Phase-space theory of the Dimits shift and cross-scale interactions in drift-wave turbulence

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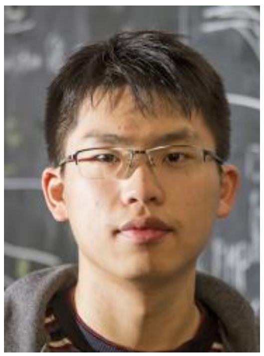
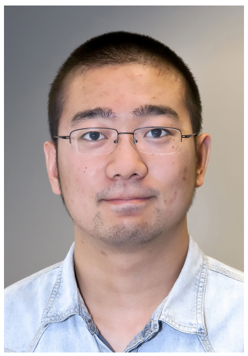


in collaboration with:

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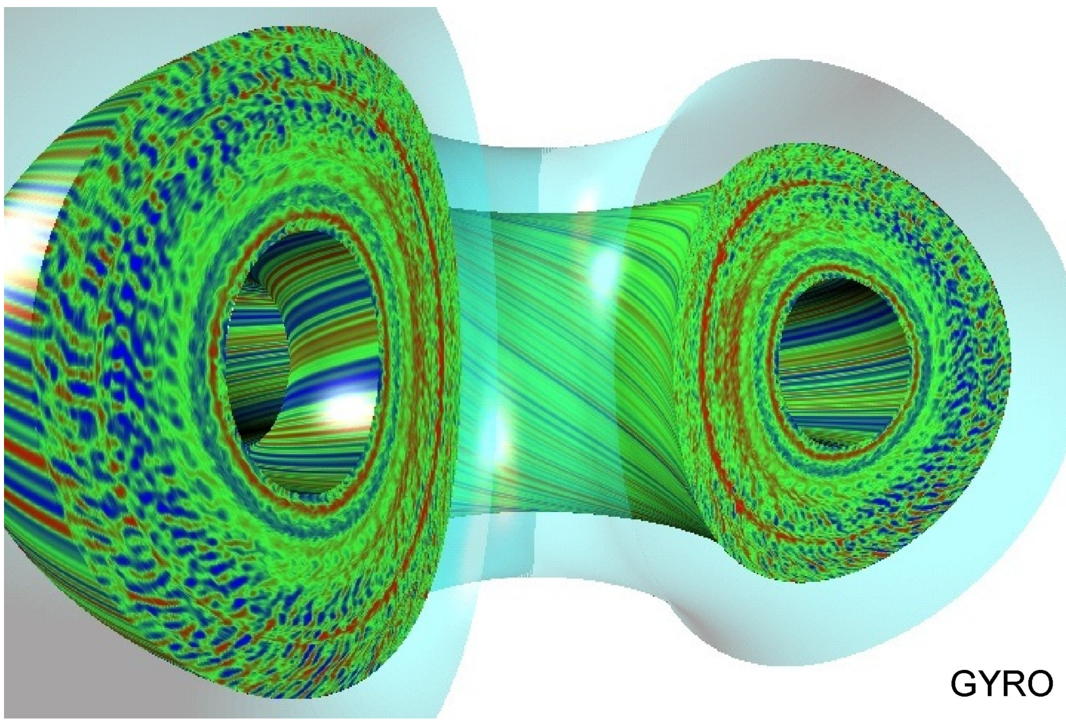
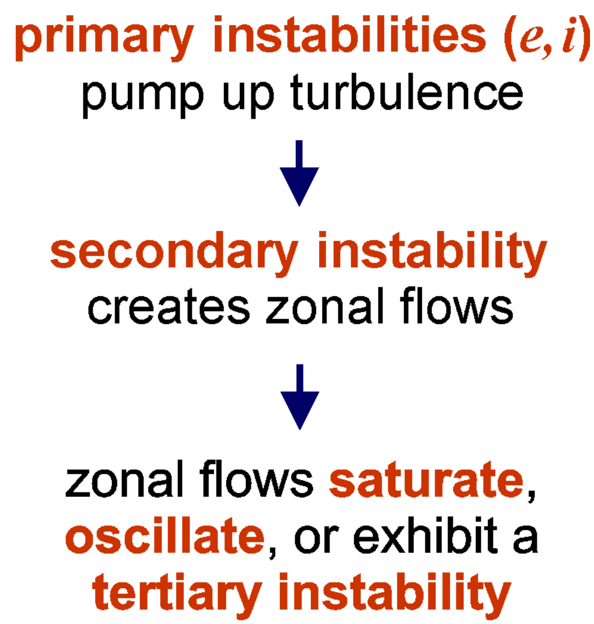
Introduction

- Drift-wave (DW) turbulence is ubiquitous in magnetized plasmas. In fusion science, DW turbulence is actively studied because it affects plasma confinement.
- DW turbulence can spontaneously generate zonal flows (ZF), which are banded shear flows with $k_{\parallel} = 0$. ZFs reduce turbulent transport but can be unstable.

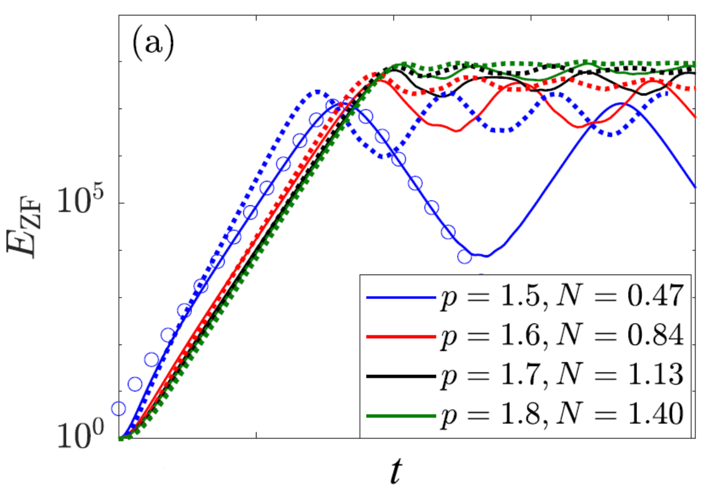


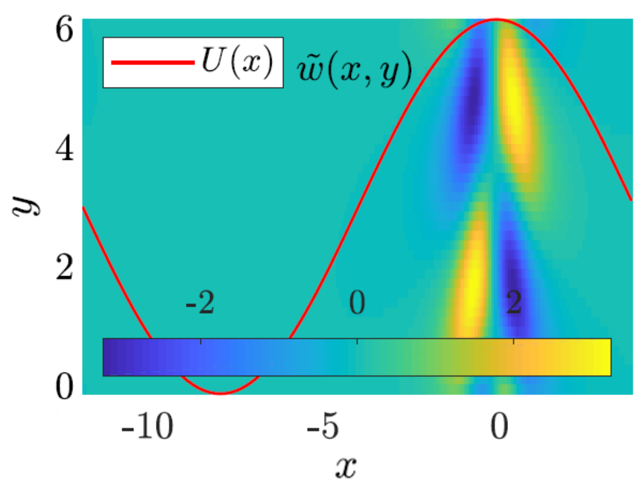


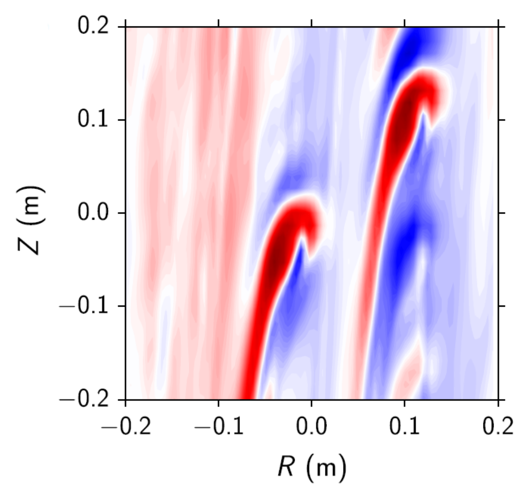
Simple questions are still awaiting simple answers.

- Gyrokinetic simulations provide numerical data but basic physics is not entirely clear.
  - *What determines the ZF saturation/oscillations/merging, amplitudes, scales?*
  - *What determines the ZF stability? How do ZFs suppress turbulence?*
  - *What determines the propagating zonal structures seen in subcritical turbulence?*
  - *How does electron-scale turbulence interact with ion-scale turbulence?...*

- Analytic modeling is needed to develop robust qualitative understanding. Gyrokinetic calculations are not intuitive. High-level theories can be advantageous.

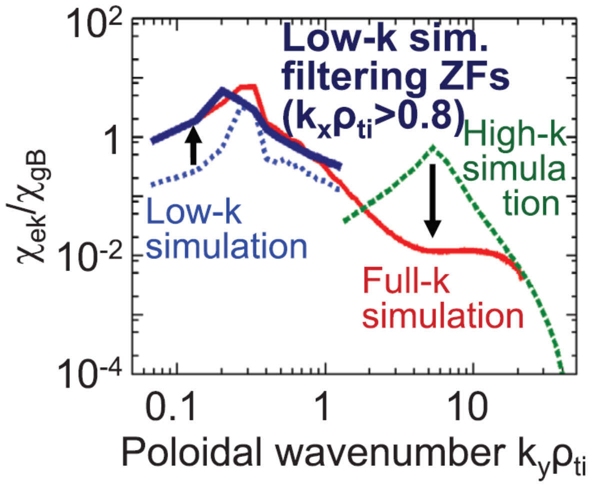






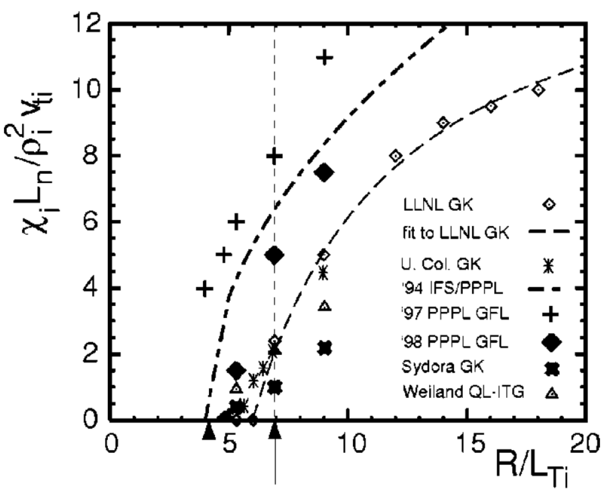
Figures taken from Zhu *et al.* (2019) (left) and van Wyk *et al.* (2016) (right).

