AVAILABLE ENERGY IN TOKAMAKS AND STELLARATORS

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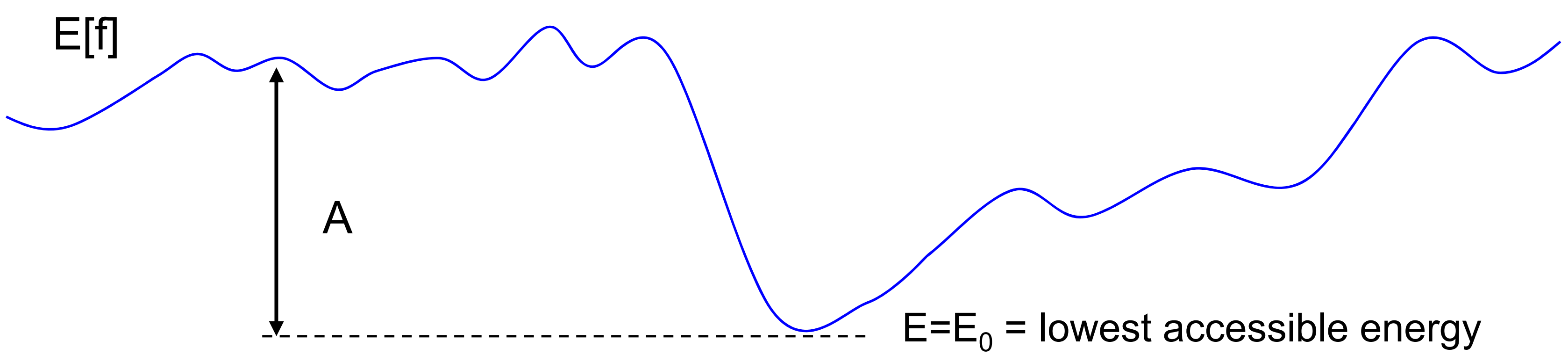
MOTIVATION

- Linear stability calculations
  - require that fluctuations be very small
  - may not say much about the eventual fate of the plasma



- Would it not also be interesting to calculate $A=E-E_0$?
  - the maximum amount of energy that can be extracted

Energy conservation

- Total energy is conserved

$$W = E + \Phi = \text{constant}$$

- For the Vlasov-Poisson system

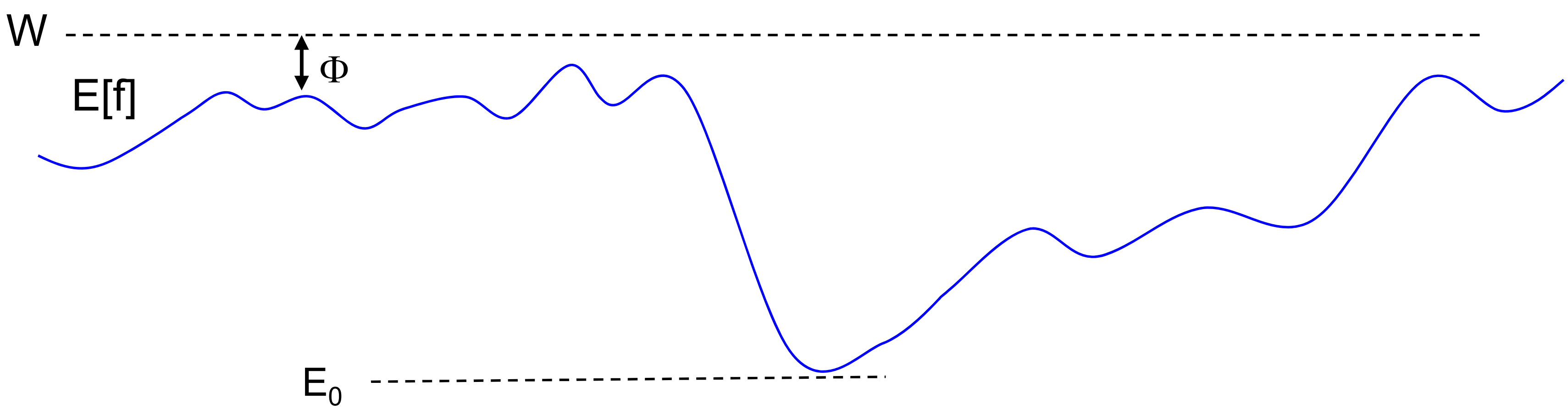
$$E = \left\langle \sum_a \int \frac{m_a v^2}{2} f_a \, d\mathbf{v} \right\rangle, \qquad \Phi = \left\langle \frac{|\nabla\phi|^2}{8\pi} \right\rangle$$

- For collisionless, electrostatic, long-wavelength gyrokinetics

$$E = \sum_a \left\langle 2\pi B \int f_a(\mathbf{R}, \mu, v_\parallel) \left(\mu B + \frac{m_a v_\parallel^2}{2} \right) d\mu dv_\parallel \right\rangle$$

$$\Phi = \left\langle \frac{|\nabla\phi|^2}{8\pi} + \sum_a \frac{m_a n_a}{2} \left| \frac{\mathbf{B} \times \nabla\phi}{B^2} \right|^2 \right\rangle$$

ENERGY CONSERVATION



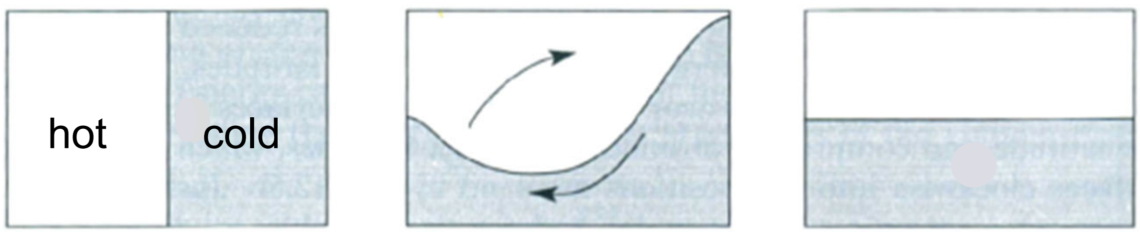
- Fluctuation energy Φ is bounded by

$$\Phi = W - E \leq W - E_0$$

- Ground state nonlinearly stable.

