A gyrokinetic model for the tokamak periphery

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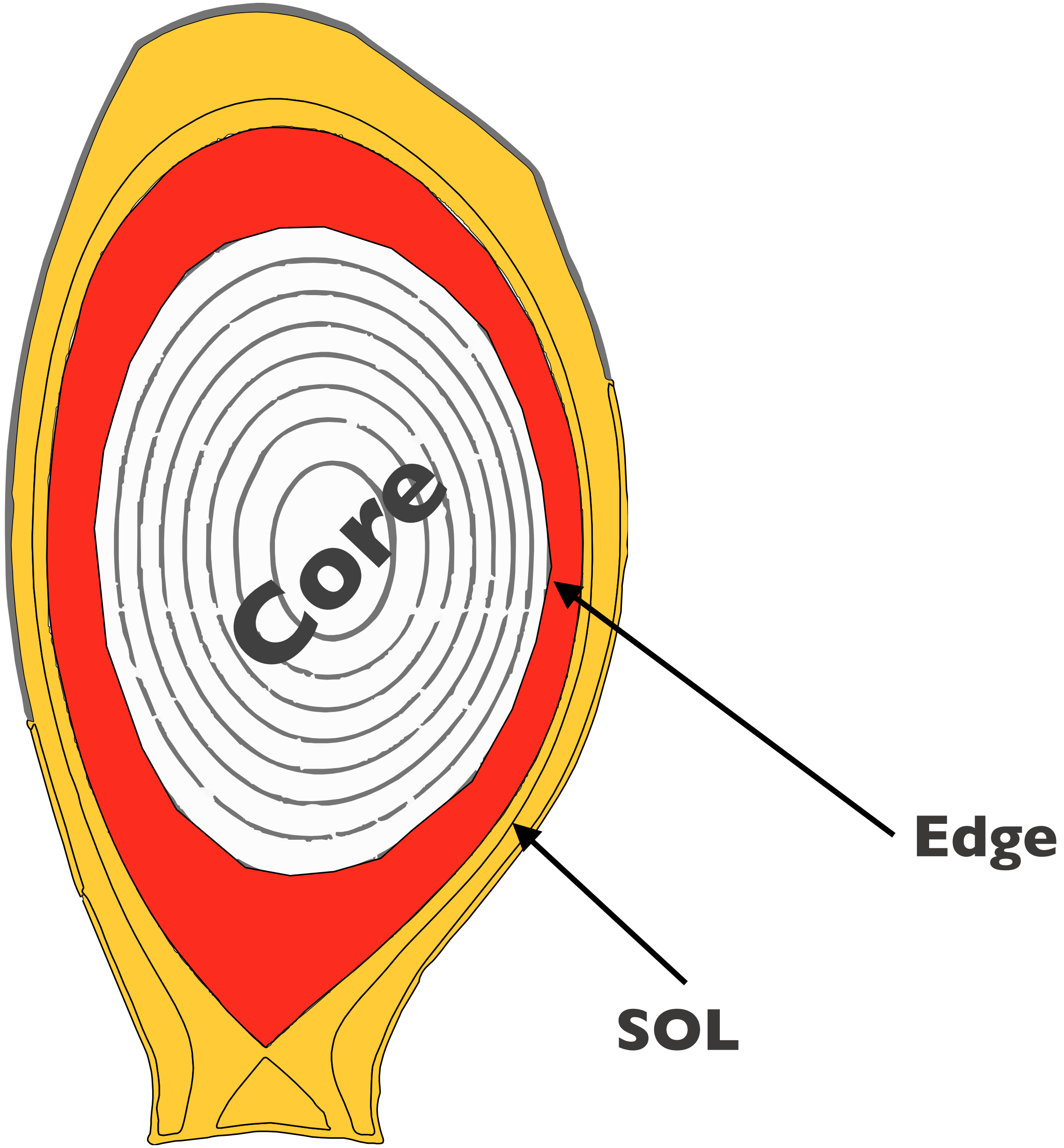
The Tokamak Periphery = Edge + SOL



- ❑ Core boundary conditions
- ❑ Heat exhaust
- ❑ Plasma fueling and ashes removal
- ❑ Impurity control