Two high-level theories of specific effects will be presented.



Part 1: Interactions of electron-scale and ion-scale turbulence ("cross-scale interactions")

Why does ITG turbulence suppress ETG turbulence, as seen in gyrokinetic simulations?



Part 2: Stability of zonal flows, nonlinear suppression of DW turbulence, and the Dimits shift

What determines the stability of ZFs in collisionless and collisional turbulence? Minimal model of the tertiary instability and the Dimits shift.

Figures taken from Maeyama *et al.* (2017) (upper) and Dimits *et al.* (2000) (lower).

The bigger project includes many papers not covered in this talk.

- I. Y. Dodin, H. Zhu, Y. Zhou, and D. E. Ruiz, *Modeling drift-wave turbulence as quantumlike plasma*, in Proceedings of the 46th EPS Conference on Plasma Physics (Milan, Italy, 2019).
- H. Zhu, Y. Zhou, and I. Y. Dodin, *Nonlinear saturation and oscillations of collisionless zonal flows*, New J. Phys. 21, 063009 (2019).
- Y. Zhou, H. Zhu, and I. Y Dodin, *Formation of solitary zonal structures via the modulational instability of drift waves*, Plasma Phys. Control. Fusion 61, 075003 (2019).
- D. E. Ruiz, M. E. Glinsky, and I. Y. Dodin, *Wave kinetic equation for inhomogeneous drift-wave turbulence beyond the quasilinear approximation*, J. Plasma Phys. 85, 905850101 (2019).
- H. Zhu, Y. Zhou, and I. Y. Dodin, *On the Rayleigh–Kuo criterion for the tertiary instability of zonal flows*, Phys. Plasmas 25, 082121 (2018).
- H. Zhu, Y. Zhou, and I. Y. Dodin, *On the structure of the drifton phase space and its relation to the Rayleigh–Kuo criterion of the zonal-flow stability*, Phys. Plasmas 25, 072121 (2018).
- H. Zhu, Y. Zhou, D. E. Ruiz, and I. Y. Dodin, *Wave kinetics of drift-wave turbulence and zonal flows beyond the ray approximation*, Phys. Rev. E 97, 053210 (2018).
- D. E. Ruiz, J. B. Parker, E. L. Shi, and I. Y. Dodin, *Zonal-flow dynamics from a phase-space perspective*, Phys. Plasmas 23, 122304 (2016).

Background

- Gyrokinetic simulations show that low-k ITG turbulence can suppress high-k ETG turbulence. For example, see Maeyama *et al.*, PRL (2015):

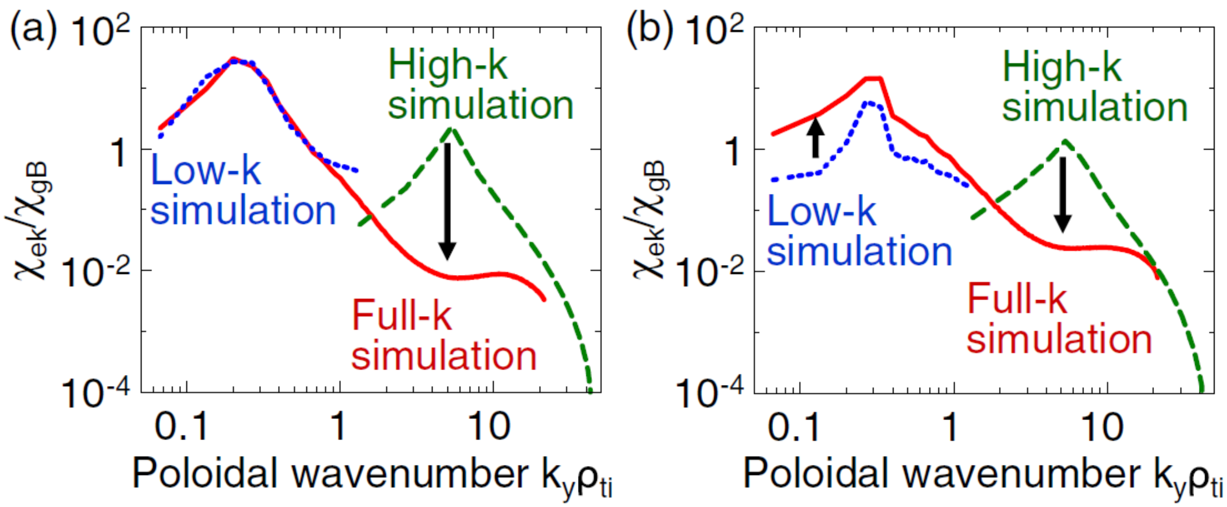


FIG. 2 (color online). Poloidal wave number spectrum of the time-averaged electron energy diffusivity χ_{ek} for (a) electrostatic ($\beta = 0.04\%$) and (b) electromagnetic ($\beta = 2.0\%$) cases. The solid (red), dotted (blue), and dashed (green) lines plot χ_{ek} as obtained from the full-k, low-k ($k_y\rho_{\mathrm{ti}} < 1.3$), and high-$k$ ($k_y\rho_{\mathrm{ti}} > 1.3$) simulations, respectively.

Our goal is to explain this effect within the simplest meaningful model.

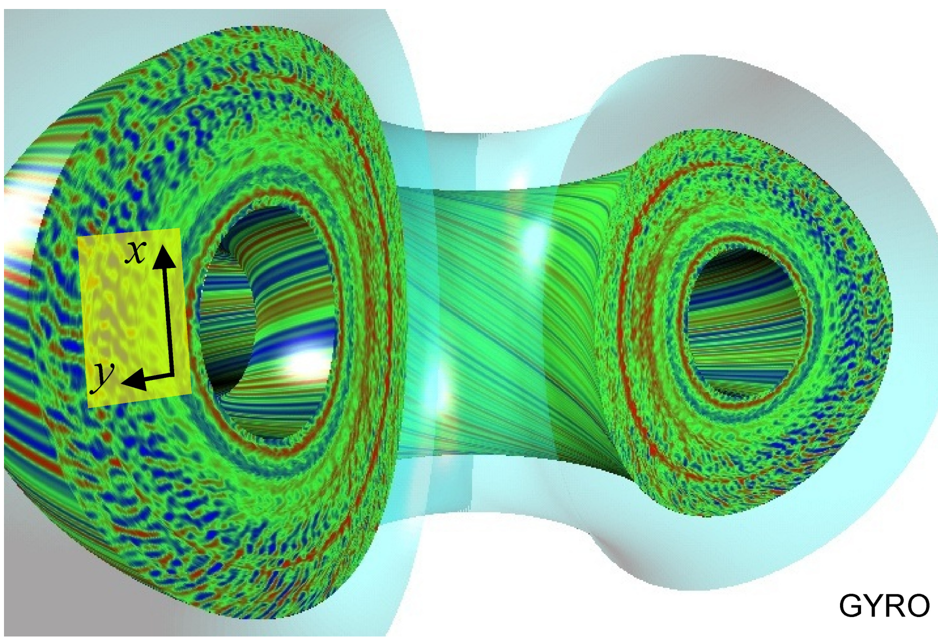
As a starting point, consider the Hasegawa–Mima model.

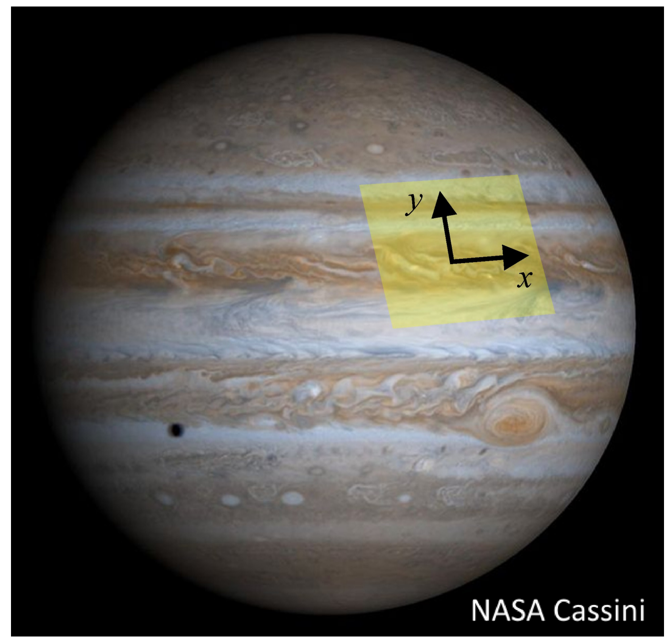
Basic physics of DW turbulence is often studied within the Hasegawa–Mima model:

$$\partial_t w + \{\varphi, w\} + \beta \partial_x \varphi = 0, \quad w = (\nabla^2 - \widehat{a})\varphi, \quad \beta \sim \partial_y N$$

Electrons respond adiabatically to drift waves ($k_\parallel \neq 0$) and do not respond to zonal flows (ZFs), which are spontaneously-generated banded shear flows with $k_\parallel = 0$.

$$w_{\mathrm{dw}} = (\nabla^2 - 1)\varphi_{\mathrm{dw}}, \quad w_{\mathrm{zf}} = \nabla^2 \varphi_{\mathrm{zf}}$$