Available Potential Energy

- Consider the atmosphere. How much could its potential energy be lowered by adiabatic redistribution of air?
  - Available potential energy (Lorenz 1955)
  - Usually a few percent



Ground states of the Vlasov equation

- Consider any kinetic equation satisfying Liouville's theorem

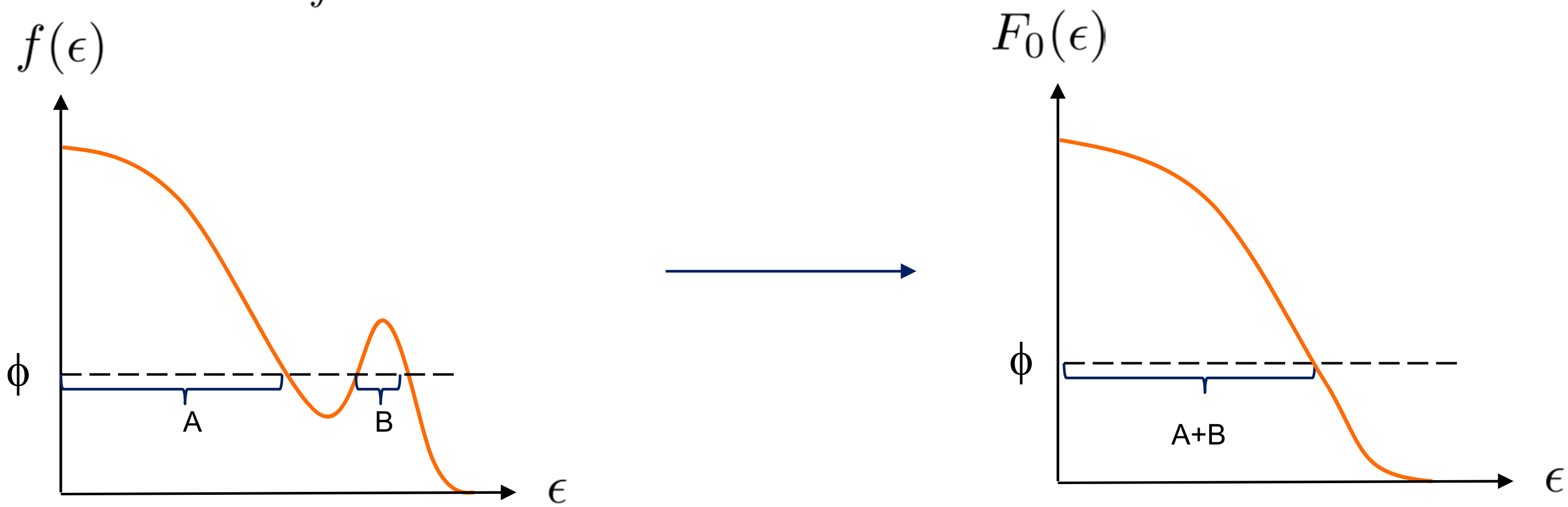
$$\frac{\partial(\sqrt{g}\, f)}{\partial t} + \nabla \cdot (\sqrt{g}\, \dot{\mathbf{x}} f) = 0,$$

$$\frac{\partial(\sqrt{g})}{\partial t} + \nabla \cdot (\sqrt{g}\, \dot{\mathbf{x}}) = 0$$

- What is the lowest accessible energy?

Gardner, Phys. Fluids 196

$$E = \int \epsilon f \sqrt{g}\, d\mathbf{x}$$



GROUND STATE

- Energy minimised by $F_0[\epsilon(\mathbf{x})]$ with F_0 given by the ground-state equation

$$H\left[F_0(y)\right] = \Omega(y)$$

a nonlinear integral equation, where

$$\Omega(\kappa) = \int \Theta\left[\kappa - \epsilon(\mathbf{x})\right] \sqrt{g}\, d\mathbf{x}$$

$$H(\phi) = \int \Theta[f(\mathbf{x}) - \phi] \sqrt{g}\, d\mathbf{x}$$

- Any monotonically decreasing function of energy alone is a ground state.

- Available energy

$$A = E - E_0 = \int \epsilon(f - F_0) \sqrt{g}\, d\mathbf{x}$$

Helander, JPP 201

EXAMPLE 1: BI-MAXWELLIAN

- For a simple Vlasov plasma

$$\Omega(\epsilon) = \int d\mathbf{r} \int_0^\infty \Theta\left(\epsilon - \frac{mv^2}{2}\right) 4\pi v^2 dv = \frac{4\pi V}{3}\left(\frac{2\epsilon}{m}\right)^{3/2}$$

- Bi-Maxwellian initial condition

$$f(\mathbf{v}) = Me^{-\frac{mv_\perp^2}{2T_\perp} - \frac{mv_\parallel^2}{2T_\parallel}} \qquad M = n\left(\frac{m}{2\pi\bar{T}}\right)^{3/2} \qquad \bar{T} = T_\perp^{2/3} T_\parallel^{1/3}$$

$$n(\mathbf{r}) = \int f d\mathbf{v}, \qquad T(\mathbf{r}) = \frac{2}{3n}\int \epsilon f d\mathbf{v} = \frac{2T_\perp + T_\parallel}{3}$$

BI-MAXWELLIAN, CONT'D

- Level curve $f(\mathbf{v}) = \phi$ is an ellipsoid in velocity space

$$f(\mathbf{v}) = M e^{-\frac{m v_\perp^2}{2T_\perp} - \frac{m v_\parallel^2}{2T_\parallel}} = \phi$$

enclosing the volume

$$H(\phi) = \frac{4\pi V}{3} \left(\frac{2\bar{T}}{m} \ln \frac{M}{\phi} \right)^{3/2}$$

- Ground-state equation becomes

$$\frac{4\pi V}{3} \left(\frac{2\bar{T}}{m} \ln \frac{M}{F_0} \right)^{3/2} = \frac{4\pi V}{3} \left(\frac{2\epsilon}{m} \right)^{3/2} \quad \Rightarrow \quad F_0(\epsilon) = M e^{-\epsilon/\bar{T}}$$