Properties of Periphery Turbulence

- Low-frequency $\omega \ll \Omega_i$
- Large Scale Fluctuations

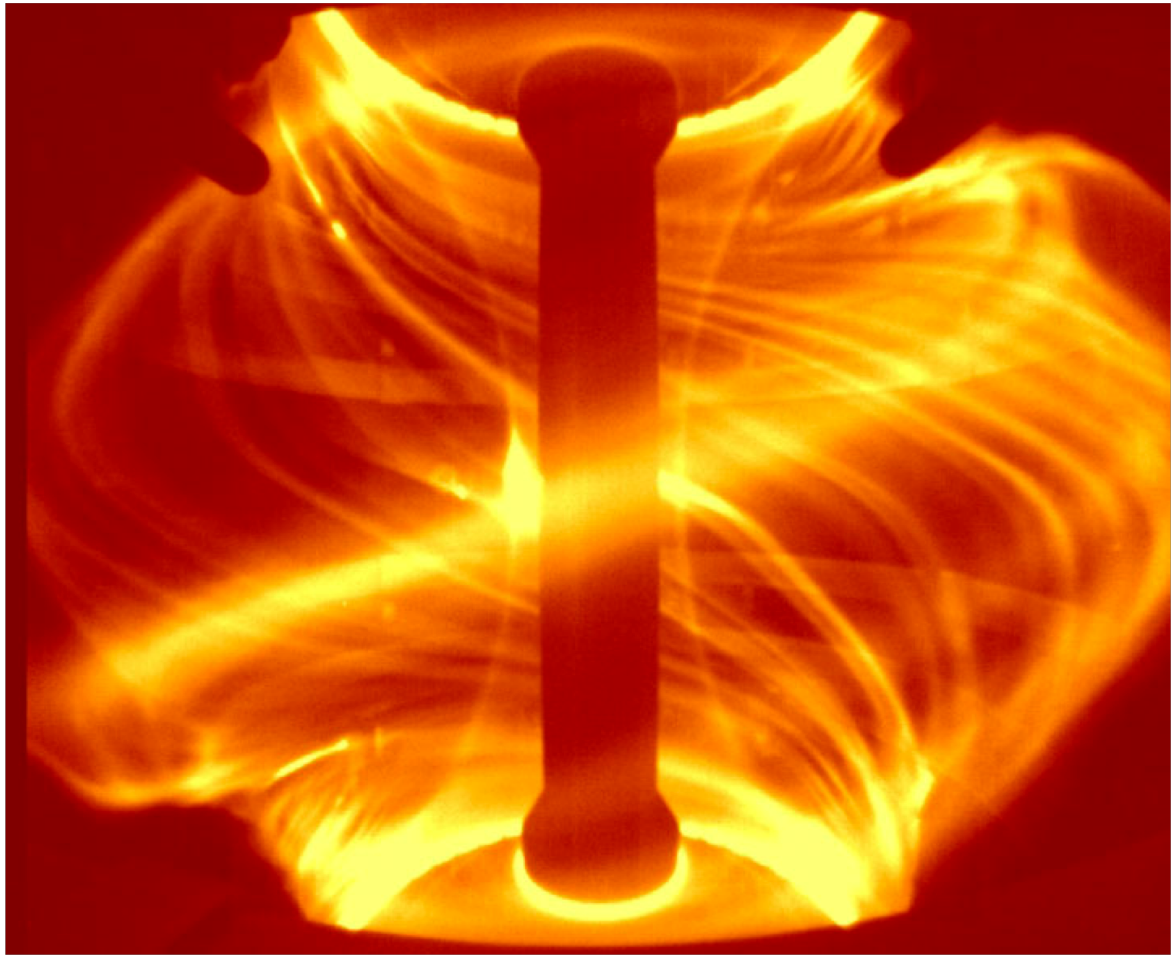
$$k_\perp \rho_i \ll 1 \qquad \frac{e\phi}{T_e} \sim 1$$

- Small Scale Fluctuations

$$k_\perp \rho_i \sim 1 \qquad \frac{e\phi}{T_e} \ll 1$$

- Wide range of temperatures and densities

$$T_e \sim 1 - 10^3 \text{ eV} \qquad n \sim 10^{18} - 10^{20} \text{ m}^{-3}$$



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- Low-frequency $\omega \ll \Omega_i$
- Large Scale Fluctuations

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→ **Gyrokinetic Theory**

$$\boxed{\epsilon = k_\perp \rho_i \frac{e\phi}{T_e} \ll 1}$$

- Small Scale Fluctuations

$$k_\perp \rho_i \sim 1 \qquad \frac{e\phi}{T_e} \ll 1$$

- Wide range of temperatures

  and densities

$$T_e \sim 1 - 10^3 \text{ eV} \qquad n \sim 10^{18} - 10^{20} \text{ m}^{-3}$$

→ Arbitrary Collisionality

First Heroic Steps On The Way To An Edge Model

	Qin et al. 2007[1]	Hahm et al. 2009[2]	Dimits et al. 2012[3]
Small Scale Fluctuations	EM $\mathrm{O}(\epsilon^2)$	EM $\mathrm{O}(\epsilon^2)$	EM $\mathrm{O}(\epsilon^2)$
Large Scale Fluctuations	EM $\mathrm{O}(\epsilon)$	ES $\mathrm{O}(\epsilon^2)$	ES $\mathrm{O}(\epsilon^2)$
Poisson's Eq.	$\int (\dots)\, d^3v$	Long-Wavelength Limit	$\int (\dots)\, d^3v$
Collisions	No	No	No

[1]Qin et al., Physics of Plasmas **14**, 056110 (2007)

[2]Hahm et al., Physics of Plasmas **16**, 022305 (2009)

[3]Dimits, Physics of Plasmas **19**, 022504 (2012)

Our Model

Retain

- Both large scale (full-F) and small scale fluctuations
- Second order
- Full Coulomb collisions
- Numerical efficiency

How we proceed

Single Particle Dynamics → Gyrokinetic Theory → Moment Hierarchy

Single Particle Dynamics - Ordering Assumptions

Magnetized Plasma

$$\frac{\omega}{\Omega_i} \sim \epsilon\, k_\perp \rho_i \qquad \frac{k_\parallel}{k_\perp} \sim \epsilon \qquad \frac{\rho_i}{L_p} \sim \epsilon$$