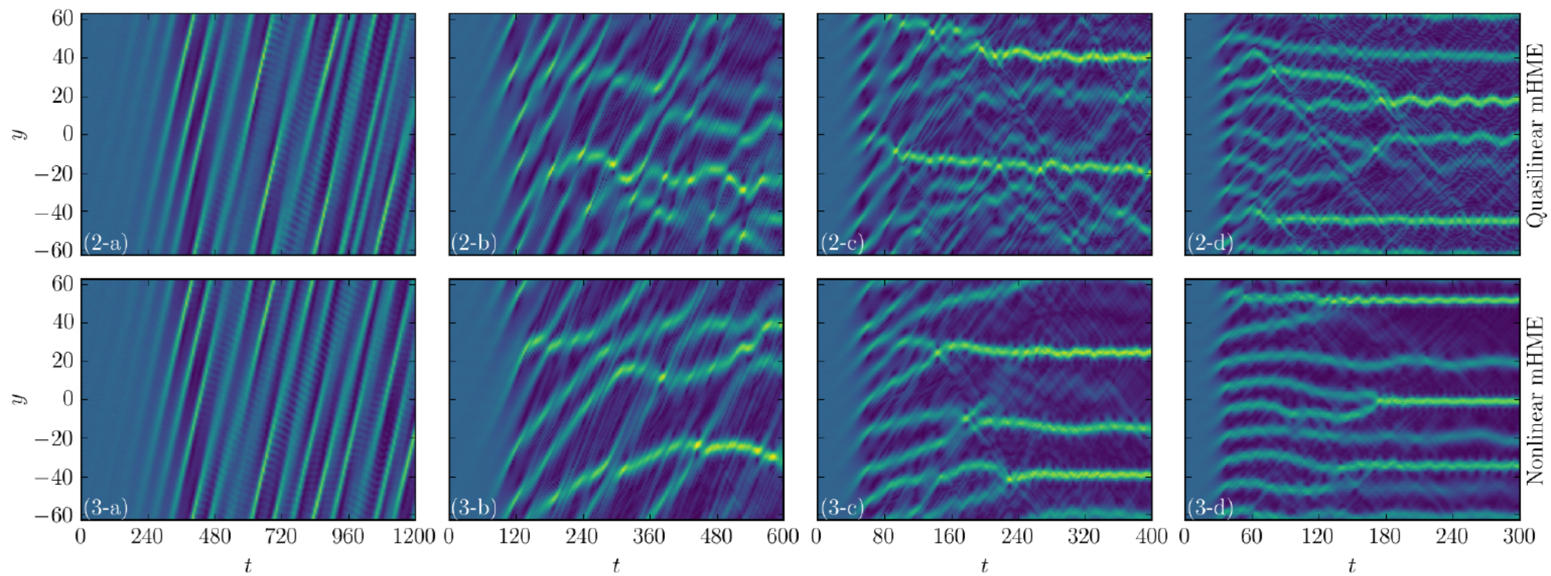


We further reduce the model using the quasilinear approximation.

- The quasilinear approximation is sufficiently accurate to capture basic effects.

$$\textit{average:}\quad \partial_t U + \partial_y \overline{\tilde{v}_x \tilde{v}_y} = 0, \quad \tilde{\mathbf{v}} = \widehat{\mathbf{z}} \times \nabla\tilde{\varphi}, \quad \tilde{w} = (\nabla^2 - 1)\tilde{\varphi}$$

$$\textit{fluctuations:}\quad \partial_t \tilde{w} + U\partial_x \tilde{w} + [\beta - (\partial_y^2 U)]\partial_x \tilde{\varphi} = \underbrace{\overline{\tilde{\mathbf{v}} \cdot \nabla \tilde{w}} - \tilde{\mathbf{v}} \cdot \nabla \tilde{w}}_{\text{neglected (QL model)}}$$



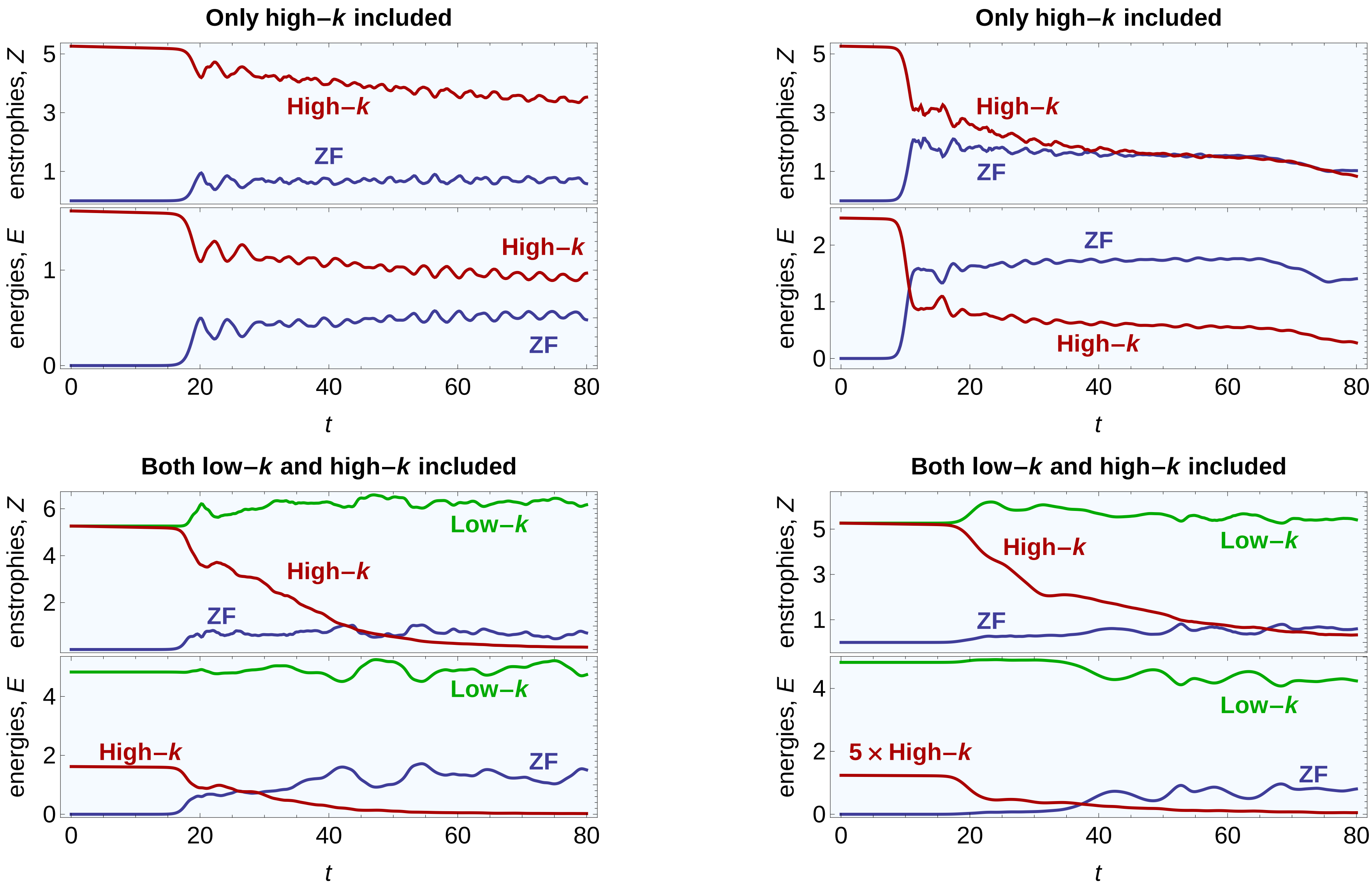
- The equation for $\tilde{w}$ can be expressed as a Schrödinger equation for "driftons":

$$\boxed{i\partial_t \tilde{w} = \widehat{\mathcal{H}}\tilde{w} + \cancel{NL}, \quad \widehat{\mathcal{H}} = \widehat{k}_x \widehat{U} - \widehat{k}_x(\beta - \widehat{U}'')(1 + \widehat{k}_\perp^2)^{-1}, \quad \widehat{\mathbf{k}} \doteq -i\nabla}$$

Ruiz *et al.* (2016); Zhou *et al.* (2019)

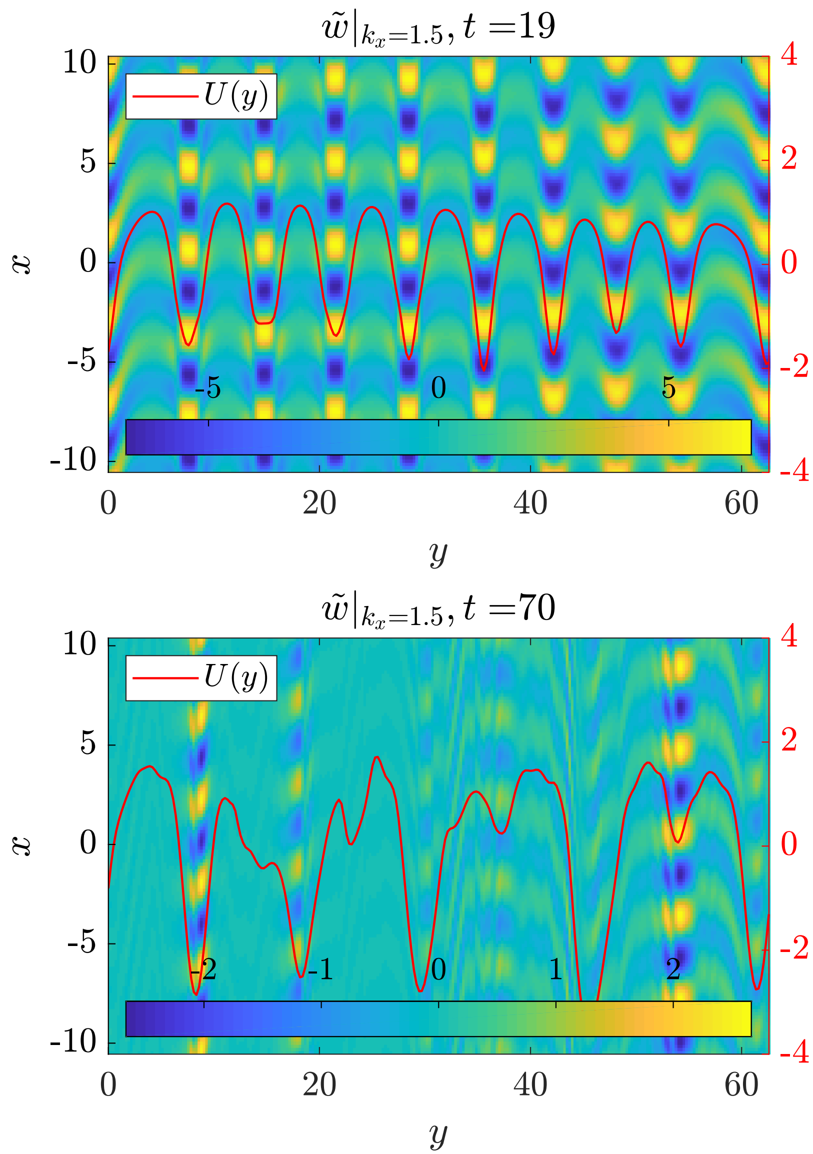
The quasilinear HM model captures cross-scale interactions.