NOTA BENE

- The bi-Maxwellian is an equilibrium state but not a ground state.
- The ground state is Maxwellian although we have neglected collisions.
- The available energy is always positive:

$$A = \frac{3nV}{2}\left(T - \bar{T}\right) \geq 0$$

EXAMPLE 2: MAXWELLIAN

- Now choose

$$f(\mathbf{r},\mathbf{v}) = n(\mathbf{r})\left(\frac{m}{2\pi T(\mathbf{r})}\right)^{3/2} e^{-mv^2/2T(\mathbf{r})}$$

$$n(\mathbf{r}) = \langle n\rangle[1+\nu(\mathbf{r})]$$

$$T(\mathbf{r}) = \langle T\rangle[1+\tau(\mathbf{r})]$$

so that

$$E = \frac{3V}{2}\langle n\rangle\langle T\rangle\langle 1+\nu\tau\rangle$$

- If $\nu \sim \tau << 1$, then after some algebra we find

$$\frac{A}{E} = \left\langle \frac{\nu^2}{3} + \frac{\tau^2}{2} \right\rangle$$

CONSERVED QUANTITIES

- If there is a conserved quantity, μ, use phase-space coordinates

$$\mathbf{x} = (\mathbf{z}, \mu)$$

- Then for each μ, the ground-state equation is as before

$$H\left[F_0(\epsilon, \mu), \mu\right] = \Omega(\epsilon, \mu)$$

where

$$H(\phi, \mu) = \int \Theta[f(\mathbf{z}, \mu) - \phi]\sqrt{g}\, d\mathbf{z}$$

$$\Omega(w, \mu) = \int \Theta\left[w - \epsilon(\mathbf{z}, \mu)\right]\sqrt{g}\, d\mathbf{z}$$

- Any distribution function depending only on ε and μ is a ground state if

$$\left(\frac{\partial F_0}{\partial \epsilon}\right)_\mu \leq 0$$

MAGNETIC-MOMENT CONSERVATION

- The density and temperature of ground states are in general not constant if B = |**B**| varies

$$\begin{pmatrix} n \\ T \end{pmatrix} = \pi B \left(\frac{2}{m}\right)^{3/2} \int_0^\infty \begin{pmatrix} 1 \\ \frac{2\epsilon}{3n} \end{pmatrix} d\epsilon \int_0^{\epsilon/B} \frac{F_0(\epsilon,\mu)\, d\mu}{\sqrt{\epsilon - \mu B}}$$

- Spontaneous peaking of the density and temperature profile may be expected.

PARALLEL ADIABATIC INVARIANT

- If fluctuation frequencies << bounce frequency, the second adiabatic invariant is conserved

$$J = \int m v_{||} \, dl = \text{constant}$$

- Ground states:

$$\left(\frac{\partial F_0}{\partial \epsilon} \right)_{\mu, J} \leq 0$$

- Now suppose **B** traces out flux surfaces, ψ, and

$$F_0 = f_M(\psi, \epsilon) = n(\psi) \left[\frac{m}{2\pi T(\psi)} \right]^{3/2} e^{-\epsilon / T(\psi)}$$

GROUND-STATE CRITERION

- Now

$$\left(\frac{\partial f_M}{\partial \epsilon}\right)_{\mu,J} = \left(\frac{\partial f_M}{\partial \psi}\right)_{\epsilon}\left(\frac{\partial \psi}{\partial \epsilon}\right)_{\mu,J} - \frac{f_M}{T} = \frac{f_M}{T}\left(\frac{\omega_*^T}{\overline{\omega}_d} - 1\right)$$

where

$$\omega_* = \frac{T}{q}\frac{d\ln n}{d\psi} \qquad \omega_*^T = \omega_*\left[1 + \eta\left(\frac{\epsilon}{T} - \frac{3}{2}\right)\right] \qquad \eta = \frac{d\ln T/d\psi}{d\ln n/d\psi}$$

$$\omega_d = \mathbf{v}_d \cdot \nabla\alpha \qquad \overline{\omega}_d = \overline{\mathbf{v}_d \cdot \nabla\alpha}$$

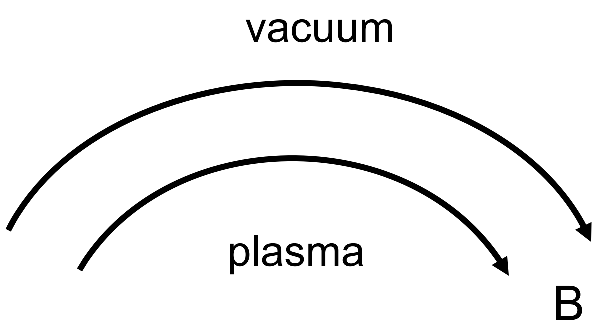
$$\mathbf{B} = \nabla\psi \times \nabla\alpha$$

- Ground state if, for all orbits,

$$\frac{\omega_*^T}{\overline{\omega}_d} < 1$$

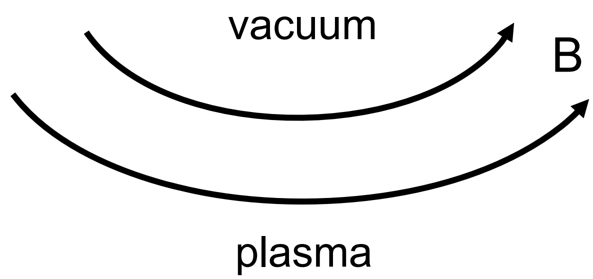
PRECESSION FREQUENCY

- The precession frequency is geometry-dependent.
  - related to average magnetic curvature



$$\omega_* \overline{\omega}_d > 0$$

bad curvature



$$\omega_* \overline{\omega}_d < 0 \qquad \frac{\partial J}{\partial \psi} < 0$$

good curvature

- Tokamak
  - bad curvature on the outboard side
  - good on the inboard side
  - cannot satisfy