Fluctuation Scales and Amplitude

$$k_\perp \rho_i \frac{e\phi}{T_e} = \epsilon \ll 1$$

Electromagnetic

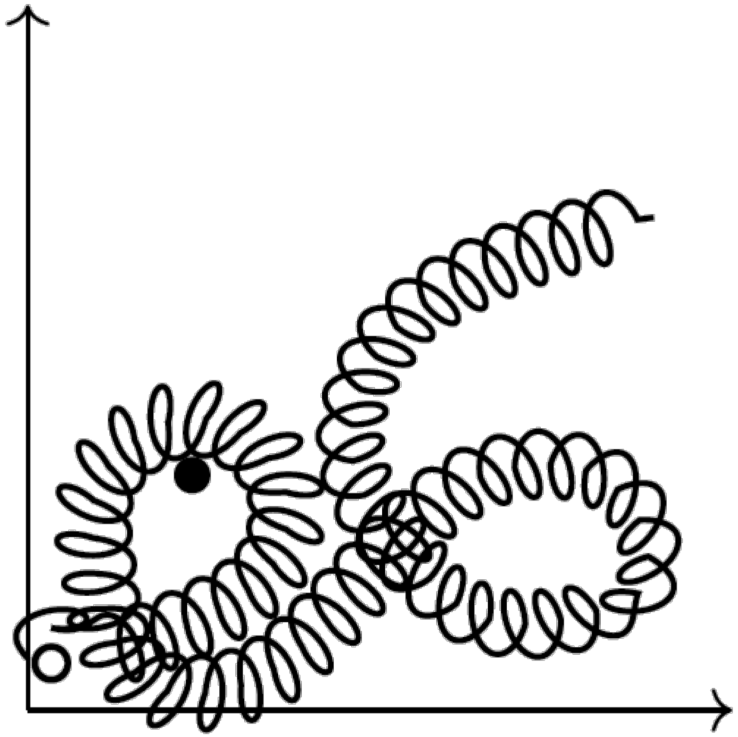
$$v_\parallel A_\parallel \sim \phi$$

Arbitrary Collisionalities

(still magnetized)

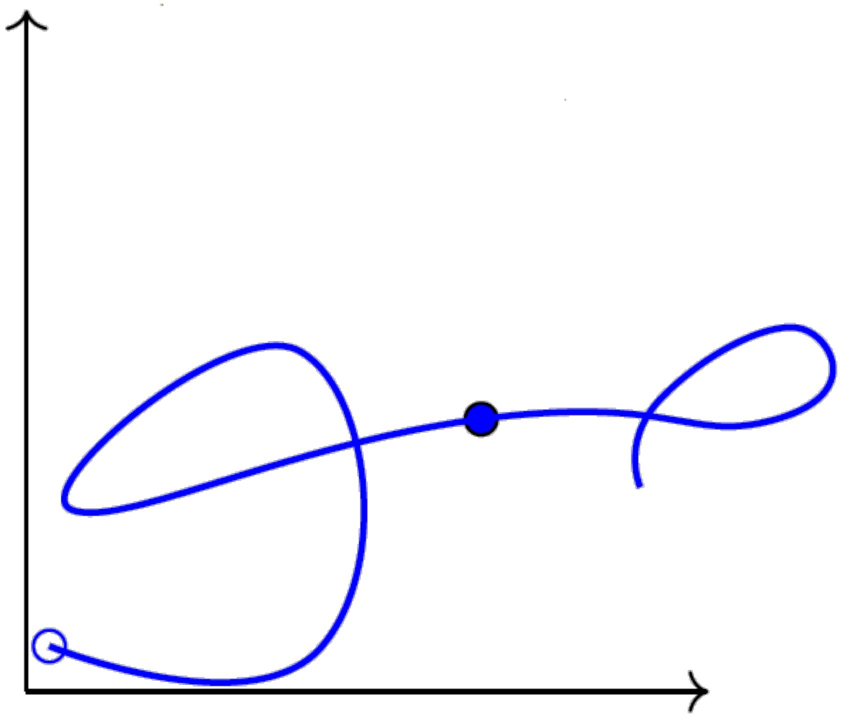
$$\frac{\nu_{ei}}{\Omega_i} \lesssim \epsilon$$

Single Particle Dynamics – Hamiltonian Perturbation Theory



2-Step Phase-Space

Reduction



Particle Lagrangian

$L(\mathbf{x}, \mathbf{v})$

Lagrangian

$\Gamma(\mathbf{R}, v_{\|}, \mu)$

(cyclic coordinate θ, conserved μ)

Single Particle Dynamics – First Step

Large Scales $\epsilon \sim k_\perp \rho_i \ll 1\,,\ \dfrac{e\phi}{T_e} \sim 1$

Start from

$$L = q\mathbf{A}\cdot\dot{\mathbf{x}} - q\phi - \frac{mv^2}{2}$$

Split between parallel and perpendicular velocity

Describe guiding center motion $O(\epsilon^2)$

$$L = q\mathbf{A}^*\cdot\dot{\mathbf{R}} - q\phi^* - \frac{mv_\parallel^2}{2} + \mu\frac{m\dot{\theta}}{q}$$

Single Particle Dynamics – Second Step

Small Scales $\epsilon \sim \frac{e\phi}{T_e} \ll 1\,,\; k_\perp \rho_i \sim 1$

Start from

$$L = q\mathbf{A}^* \cdot \dot{\mathbf{R}} - q\phi^* - \frac{mv_\parallel^2}{2} + \mu \frac{m\dot{\theta}}{q}$$