$k_{x2}/k_{x1} = 5$ | $k_{x2}/k_{x1} = 15$



$Z_{\mathrm{dw}} \doteq \frac{1}{2}\int \mathrm{d}^2x\, \tilde{w}^2, \quad Z_{\mathrm{zf}} \doteq \frac{1}{2}\int \mathrm{d}y\, (U')^2, \quad E_{\mathrm{dw}} = -\frac{1}{2}\int \mathrm{d}^2x\, \tilde{w}\tilde{\varphi}, \quad E_{\mathrm{zf}} \doteq \frac{1}{2}\int \mathrm{d}y\, U^2$

The main claim



- Simulations show that ZFs exhibit substantial merging in multi-scale turbulence.

Claim: this merging is the cause of the high-k turbulence demise and a generic property of multi-scale turbulence.

- To show this, some theory will be needed:
  - general wave-kinetic theory,
  - topology of the drifton phase-space,
  - approximate closure for U,
  - conditions for ZF merging.
- We will also argue that the physics beyond the quasilinear approximation is not very different.

A statistical theory is constructed by analogy with that in QM.

- The Wigner function $W(t, y, \mathbf{k}) \doteq \int \mathrm{d}^2 s\, e^{-i\mathbf{k}\cdot\mathbf{s}} \langle \tilde{w}(t, \mathbf{x}+\mathbf{s}/2)\tilde{w}(t, \mathbf{x}-\mathbf{s}/2)\rangle$ (i.e., the spectrum of the two-point correlator, or "quasiprobability distribution") satisfies

$$\boxed{\frac{\partial W}{\partial t} = \{\{\mathcal{H}_H, W\}\} + [\![\mathcal{H}_A, W]\!], \quad \frac{\partial U}{\partial t} = \frac{\partial}{\partial y}\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{1+k_\perp^2} \star k_x k_y W \star \frac{1}{1+k_\perp^2}}$$

$$\mathcal{H}_H = k_x U - \frac{\beta k_x}{1+k_\perp^2} + \frac{1}{2}\left[\!\!\left[U'', \frac{k_x}{1+k_\perp^2}\right]\!\!\right], \quad \mathcal{H}_A = \frac{1}{2}\left\{\!\!\left\{U'', \frac{k_x}{1+k_\perp^2}\right\}\!\!\right\}$$

- Geometrical-optics limit: improved wave kinetic equation (iWKE) **with new terms**:

$$\widehat{\mathcal{L}} = O(\partial_x \partial_k) \sim (k L_{\mathrm{zf}})^{-1} \ll 1, \quad A e^{i\widehat{\mathcal{L}}/2} B = 1 + {}^{i}\!/_{2}\{A, B\} + \dots$$

$$\boxed{\frac{\partial W}{\partial t} = \{\mathcal{H}_H, W\} + \boldsymbol{2\mathcal{H}_A W}, \quad \frac{\partial U}{\partial t} = \frac{\partial}{\partial y}\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{k_x k_y W}{(1+k_\perp^2)^2}}$$

$$\mathcal{H}_H \approx k_x U - k_x(\beta - \boldsymbol{U''})/(1+k_\perp^2), \quad \mathcal{H}_A \approx -\boldsymbol{U'''} k_x k_y/(1+k_\perp^2)^2$$

$\widehat{\mathcal{L}} = \{\cdot, \cdot\}$ | $\{\{A, B\}\} = 2A\sin(\widehat{\mathcal{L}}/2)B$ | $[\![A, B]\!] = 2A\cos(\widehat{\mathcal{L}}/2)B$ | $A \star B = A e^{i\widehat{\mathcal{L}}/2} B$

Quasistatic approximation in the limit $U'' \ll \beta$

- Let us rewrite the iWKE in the following form using the group velocity v_g:

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial y}(W v_g) = \frac{\partial}{\partial k_y}\left(W \frac{\partial \mathcal{H}_H}{\partial y}\right) - \frac{U'''}{\beta - U''} W v_g, \quad v_g = \frac{2k_x k_y}{(1+k_\perp^2)^2}(\beta - U'')$$

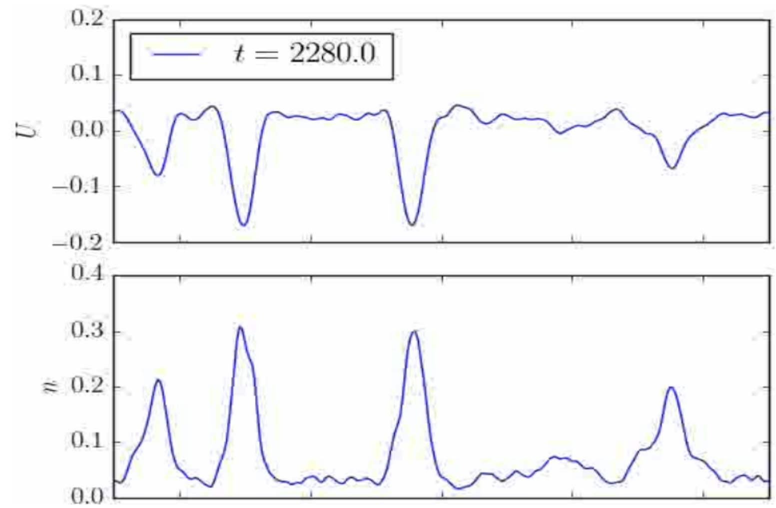
- By integrating the iWKE over $\mathbf{k}$, one obtains an equation for the drifton density. The **term on the right** can be neglected compared to $\partial_y J$ when $U'' \ll \beta$.

$$\partial_t N + \partial_y J = -\boldsymbol{J U'''/(\beta - U'')}, \quad N \doteq \int W \, \mathrm{d}^2 k, \quad J \doteq \int W v_g \, \mathrm{d}^2 k$$

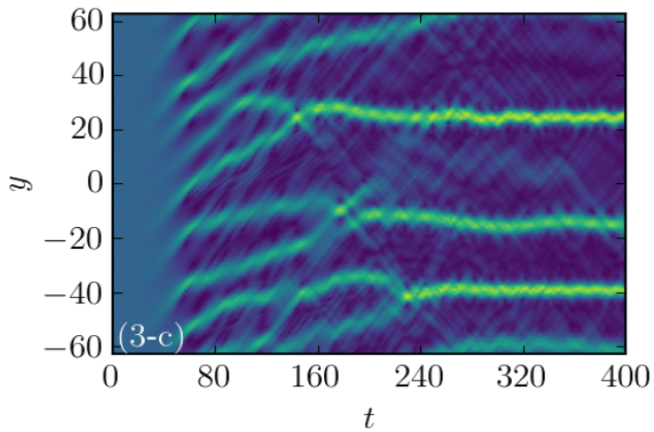
- In this "quasistatic" limit, U becomes a local function of N ("equation of state"):

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial y}\left[\frac{J}{2(\beta - U'')}\right] \approx \frac{\partial_y J}{2\beta} \approx -\frac{\partial_t N}{2\beta}$$

$$\boxed{U \approx -\frac{N}{2\beta} + \text{const}}$$



Even at small U, the ZF stability depends on the ZF wavenumber q.