$$\frac{\omega_*^T}{\overline{\omega}_d} < 1$$

TWO KINDS OF GROUND STATES

There are two ways of satisfying the ground-state criterion

$$\frac{\omega_*^T}{\overline{\omega}_d} < 1$$

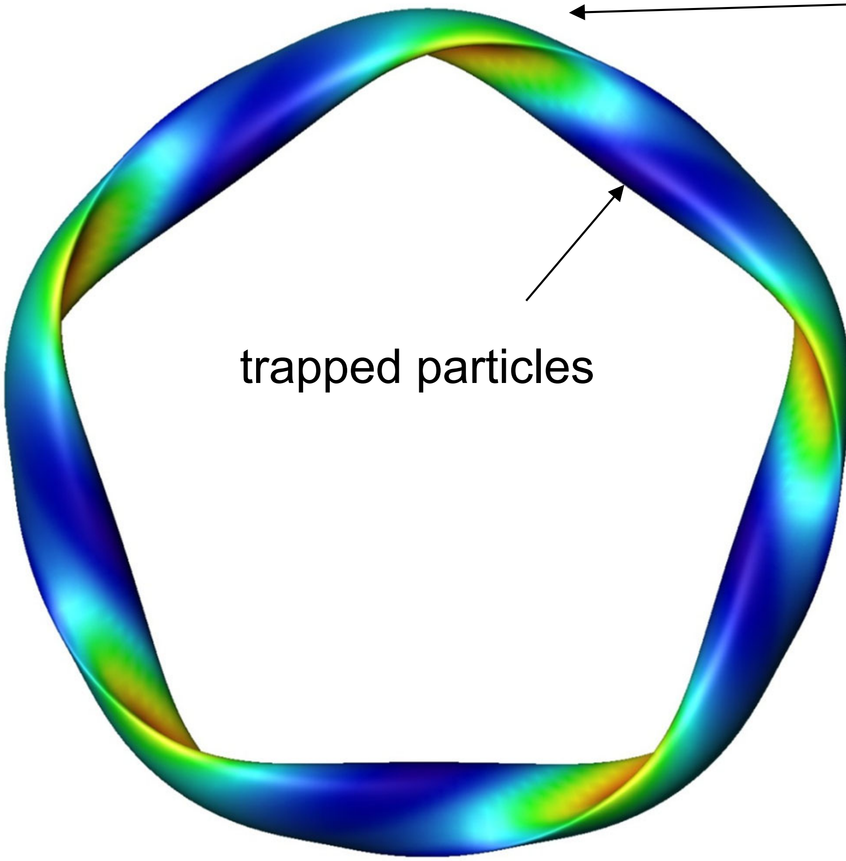
Either

$$\omega_* \overline{\omega}_d \leq 0, \qquad 0 \leq \eta < \frac{2}{3}$$

or

$$0 < \frac{\eta \omega_*}{\tilde{\omega}_d} < 1, \qquad \eta > \frac{2}{3}$$

First possbility: Wendelstein 7-X



Bad curvature

trapped particles

Average good curvature for (almost) all trapped particles

$$\omega_* \overline{\omega}_d \leq 0$$

PHYSICAL PICTURE

- In a device such that

$$\frac{\partial J}{\partial \psi} < 0$$

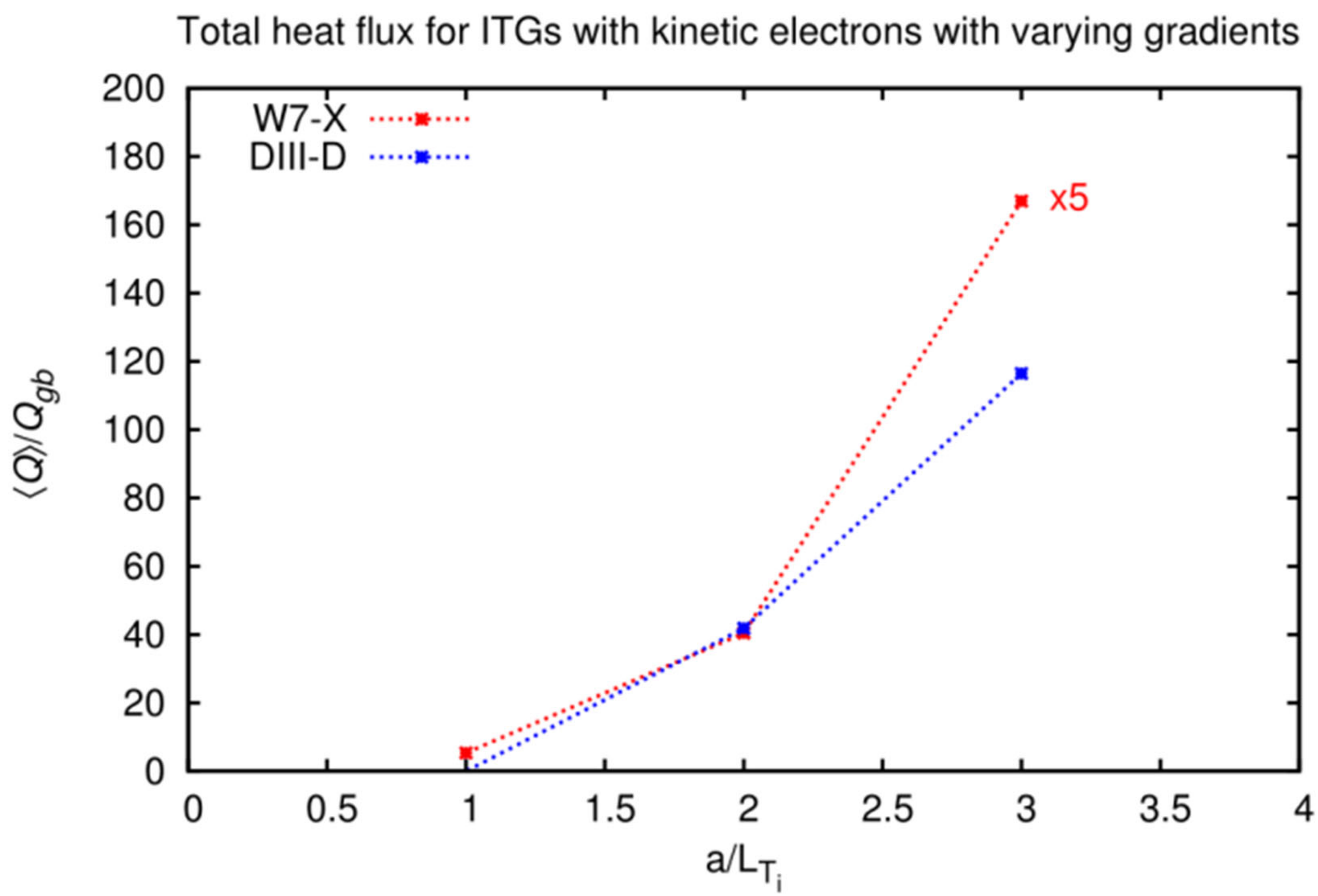
it <u>costs energy</u> to move a particle radially outward:

$$\Delta J = \frac{\partial J}{\partial \psi}\Delta\psi + \frac{\partial J}{\partial E}\Delta E \qquad \Rightarrow \qquad \Delta E = -\frac{\partial J/\partial\psi}{\partial J/\partial E} > 0$$

Linear density-gradient-driven instabilities are then absent.