Introduce small scale fluctuations

Describe gyro center motion $O(\epsilon^2)$

$$L = q\mathbf{A}^* \cdot \dot{\mathbf{R}} - q\phi^* - \frac{mv_\parallel^2}{2} + \mu \frac{m\dot{\theta}}{q} - q\left\langle \phi - v_\parallel A_\parallel \right\rangle - q\frac{\partial \left\langle (\phi - v_\parallel A_\parallel)^2 \right\rangle}{\partial \mu}$$

Single Particle Dynamics – Equations of Motion

$$\dot{\mathbf{R}} = v_{\|}\mathbf{b} + \mathbf{v}_{\mathbf{E}\times\mathbf{B}} + \text{Other Drifts}$$

Including large scale $\dfrac{\nabla\phi(\mathbf{R}) \times \mathbf{B}}{B^2}$

and small scale $\dfrac{\nabla\left\langle\phi(\mathbf{x})\right\rangle \times \mathbf{B}}{B^2}$ fluctuations

Curvature Drift

Polarization Drift

Non-Linear Drifts

Single Particle Dynamics – Equations of Motion

$$\dot{\mathbf{R}} = v_{||}\mathbf{b} + \mathbf{v}_{\mathbf{E}\times\mathbf{B}} + \text{Other Drifts}$$

$$\dot{v}_{||} = qE_{||} + \mu\nabla_{||}B + \text{Non-Linear Forces}$$

Including large scale $\nabla_{||}\phi(\mathbf{R})$

and small scale $\nabla_{||}\langle\phi(\mathbf{x})\rangle$ fluctuations

Single Particle Dynamics – Equations of Motion

$$\dot{\mathbf{R}} = v_{||}\mathbf{b} + \mathbf{v}_{\mathbf{E}\times\mathbf{B}} + \text{Other Drifts}$$

$$\dot{v}_{||} = qE_{||} + \mu\nabla_{||}B + \text{Non-Linear Forces}$$

$$\dot{\mu} = 0$$

Conserved Adiabatic Invariant

From Single Particle to Particle Distribution

Gyrokinetic Equation

$$\frac{\partial F}{\partial t} + \dot{\mathbf{R}} \cdot \nabla F + \dot{v}_{\|} \frac{\partial F}{\partial v_{\|}} = \langle C(F) \rangle$$

Challenges

- 5-D + time
- Full Coulomb Collisions
- Coupling to Maxwell's equations (integro-differential system)

These challenges can be successfully approached by using a moment hierarchy

Our Goal – Turn Gyrokinetic Eq. Into a Hierarchy of Fluid-Like Eqs.

Expand GK Equation into a Set of 3D

Moment Hierarchy Equations

$$\frac{dn}{dt} = \ldots$$

$$\frac{dv}{dt} = \ldots$$

$$\frac{dT}{dt} = \ldots$$

$$\ldots$$

Retain necessary kinetic effects and no more

Advantages of a Moment Hierarchy Model

Set of fluid-like equations with reasonable computational cost

Tune the number of moments according to the desired level of accuracy (function of collisionality)

Reduce to the fluid model at high collisionality

3 Steps To Build a Moment Hierarchy Model

1. Choose an orthogonal polynomial basis for F

$$F = F_M \sum_{p,j} N^{pj}(\mathbf{R}) H_p(v_{||}) L_j(\mu)$$

2. Project kinetic equation onto basis

$$\int (\text{GK Eq.})\, H_p L_j dv_{||} d\mu$$

3. Obtain evolution equation for the basis coefficients

$$\frac{\partial N^{pj}}{\partial t} = \ldots$$

From Gyrokinetic Equation to Moment Hierarchy

$$\int (\mathrm{GK\ Eq.})\, H_p L_j \, dv_\parallel \, d\mu$$

$$\underbrace{\boxed{\frac{\partial N^{pj}}{\partial t}}}_{\text{Time Evolution}} + \overbrace{\boxed{\nabla \cdot \dot{\mathbf{R}}^{pj}}}^{\text{Spatial evolution of Moments + Fields}} - \underbrace{\boxed{\frac{\sqrt{2p}}{v_{th}} \dot{v}_\parallel^{p-1j}}}_{\text{Forces included at p>0}} + \overbrace{\boxed{\mathcal{F}^{pj}}}^{\text{Fluid Operator (density, velocity, temperature)}} = \underbrace{\boxed{C^{pj}}}_{\text{Collisions}}$$

Example – 1D Linear Gyrokinetic Moment Hierarchy

Phase Mixing
(coupling with other moments)

Collisions

$$\boxed{\frac{\partial N^{pj}}{\partial t}} + \boxed{\frac{1}{2}\frac{\partial N^{p+1j}}{\partial z} + p\frac{\partial N^{p-1j}}{\partial z}} = \boxed{2\delta_{p,1}K_j(k_\perp \rho)E_\parallel} + \boxed{C^{pj}}$$

Time Evolution

Electric Field Drive

Example – 1D Linear Gyrokinetic Moment Hierarchy

Phase Mixing
(coupling with other moments)

Collisions

$$\boxed{\frac{\partial N^{pj}}{\partial t}} + \boxed{\frac{1}{2}\frac{\partial N^{p+1j}}{\partial z} + p\frac{\partial N^{p-1j}}{\partial z}} = \boxed{2\delta_{p,1}K_j(k_\perp \rho)E_\parallel} + \boxed{C^{pj}}$$