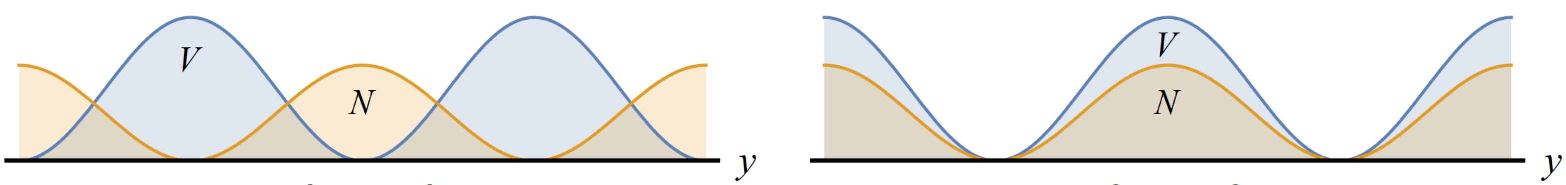


- The iWKE is only marginally applicable to ZF formation but can explain it qualitatively.

$$\mathcal{H} = \frac{k_x(-\beta + U'')}{1 + k_x^2 + k_y^2} + k_x U, \quad U \approx -\frac{N}{2\beta} + \text{const}^*$$

$$k_y^2 \ll 1 + k_x^2, \quad q^2 \doteq -U''/U, \quad k_x = \text{const}$$

$$\boxed{\mathcal{H} \approx \frac{k_x}{\beta}\left(\frac{k_y^2}{2m} + V\right) + \text{const}, \quad \frac{1}{m} \doteq \frac{2\beta^2}{(1+k_x^2)^2}, \quad V \doteq \left(\frac{q^2}{1+k_x^2} - 1\right)\frac{N}{2}}$$



$q^2 < 1 + k_x^2$ $\qquad$ $q^2 > 1 + k_x^2$

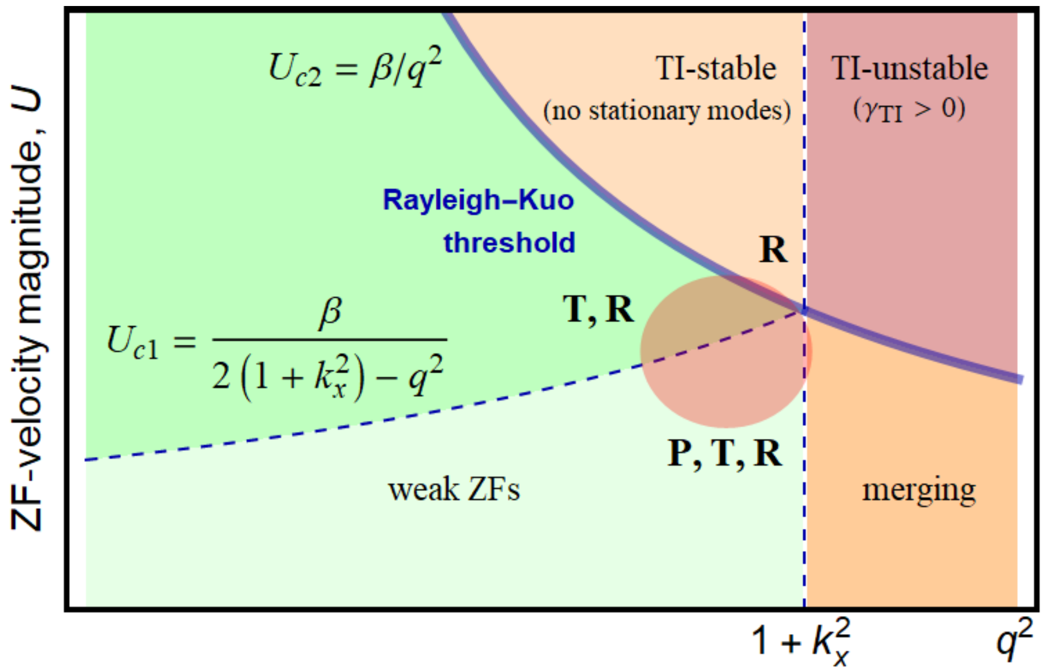
- If $q^2 < 1 + k_x^2$, driftons reside near minima of V, so the system is stable.
- If $q^2 > 1 + k_x^2$, driftons reside near maxima of V. The system can lower the energy by bifurcating to a lower-q state, so it is unstable to ZF merging.

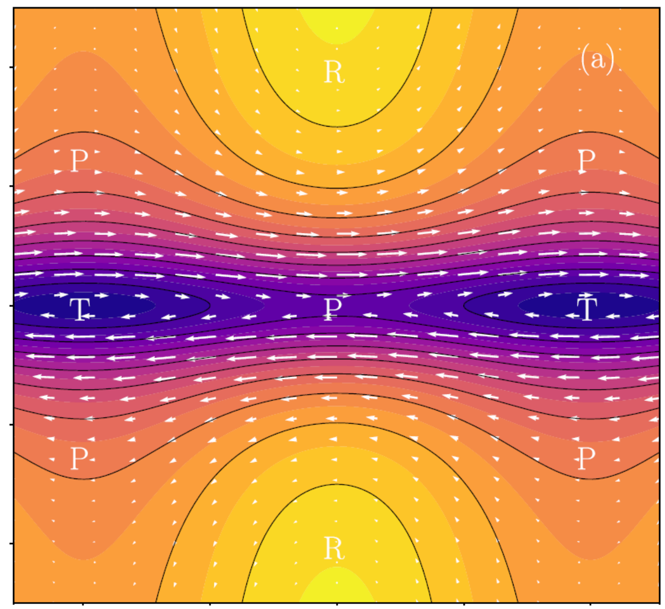
* Here, we assume $U'' \lesssim \beta$. Unlike in single-scale turbulence, this does not rule out $q \gtrsim k_x$.

ZFs have many regimes but not all of them are realized naturally.

- In single-scale turbulence, q corresponds to the maximum of $\gamma_{\text{secondary}}$. Then, there are passing and/or trapped trajectories, and many driftons survive.

$$q \sim \min\left\{k_x^2\sqrt{N}/\beta, \sqrt{1+k_x^2}\right\}, \quad U \lesssim U_{c1}$$





- Low-k waves cause ZFs to merge down to $q^2 \sim 1 + k_x^2$. High-k waves become runaways and dissipate. Low-k waves remain passing because $U_{c1} = U_{c1}(k_x)$.

Zhu *et al.* (2019)

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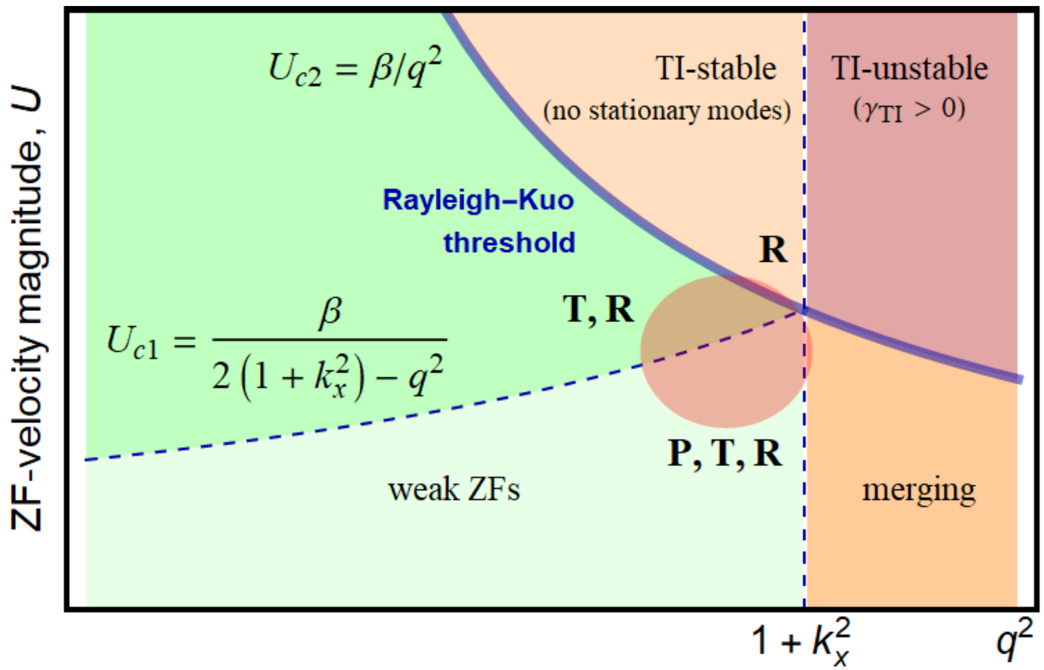