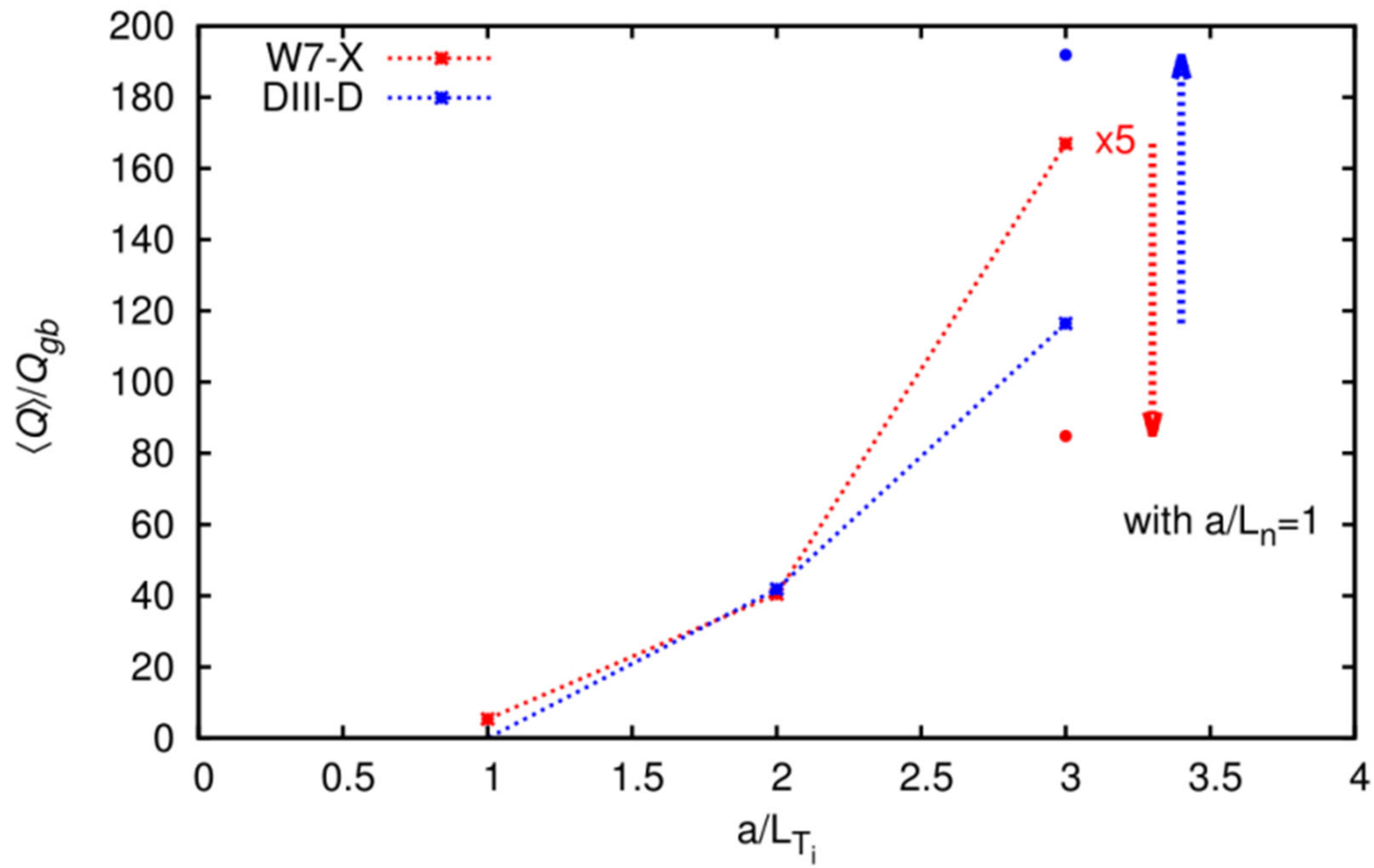
Rosenbluth (Phys Fluids 1968); Proll, Helander, Connor and Plunk (PR

NONLINEAR GK SIMULATIONS (PROLL)



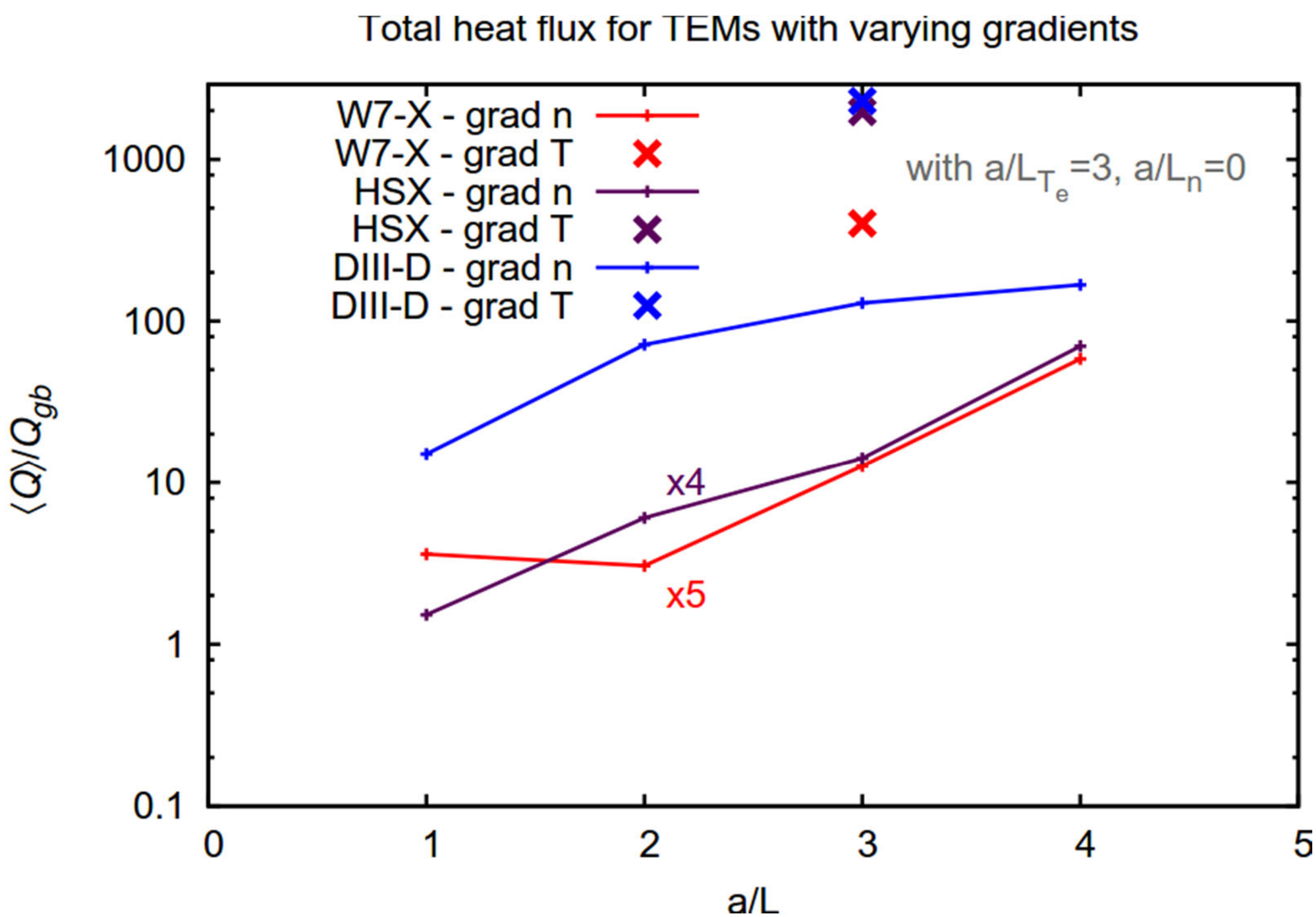
NONLINEAR GK SIMULATIONS (PROLL)