Time Evolution

Kernel $K_j(x) = \frac{1}{j!}\left(\frac{x}{2}\right)^{2j} e^{-x^2/4}$

Analytical Closed formula for the Gyroaverage operation

Full finite Larmor radius effects

Projection of the Full Coulomb Collision Operator

$$C^{pj} = \int \langle C(F) \rangle \, H_p L_j dv_{||} d\mu$$

$$\text{with} \;\; C(F) = \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{H}(F)F] + \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : [\mathbf{G}(F)F]$$

- Bilinear
- Tensorial Nature
- Gyroaveraging Operation
- No parallel/perpendicular velocity symmetries

$$H(F) \sim \int \frac{F(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} d\mathbf{v}'$$

Not immediate…

Collision Operator - Spherical Harmonic Decomposition

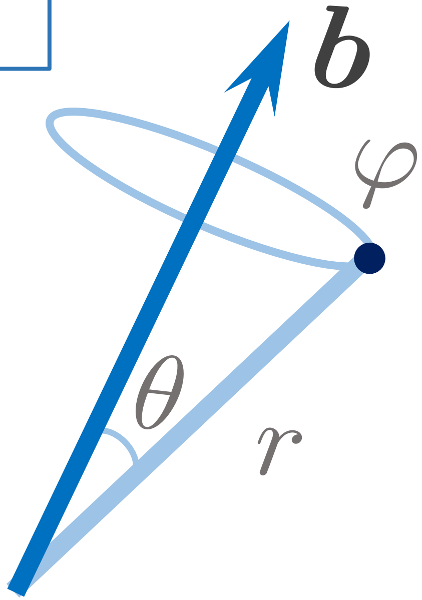
Expand the Rosenbluth potentials

$$H(F) \sim \int \frac{F(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|} d\mathbf{v}'$$

Electrostatic potential of a charge distribution F in velocity space

in a Taylor series

$$\frac{1}{|\mathbf{v}-\mathbf{v}'|} = \begin{cases} \sum_l \frac{(-\mathbf{v}')^l}{l!} \cdot \partial^l_{\mathbf{v}} \left(\frac{1}{v}\right), & v' \leq v \\ \sum_l \frac{(-\mathbf{v})^l}{l!} \cdot \partial^l_{\mathbf{v}'} \left(\frac{1}{v'}\right) & v < v' \end{cases}$$



with spherical harmonics as coefficients

$$(-1)^l v^{l+1} \left(\frac{\partial}{\partial \mathbf{v}}\right)^l \frac{1}{v} \sim \sum_{m=-l}^{l} \boxed{Y_{lm}}(\varphi, \boxed{\theta})\hat{\mathbf{e}}^{lm}$$

Basis Tensors $\hat{\mathbf{e}}^{lm}$

Spherical Harmonics

Gyroangle

Collision Operator – Expansion in Spherical Harmonics

$$C[F] \sim C[Y_{lm}] \sim Y_{lm}$$

Gyroaveraging Procedure

Large Scales $k_\perp \rho_i \ll 1$

$$\langle Y_{lm}(\varphi, \theta) \rangle \longrightarrow H_p(v_\parallel) L_j(\mu)$$

Small Scales $\dfrac{e\phi}{T_e} \ll 1$

$$\left\langle Y_{lm}(\varphi, \theta) e^{in\theta} \right\rangle \longrightarrow f(n) K_j(k_\perp \rho) H_p(v_\parallel) L_j(\mu)$$

Moments of the Collisional Operator

$$C^{pj} = \int \langle C(F) \rangle H_p L_j dv_{||} d\mu$$

After integration

$$C^{pj} = \text{ Kernel} \times \mathrm{f(n)} \times \text{moments } N^{pj} \text{ of } \mathrm{F}$$

Physics That We Are Able To Capture – High Collisionality

$$F = F_M(1 + \delta F) \quad \text{with} \quad \delta F = \sum_{p,j} N^{pj} H_p L_j$$

Semi-collisional closure

$$\delta F \sim \frac{\lambda_{\text{mfp}}}{L_{\|}}$$

Drift-Reduced Braginskii equations retrieved

$$N^{30} \sim q_{\|} = -\chi_{\|} \nabla_{\|} T$$

Jorge *et al.*, JPP **83**, 6 (2018)