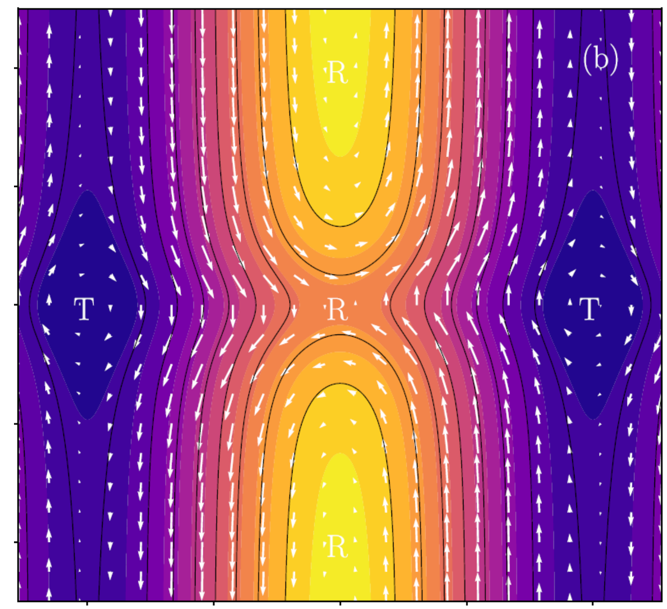

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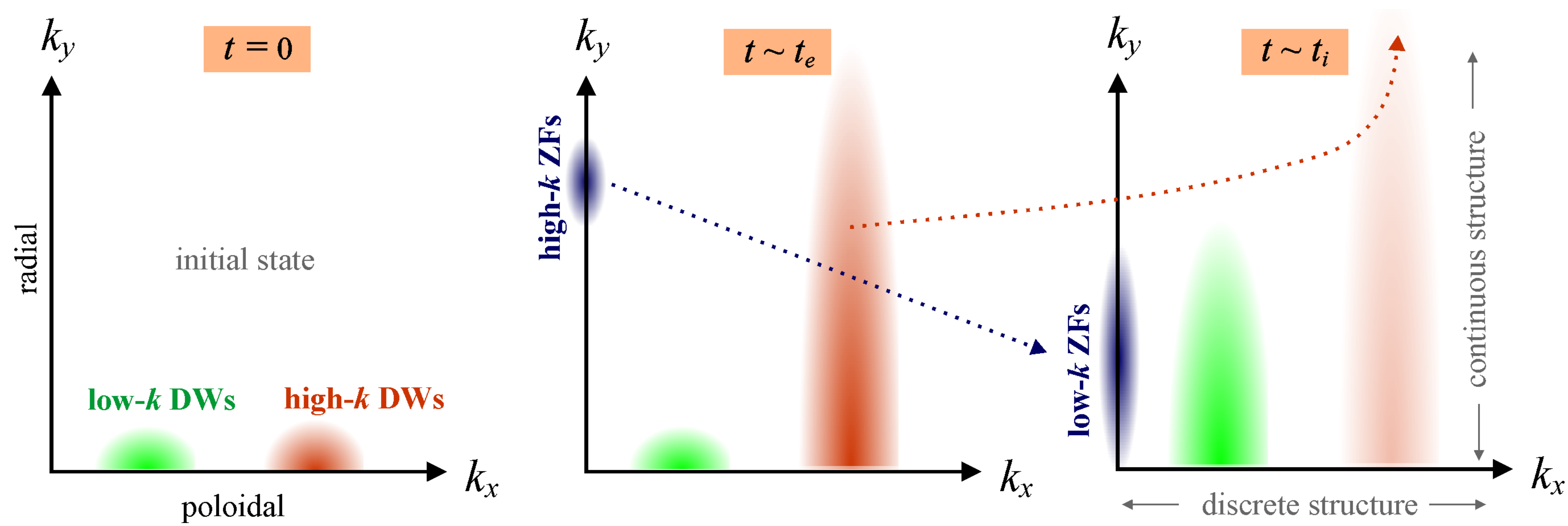
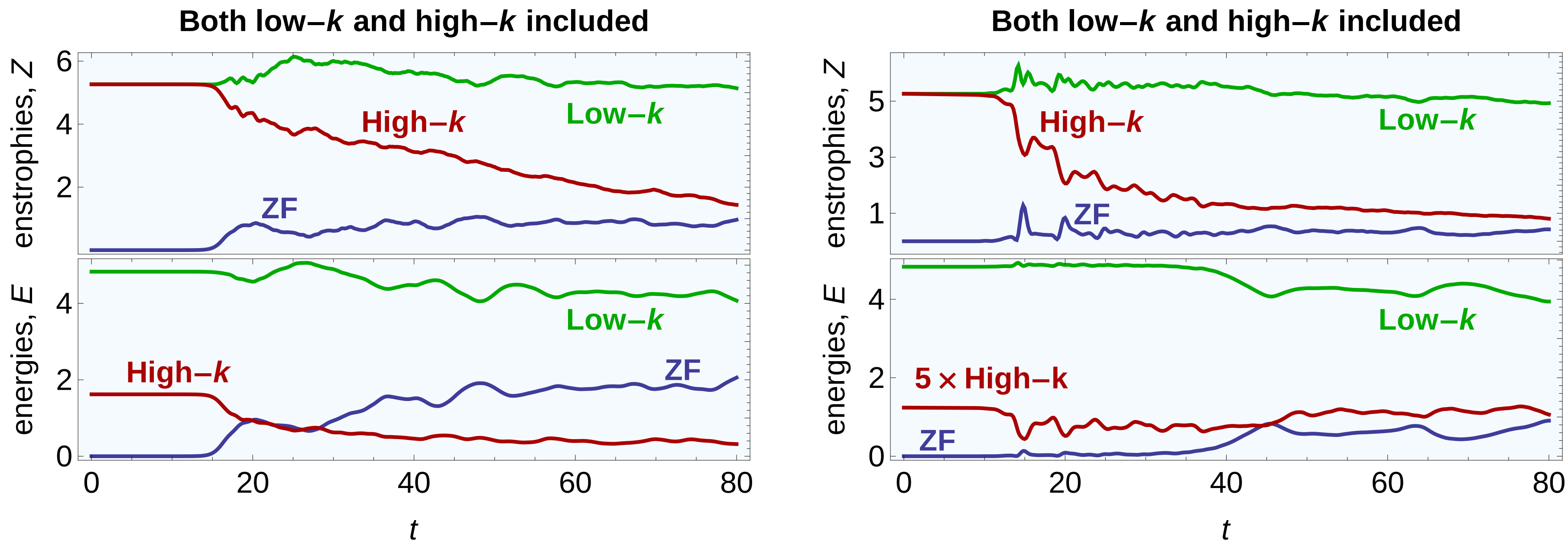
Zhu *et al.* (2019)

Cross-scale interactions in a nutshell

- The scale separation persists in the poloidal-k space.
- In the radial-k space, the scales are determined by the ZFs, which start at the electron scale and then merge down to the ion scale.
- During the ZF merging, most high-k driftons become runaways and dissipate.



Nonlinear HM simulations demonstrate similar behavior.



$k_{x2}/k_{x1} = 5$

$k_{x2}/k_{x1} = 15$

- Drifton collisions serve as an additional channel though which high-k driftons become runaways. Other than that, the effect remains the same.

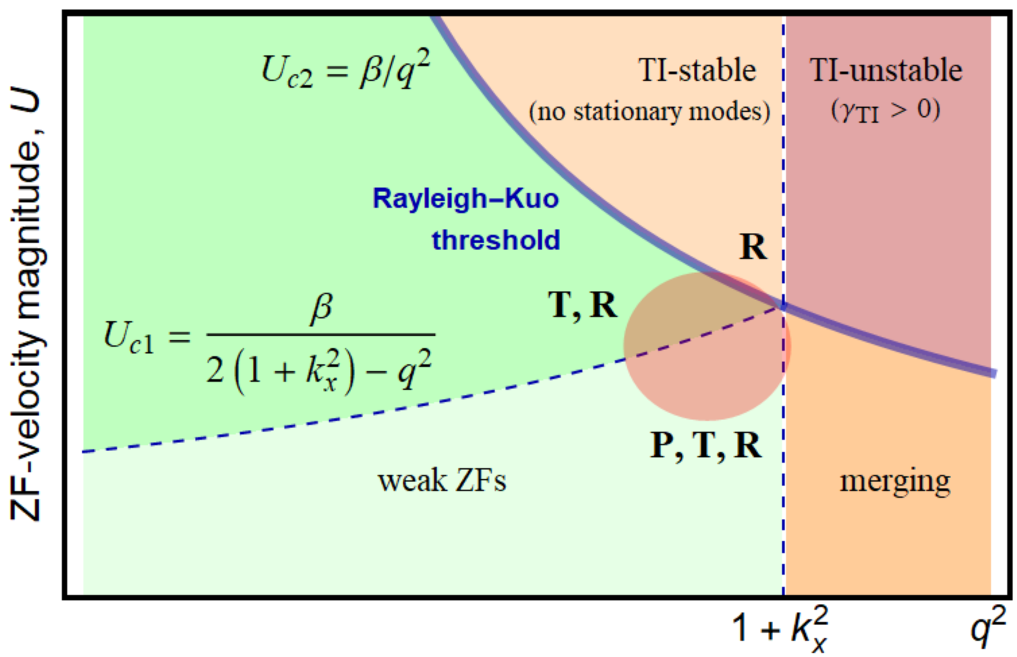
Summary: the demise of high-k turbulence via cross-scale interactions is robustly explained as a phase-space effect in the Hasegawa–Mima model.

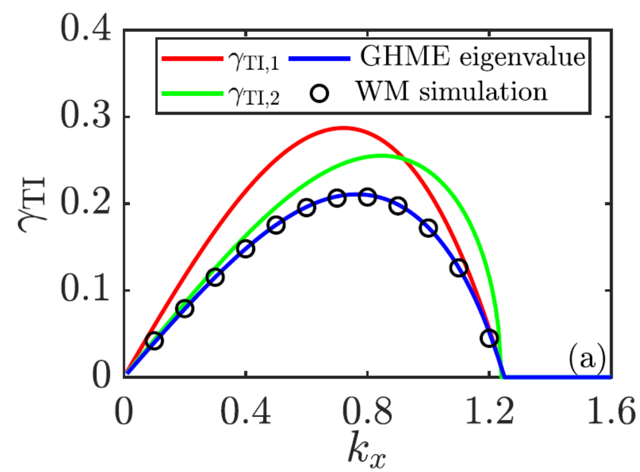
Tertiary instability in the conservative Hasegawa–Mima model

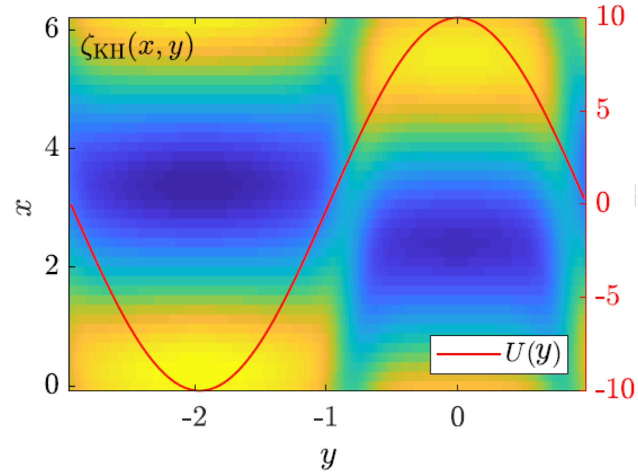
- In the Hasegawa–Mima model, the drifton Hamiltonian is pseudo-Hermitian, resulting in an instability of the Kelvin–Helmholtz type. (But is it relevant?)

$$i\partial_t \tilde{w} = \widehat{\mathcal{H}}\tilde{w}, \quad \widehat{\mathcal{H}} = k_x \widehat{U} - k_x(\beta - \widehat{U}'')(1 + k_x^2 + \widehat{k}_y^2)^{-1}$$

$$\boxed{\gamma_{\text{TI}} = |k_x U_0| \left(1 - \frac{1 + k_x^2}{q^2}\right) \sqrt{1 - \frac{\beta^2}{U_0^2 q^4}}}$$

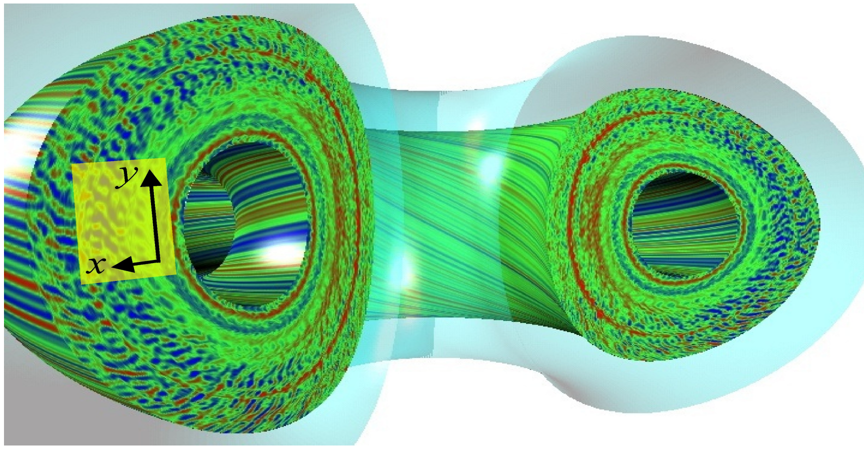






Zhu *et al.* (2018)a; Zhu *et al.* (2018)c; cf. Kuo (1949); Numata *et al.* (2007); Kim and Diamond (2002)

The dissipative TI is different. Consider the Terry–Horton model...



