}\,\Xi^{\dagger}\mathbf{P}\Xi\Big|_{-\delta}^{\delta}$$

Neglect boundary terms, subtract from $\delta W$ without changing its value.

$$\delta W = \frac{V'}{2}\int_{-\delta}^{\delta} ds \left[\Xi^{\dagger\prime}\mathbf{F}\Xi' + \Xi^{\dagger\prime}\left(\mathbf{K}-\mathbf{P}\right)\Xi + \Xi^{\dagger}\left(\mathbf{K}^{\dagger}-\mathbf{P}\right)\Xi' + \Xi^{\dagger}\left(\mathbf{C}-\mathbf{P}'\right)\Xi\right]$$

### Completing the Square

Choose $\mathbf{P} \equiv \mathbf{U}_{22}\mathbf{U}_{12}^{-1}, \quad D_C \equiv \det \mathbf{P}^{-1}$

$\mathbf{P}$ is self-adjoint by the symplectic symmetry of $\mathbf{U}$
It exists if and only if $D_C \neq 0$ for $s \in (0, a)$.

$$\mathbf{P}' = \mathbf{C} - \left(\mathbf{P}-\mathbf{K}\right)\mathbf{F}^{-1}\left(\mathbf{P}-\mathbf{K}^{\dagger}\right)$$

$$\delta W = \frac{V'}{2}\int_{-\delta}^{\delta} \left[\Xi'^{\dagger} + \Xi^{\dagger}\left(\mathbf{K}^{\dagger}-\mathbf{P}\right)\mathbf{F}^{-1}\right]\mathbf{F}\left[\Xi' + \mathbf{F}^{-1}\left(\mathbf{K}-\mathbf{P}\right)\Xi\right] ds$$

$\delta W \geq 0$. $D_C \neq 0$ is generalized newcomb criterion.
Boundary terms couple each region to neighboring regions.

# Summary of Stability Procedure

- Read equilibrium data.
- Compute Fourier expansion of metric tensor, fit to complex cubic splines.
- Compute coefficients of Euler-Lagrange equations in each interface and volume.
- Initialize in first volume near magnetic axis.
- Integrate Euler-Lagrange equations across each region, initializing from previous region.
- Monitor critical determinant $D_C(s)$, generalized Newcomb criterion; change of sign indicates fixed-boundary instability.
- If no change of sign, then construct plasma response matric $W_P$, vacuum response matrix $W_V$, total response matrix $W_T = W_P + W_V$. Negative eigenvalues determine free-boundary instability.

# Conclusions and Future Work

- The generalized Newcomb criterion has been derived for fixed-boundary ideal MHD perturbations of SPEC equilibria. The procedure consists of integrating the Euler-Lagrange Equation across each interval, matching the solutions across boundaries, and monitoring the criterion $D_C$ for change of sign.
- Following discussions with Stuart Hudson and Adelle Wright, ideal MHD stability is a necessary but not sufficient condition for SPEC equilibria. There is an energy principle for MRxMHD equilibria, Eq. (11) of S. R. Hudson, R. L. Dewar, G. Dennis, M. J. Hole, M. McGann, G. von Nessi, and S. Lazerson Phys. Plasmas 19, 112502 (2012). The Newcomb criterion can be extended to treat this energy principle.
- Experience with DCON for Tokamaks indicates that this method provides much faster and more accurate determination of stability than PEST-like codes, which compute the full ideal MHD spectrum.
- Implementation is under development.
- DCON for SPEC can be extended to treat free-boundary stability. This will require a Green's function computation of perturbed vacuum energy, equivalent to Morrell Chance's axisymmetric VACUUM code and Peter Merkel's nonaxisymmetric code.

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