Towards a Numerical Implementation

Numerical and theoretical investigation of linear modes

GK non-linear model reduced to the drift-kinetic linear limit

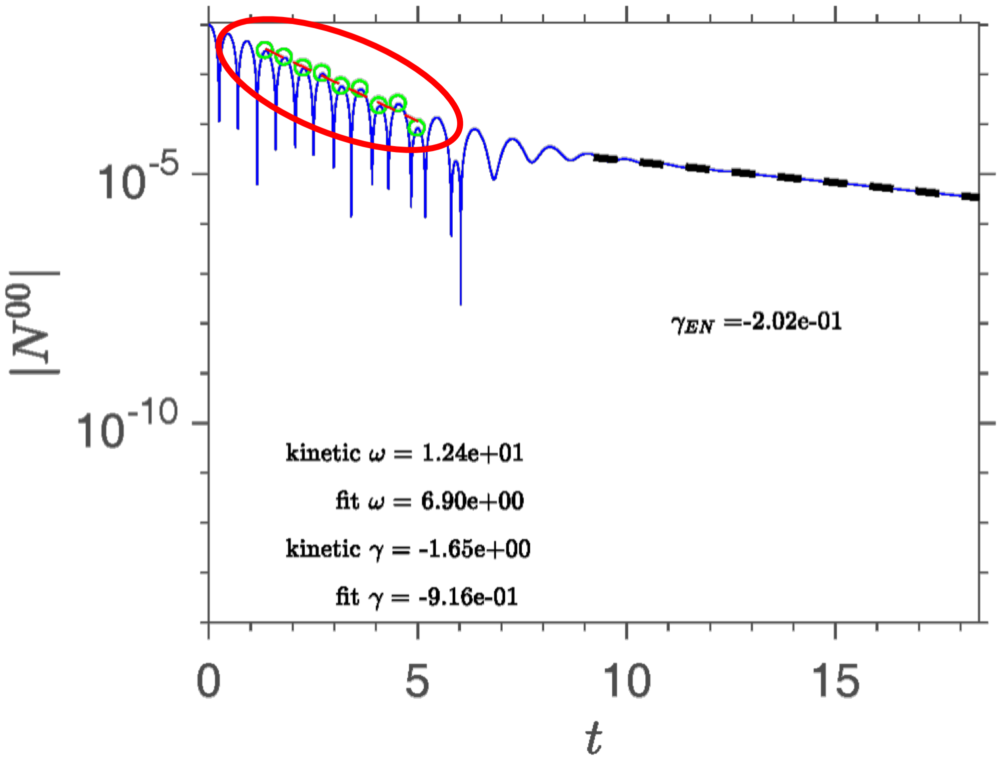
$$\frac{\partial N^{pj}}{\partial t} = \sum_{s,t} D^{pj}_{st} N^{st}$$

- Compute collisional/hierarchy coefficients D^{pj}_{st}
- Solve time evolution problem
- Perform eigenvalue analysis
- Compare with collisionless result

Electron Plasma Waves –Damping at Arbitrary Collisionality

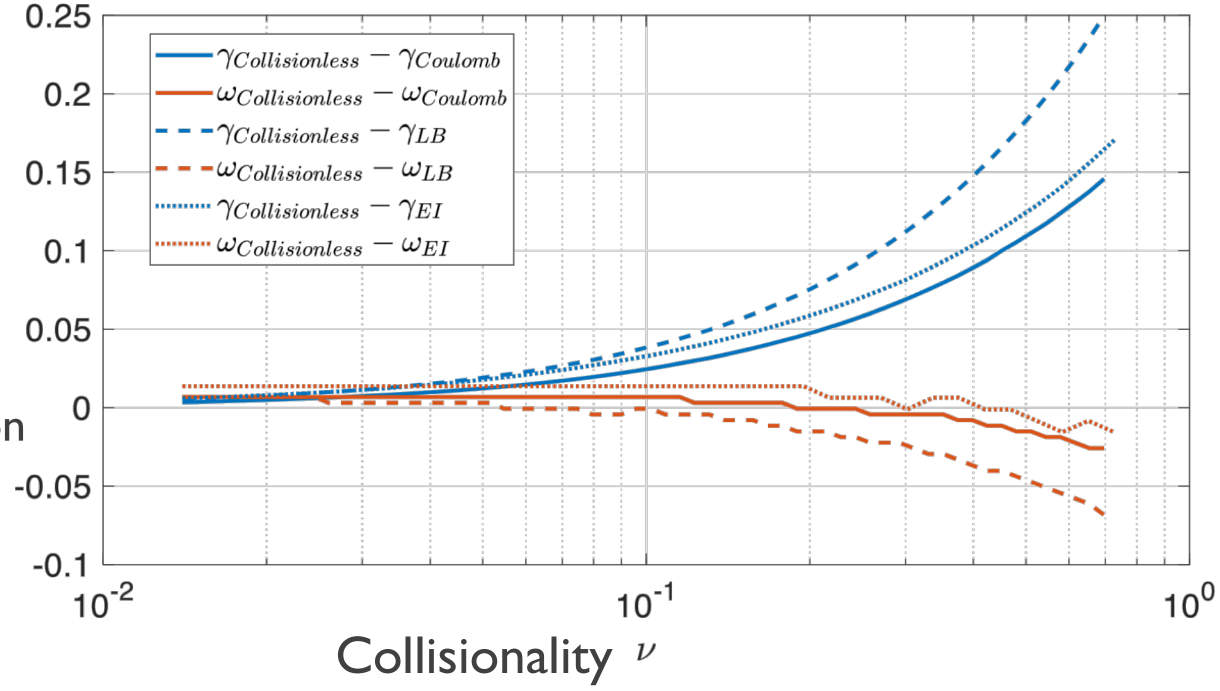
- One spatial and two velocity dimensions
- Electron perturbations only
- Time-evolution of $N^{00} \sim \phi$

- Collisional and Landau Damping computed for the first time with the full Coulomb collision operator