- Effect of density gradient on ITG transport:



NONLINEAR GK SIMULATIONS (PROLL)

- Transport from density-gradient turbulence remarkably low.



AVAILABLE ENERGY OF A FLUX SURFACE

- Ground-state:

$$\left(\frac{\partial F}{\partial \epsilon}\right)_{\mu,J} = -\frac{f_M}{T}\left|\frac{\omega_*^T}{\overline{\omega}_d} - 1\right|$$

- Available energy:

$$A = \frac{4\pi}{m^2}\int \epsilon(f_M - F)d\psi dJ d\psi d\alpha$$

- Can be evaluated by expanding

$$f_M(\psi,\mu,J) = f_M(\psi_0,\mu,J) + \frac{\partial f_M}{\partial \psi}(\psi - \psi_0) + \cdots$$

$$\epsilon(\psi,\mu,J) = \epsilon(\psi_0,\mu J) + \frac{\partial \epsilon}{\partial \psi}(\psi - \psi_0) + \cdots$$

$$F(\psi,\mu,J) = F(\psi_0,\mu,J) + \frac{\partial f_M}{\partial \psi}(\psi - \psi_0) + \cdots$$

Available Energy of a Flux Surface

- Result:

$$A = \frac{4e^2}{3T}(\Delta\psi)^3 V'(\psi) \int_{\mathrm{tr.}} f_M \overline{\omega}_d^2 h\left(\frac{\omega_*^T}{\overline{\omega}_d} - 1\right) d^3v$$

$$h(x) = x\Theta(x)$$

- Scales as

$$A_e \sim nT_e \cdot \frac{f_T(\Delta r)^2}{L_n R}$$

- If the ions are not subject to J-conservation

$$A_i \sim nT_i \left(\frac{\Delta r}{L_n}\right)^2 > A_e$$

  - Can expect ion-driven instablities (e.g. Connor, Plunk and Helander, JPP 2017).

TWO KINDS OF GROUND STATES

There are two ways of satisfying the ground-state criterion

$$\frac{\omega_*^T}{\overline{\omega}_d} < 1$$

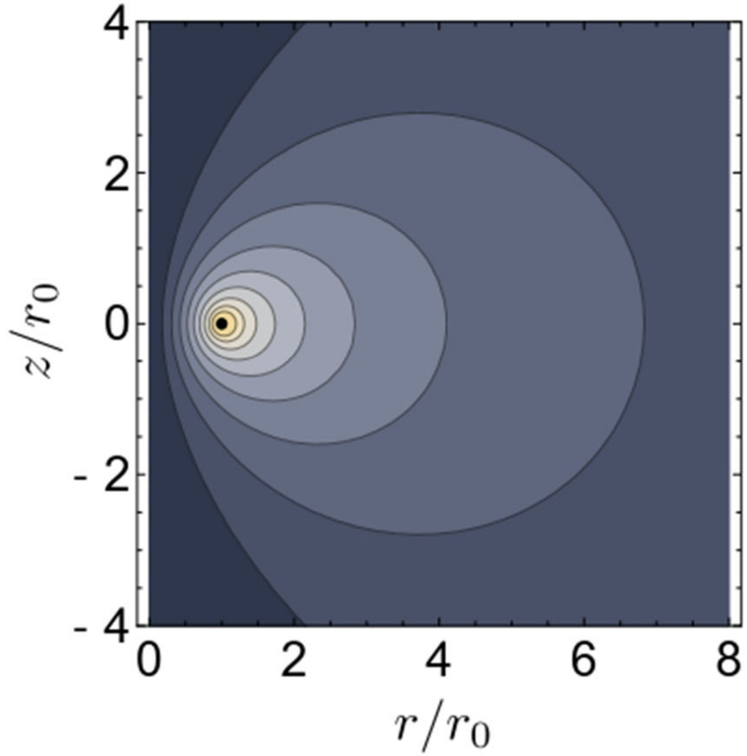
Either

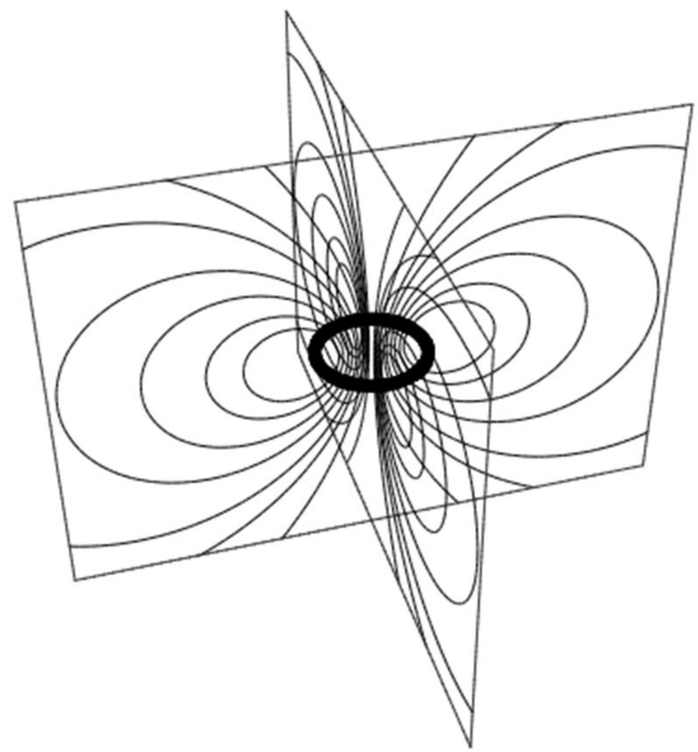
$$\omega_* \overline{\omega}_d \leq 0, \qquad 0 \leq \eta < \frac{2}{3}$$

or

$$0 < \frac{\eta \omega_*}{\tilde{\omega}_d} < 1, \qquad \eta > \frac{2}{3}$$

2ND CASE: CONFINEMENT BY A LEVITATED COIL





POINT DIPOLE LIMIT

- In a point magnetic dipole,

$$\mathbf{B} = \nabla\psi \times \nabla\alpha \qquad\qquad \psi = \frac{M}{r}\sin^2\theta, \quad \alpha = \text{toroidal angle}$$