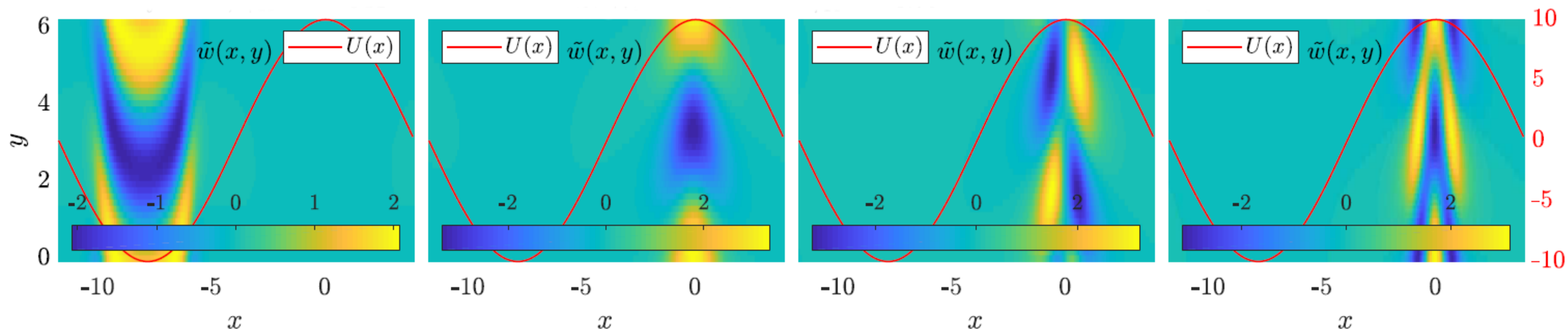
$$\partial_t w + \{\varphi, w\} = \beta \partial_y \varphi - \boldsymbol{\widehat{D}} w$$

$$w = (\nabla^2 - \hat{a} + \boldsymbol{i\widehat{\delta}})\varphi$$

$$\underbrace{\widehat{\delta} = \delta(\widehat{k}_y)}_{\text{primary instability}} \quad , \quad \underbrace{\widehat{D} = 1 - \kappa \nabla^2}_{\text{friction \& viscosity}}$$

- The "Hasegawa–Mima" TI mode becomes localized + other localized modes appear.

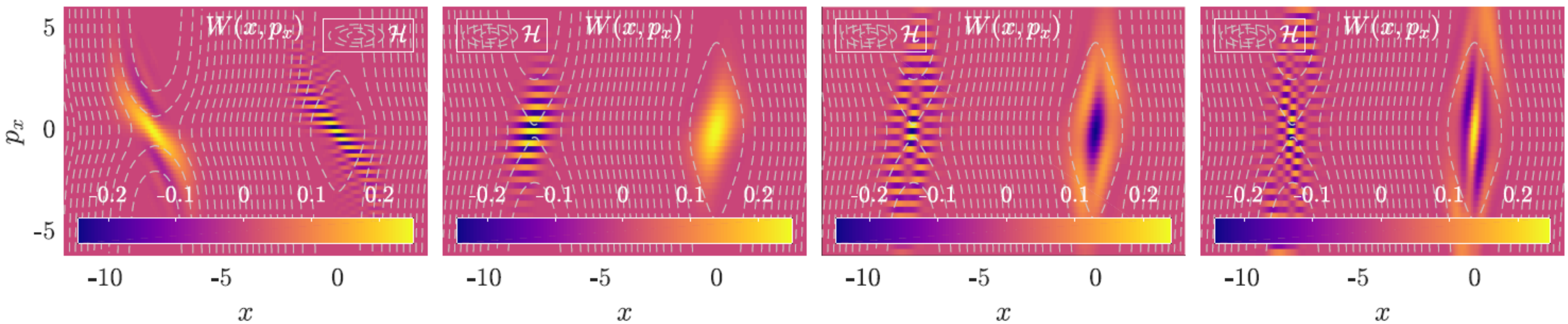
$$i\partial_t \tilde{w} = \widehat{\mathcal{H}} \tilde{w}, \qquad \widehat{\mathcal{H}} = k_y \widehat{U} + k_y(\beta + \widehat{U}'')[1 + \widehat{k}_x^2 + k_y^2 - \boldsymbol{i\delta(k_y)}]^{-1} - \boldsymbol{i\widehat{D}}$$



TI modes satisfy the equation of a quantum harmonic oscillator.

- The largest growth rates belong to the lowest-order modes. Those are localized in (x, k_x), so the drifton Hamiltonian can be approximated with its Taylor expansion:

$$\partial_t W = \{\{\mathcal{H}_H, W\}\} + [\![\mathcal{H}_A, W]\!] \quad \Rightarrow \quad \textbf{truncate } \boldsymbol{\mathcal{H}} \quad \Rightarrow \quad \widehat{\mathcal{H}} \approx c_0 + c_1 \widehat{x}^2 + c_2 \widehat{k}_x^2$$



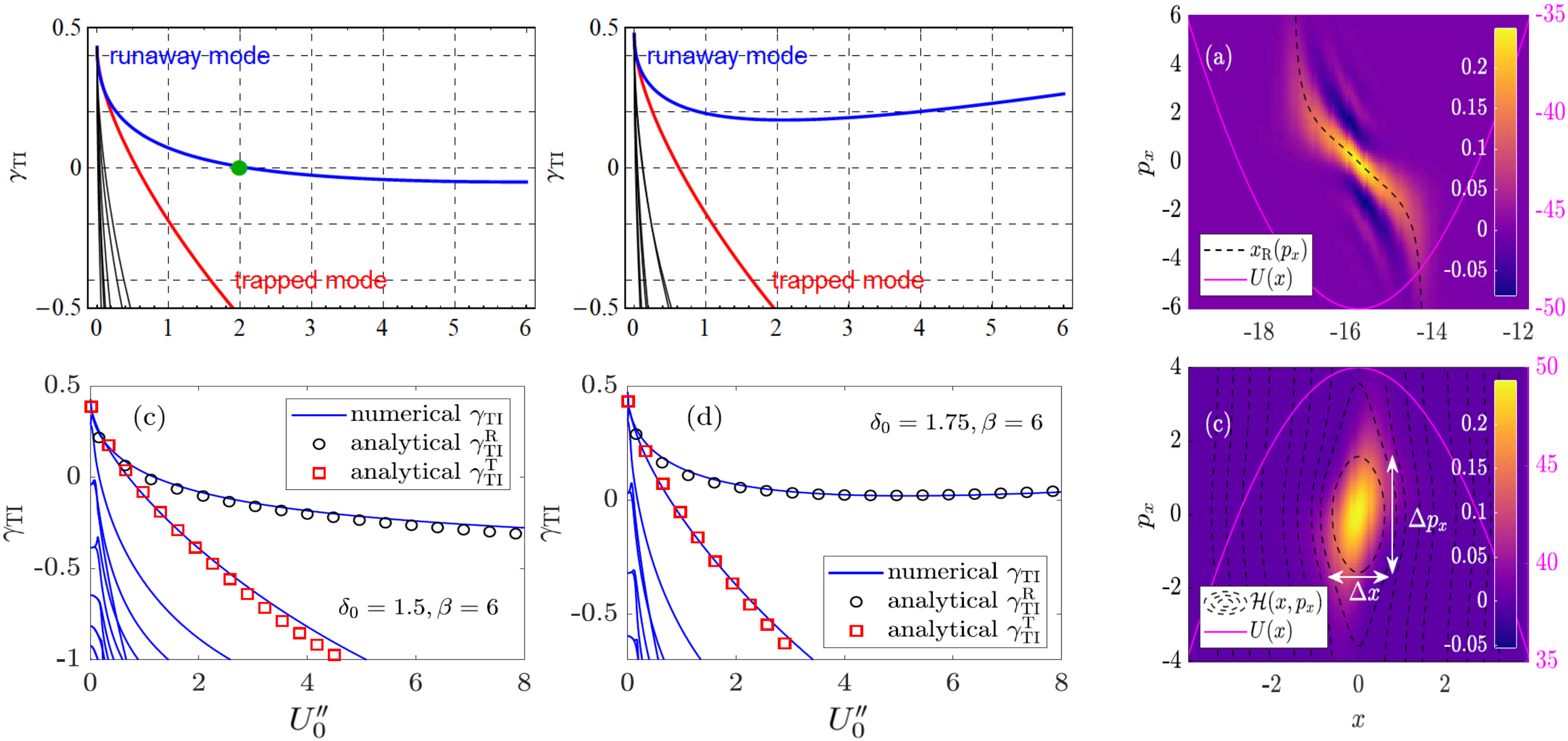
- This yields an equation of a quantum harmonic oscillator with complex coefficients:

$$\left(-\boldsymbol{\vartheta^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2\right)\tilde{w} = \varepsilon \tilde{w} \quad \Rightarrow \quad \tilde{w}_n \sim H_n\left(\frac{x}{\sqrt{\vartheta}}\right) e^{-x^2/2\vartheta}, \quad \varepsilon_n = (2n+1)\vartheta$$

$$\vartheta \doteq -\frac{i\sqrt{2(1+\beta/U_0'')}}{1+k_y^2 - i\delta}, \quad \varepsilon \doteq \frac{2}{k_y U_0''}\left[\omega_{\mathrm{TI}} - k_y U_0 + iD_0 - \frac{k_y(\beta + U_0'')}{1+k_y^2 - i\delta}\right]$$

The growth rate is obtained explicitly and agrees with simulations.