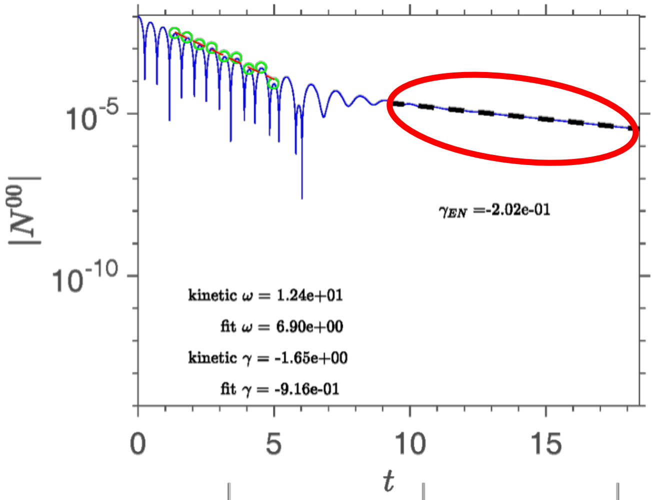


Parameter scan

Three collision operators



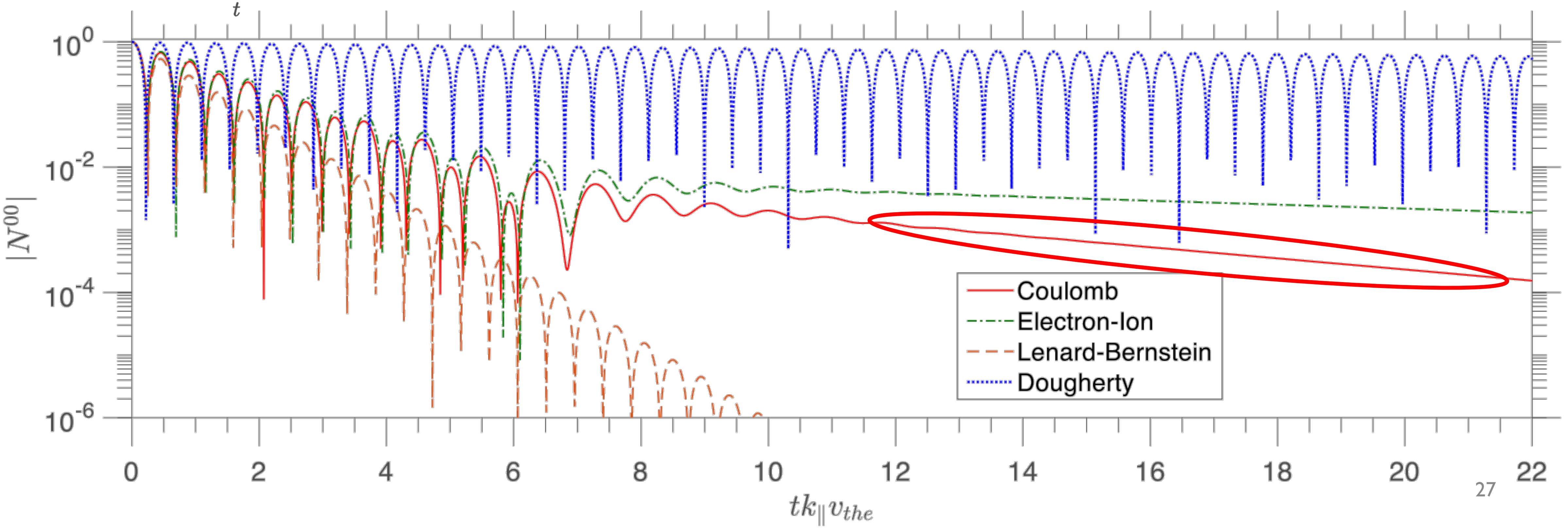
Electron Plasma Waves – Entropy Mode



Zero-frequency entropy mode

- Electron-ion collisions known to yield long-time zero-frequency behaviour
- Full Coulomb collisions reduce the entropy mode damping rate

Jorge et al., to be submitted to JPP



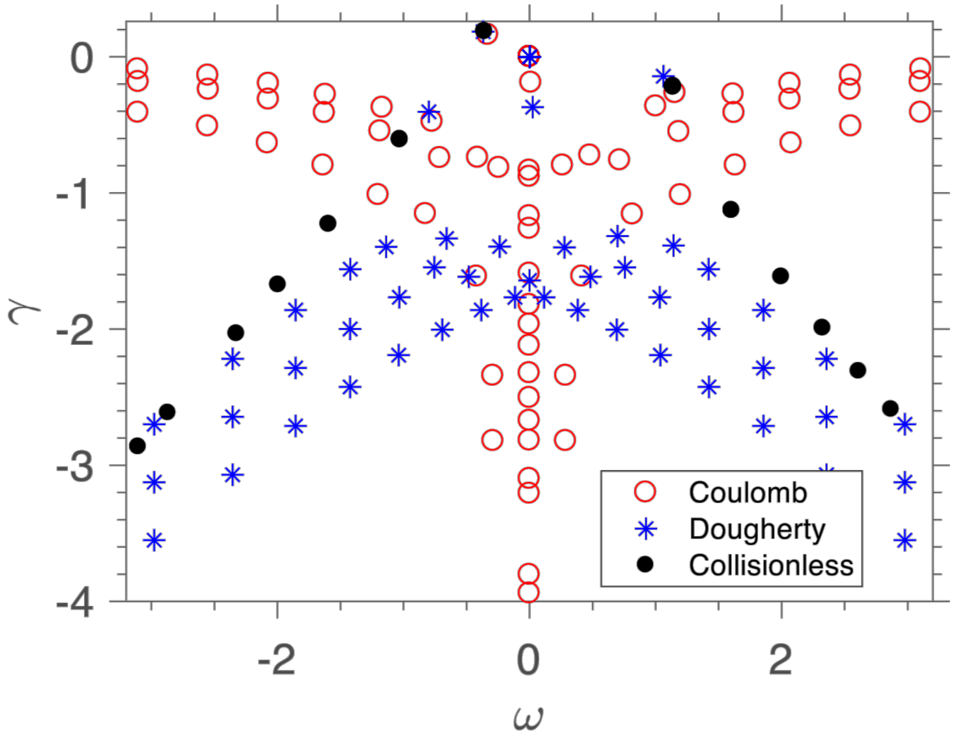
First Study on Coulomb Collision Effects on the Drift-Wave Instability