- The precession frequency is, as a function of λ=μ/ε

$$\omega_d(\lambda) \geq \omega_d(\lambda = 0) = \frac{2\epsilon}{q\psi}$$

- The stability criterion becomes

$$\frac{2}{3}\frac{d\ln n}{d\ln\psi} < \frac{d\ln T}{d\ln\psi} < 2$$

FINITE-RADIUS COIL

Two regions separated by the flux surface on which n and T peak:

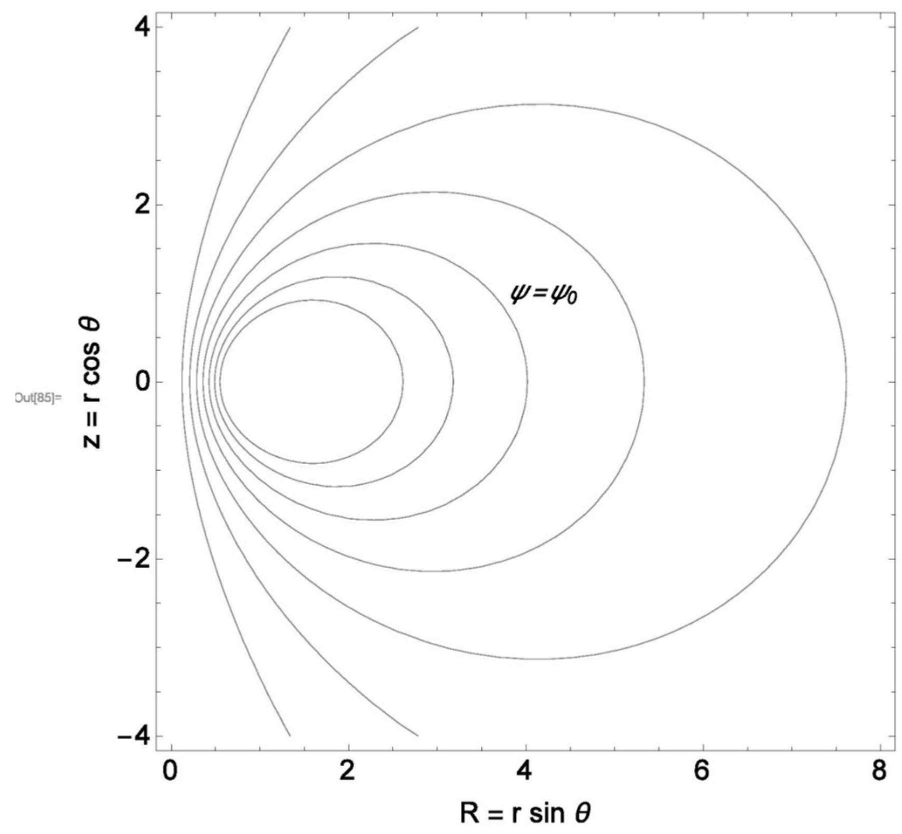
- Outer region:
  - bad curvature

$$\frac{\eta\omega_*}{\tilde{\omega}_d} < 1, \qquad \eta > \frac{2}{3}$$

- Inner region:
  - good curvature

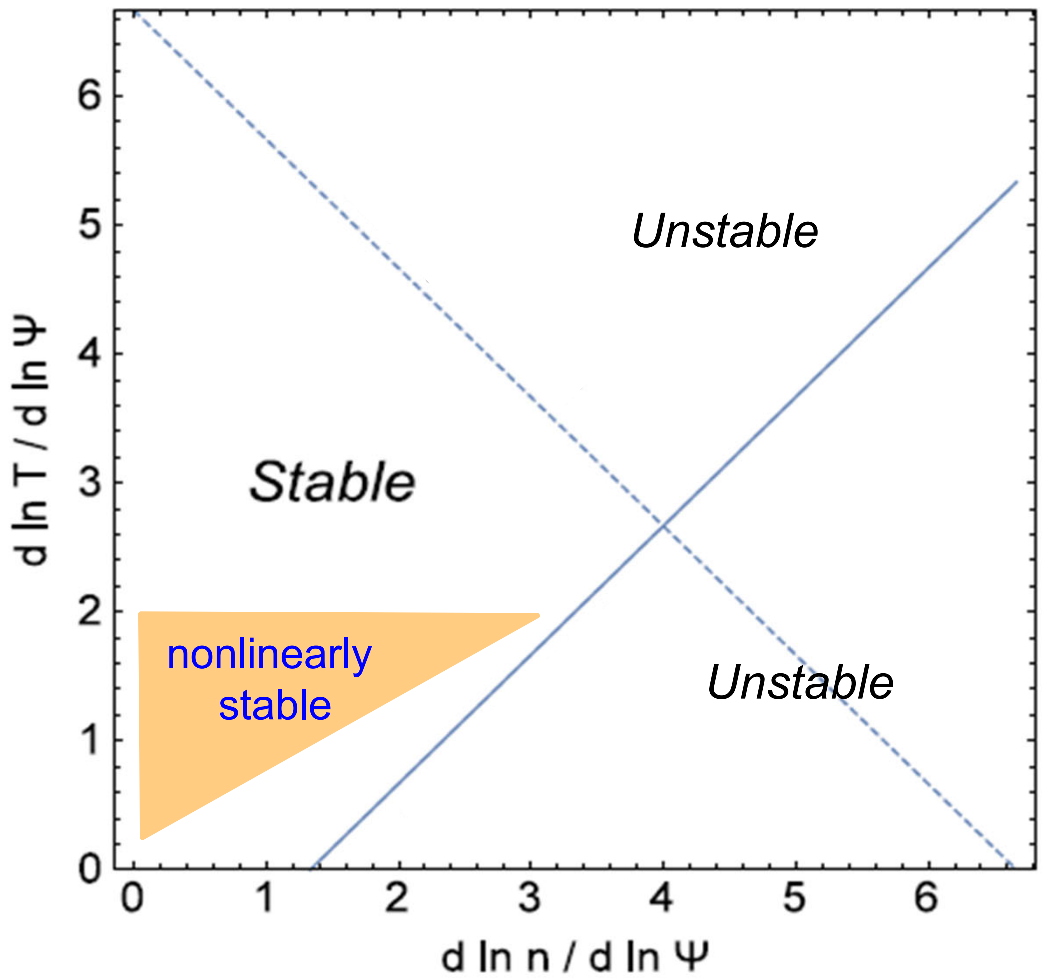
$$\omega_*\overline{\omega}_d \leq 0, \qquad 0 \leq \eta < \frac{2}{3}$$

- Nonlinearly stable plasma.



STABILITY DIAGRAM

- Linear and nonlinear stability regions



ANALOGY: ATMOSPHERE

- Stable stratification ($dT/dz > 0$) of the atmosphere occurs:
  - in the stratosphere,
  - close to the ground during temperature inversion

- No turbulence.

Summary

- Ground state = state of minimum energy under whatever constraints apply

- Available energy = actual energy – ground state energy

- Ground states generally inhomogeneous if adiabatic invariants are conserved.

- Two examples of confined plasmas in ground states:
  - Maximum-J devices (quasi-isodynamic stellarators)
  - Dipole plasmas

- Stable to low-frequency fluctuations of <u>any</u> amplitude.

PROOF OF NONLINEAR STABILITY

- The gyrokinetic equations conserve $W = E + \Phi$

$$E = \sum_a \frac{2\pi}{m_a} \int f_a(\mathbf{R}, \mu, v_\|) \kappa \sqrt{g}\, d\mathbf{x}, \qquad \kappa = \mu B + \frac{m_a v_\|^2}{2}$$

$$\Phi = \int \left(\frac{|\nabla \phi|^2}{8\pi} + \sum_a \frac{m_a n_a}{2} \left| \frac{\mathbf{B} \times \nabla \phi}{B^2} \right|^2 \right) d\mathbf{r}$$

where for slow fluctuations

$$E = \sum_a \frac{4\pi}{m_a^2} \int \frac{\kappa f_a dl}{|v_\|| \tau_b} d\mu dJ d\psi \alpha = \sum_a \frac{4\pi}{m_a^2} \int \epsilon f_a d\mu dJ d\psi \alpha$$

$$\epsilon = \frac{1}{\tau_b} \int \kappa \frac{dl}{|v_\||}$$

- Hence

$$\Phi = W - E \leq W - E_0$$

SEVERAL PARTICLE SPECIES

- In general, the minimum-energy state will be different for ions and electrons.
  - J may be conserved only for the electrons if, e.g.,

$$k_{\|} v_{Ti} \sim \omega \ll k_{\|} v_{Te}$$

- Quasineutrality adds new constraint.

- Simplest case: turbulence with adiabatic electrons
  - density profile cannot change
  - with no other constraints, the distribution function of each species is of the form

$$f_{a0}(\mathbf{x}) = F_{a0}[\epsilon(\mathbf{r}, \mathbf{v}) + \kappa(\mathbf{r})]$$

DENSITY CONSTRAINT

- These ground states have constant density.
  - The distribution function depends on energy alone.

- But sometimes, the density profile is fixed
  - e.g. in turbulence with adiabatic electrons

- Minimisation of energy subject to this constraint gives

$$f_0(\mathbf{r},\mathbf{v}) = F_0[\epsilon(\mathbf{r},\mathbf{v}) + \kappa(\mathbf{r})]$$

  - e.g., an isothermal Maxwellian:

$$f_0 = n(\mathbf{r})\left(\frac{m}{2\pi T}\right)^{3/2} e^{-\frac{mv^2}{2T}}, \qquad T = \text{constant}$$

  - General problem: given an initial state, calculate F_0 and κ.

GROUND-STATE EQ. WITH DENSITY CONSTRAINT

Given an initial state, e.g.,

$$f(\mathbf{r},\mathbf{v},0) = M(\mathbf{r})e^{-\frac{mv^2}{2T(\mathbf{r})}}$$

the ground state

$$f_0(\mathbf{r},\mathbf{v}) = F_0[\epsilon(\mathbf{r},\mathbf{v}) + \kappa(\mathbf{r})]$$

is determined by two nonlinear integral eqs:

$$\int F_0[\epsilon + \kappa(\mathbf{r})]d\mathbf{v} = n(\mathbf{r})$$

$$\int \left[w(F_0) - \kappa(\mathbf{r})\right]^{3/2} \Theta\left[w(F_0) - \kappa(\mathbf{r})\right] d\mathbf{r} = \int \left[T(\mathbf{r}) \ln \frac{M(\mathbf{r})}{F_0}\right]^{3/2} \Theta[M(\mathbf{r}) - \phi)]d\mathbf{r},$$

for $\kappa(\mathbf{r})$ and $w(F_0)$, the inverse of F_0.

AVAILABLE ENERGY WITH DENSITY CONSTRAINT

If the initial state has small density and temperature variations

$$n(\mathbf{r}) = \langle n \rangle [1 + \nu(\mathbf{r})]$$

$$T(\mathbf{r}) = \langle T \rangle [1 + \tau(\mathbf{r})]$$

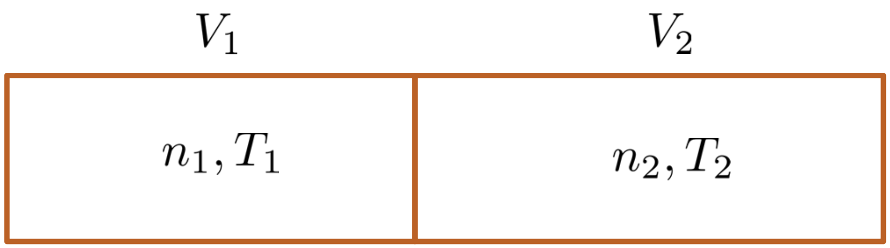
And the density is required to stay constantthe available energy becomes

$$\frac{A}{E} = \left\langle \frac{\tau^2}{2} \right\rangle$$

In general the available energy from temperature variations however depends on the density profile.

Mixing of hot and cold plasma

- Hot plasma expanding into a cold one:



- If density evolution is unconstrained and $T_2 \ll T_1$

$$A = E - E_0 = E\left[1 - \left(\frac{V_1}{V_1 + V_2}\right)^{2/3}\right]$$

- If density cannot change and $n_2 = 0$

$$A = 0$$