- Two spatial and two velocity dimensions in slab geometry
- Electron and ion perturbations
- Finite background density gradient

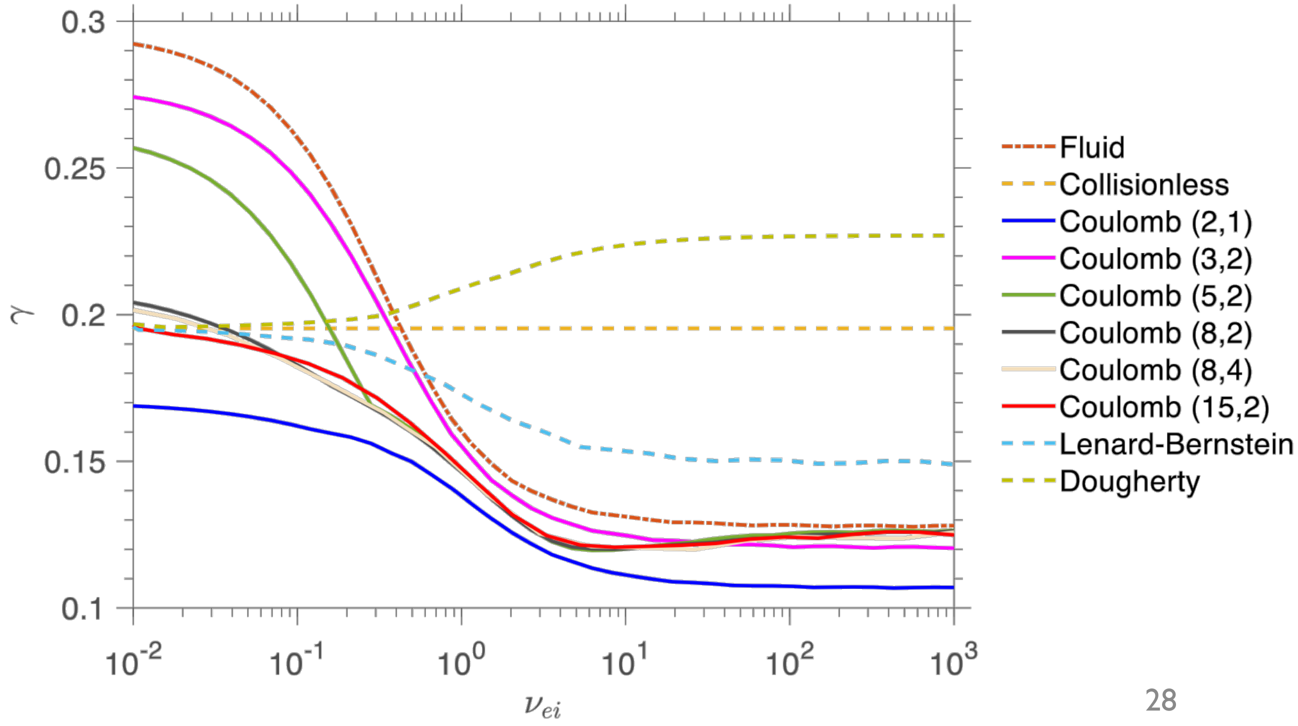
- Important deviations between full Coulomb collision operator and presently used collision operators

Jorge *et al.*, PRL **121**, 165001 (2018)

Eigenmode Spectra



Instability Growth Rate



Summary

	This Work	Qin et al. 2007[1]	Hahm et al. 2009[2]	Dimits et al. 2012[3]	Madsen et al. 2013[4]	Mandell et al. 2018[5]
Large Scales	EM $\mathrm{O}(\epsilon^2)$	EM $\mathrm{O}(\epsilon)$	ES $\mathrm{O}(\epsilon^2)$	ES $\mathrm{O}(\epsilon^2)$	ES $\mathrm{O}(\epsilon)$	ES $\mathrm{O}(\epsilon)$
Small Scales	EM $\mathrm{O}(\epsilon^2)$	EM $\mathrm{O}(\epsilon^2)$	EM $\mathrm{O}(\epsilon^2)$	EM $\mathrm{O}(\epsilon^2)$	EM $\mathrm{O}(\epsilon)$	ES $\mathrm{O}(\epsilon)$
Collisions	Yes	No	No	No	No	Simplified
Poisson's Eq.	Moments	$\int (\dots)\, d^3v$	Long-Wavelength Limit	$\int (\dots)\, d^3v$	Long-Wavelength Limit, Padé	Moments
$\mathrm{B}_{\parallel}^*$	Exact	Exact	Exact	Exact	$\mathrm{O}(\epsilon)$	B

[1] Qin *et al.*, Physics of Plasmas **14**, 056110 (2007)
[2] Hahm *et al.*, Physics of Plasmas **16**, 022305 (2009)
[3] Dimits, Physics of Plasmas **19**, 022504 (2012)
[4] Madsen, Physics of Plasmas **20**, 072301 (2013)
[5] Mandell *et al.*, J. Plasma Phys **84**, 905840108 (2018)