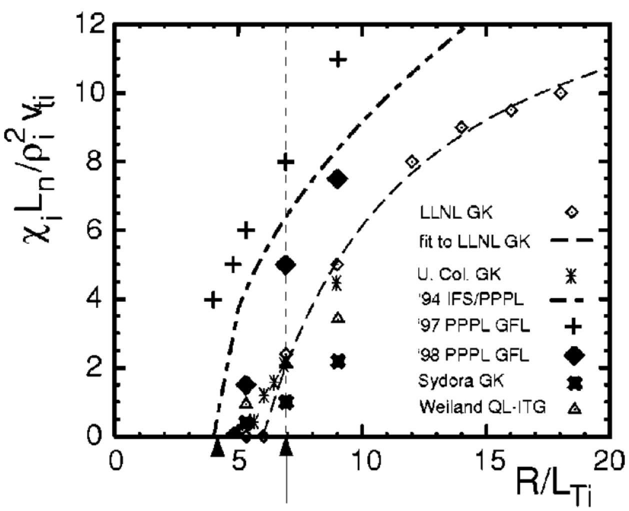
$$\gamma_{\mathrm{TI}} = -D_0 + \mathrm{Im}\left[\frac{k_y(\beta + U_0'') - ik_y U_0''\sqrt{(1+\beta/U_0'')/2}}{1 + k_y^2 - i\delta}\right]$$

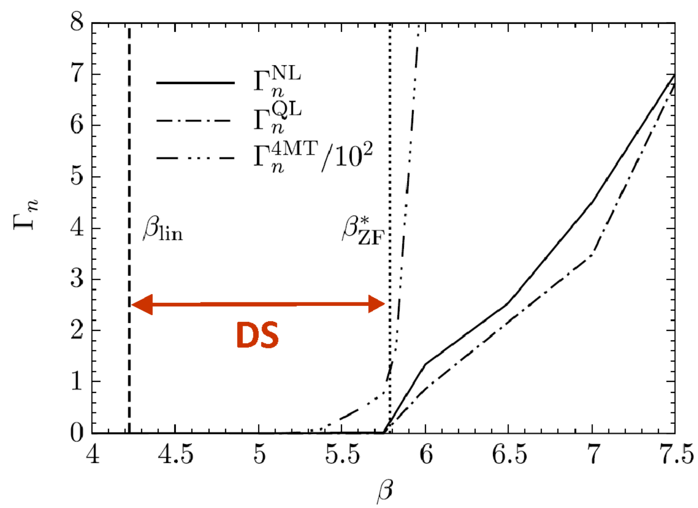


Dimits shift

$$\gamma_{\mathrm{TI}} = -D_0 + \mathrm{Im}\left[\frac{k_y(\beta + \boldsymbol{U_0''}) - ik_y\boldsymbol{U_0''}\sqrt{(1+\beta/\boldsymbol{U_0''})/2}}{1+k_y^2 - i\delta}\right] \equiv \gamma_{\mathrm{primary}}^{(\mathrm{linear})} + \Delta\gamma(\boldsymbol{U_0''})$$

- The tertiary instability can be viewed as the primary instability modified by ZFs.
  - If $\gamma_{\mathrm{TI}} < 0$, turbulence is suppressed; ZFs survive, assuming $\hat{D}$ acts only on DWs.
  - If $\gamma_{\mathrm{TI}} > 0$, the system ends up in a turbulent state.
  - Due to $\Delta\gamma$, the transition to the turbulent state occurs at plasma parameters different from those without ZFs. This is called the **Dimits shift**.

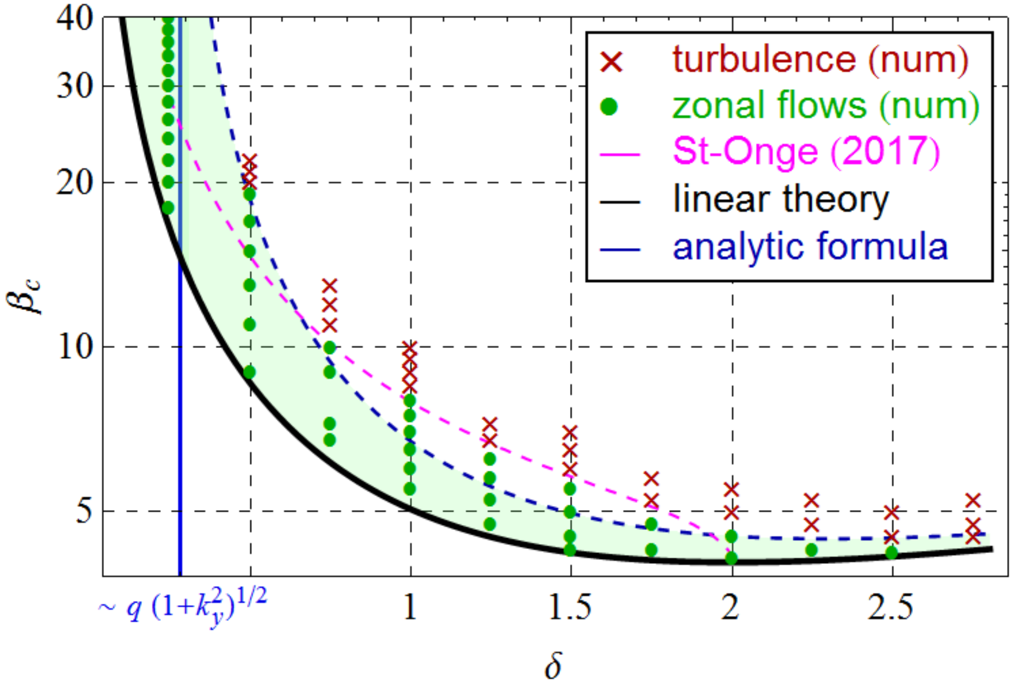


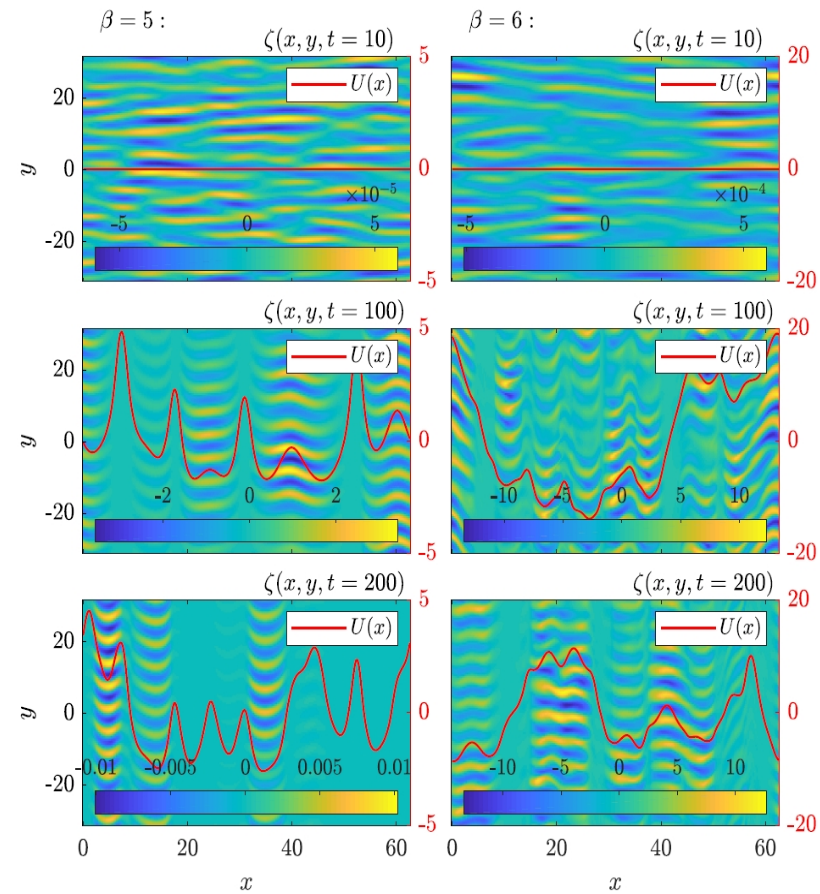


Our explicit formula for the Dimits shift agrees with simulations.

- We calculate the values of β that correspond to $\gamma_{\text{primary}}^{(\text{linear})} = 0$ and $\gamma_{\text{TI}} = 0$ using $U_0'' \sim q^2 U_{c1}$. The difference between these values is the Dimits shift (shaded).
- Compared with a related calculation by St-Onge (2017), our model is a better fit at both large and small δ. For example, it has no spurious cutoff at $\delta = 2$.

$$\boxed{\beta_c \approx \frac{D[(1+k_y^2)^2 + \delta^2]/k_y}{\delta - (1+k_y^2)\sqrt{U_0''/2\beta}}, \qquad \frac{U_0''}{\beta} \sim \frac{q^2}{k_y^2+1}}$$





Summary

- Studying drift-wave turbulence in phase space requires:
  - looking beyond geometrical optics: do not neglect $\boldsymbol{\lambda/L}$ and $\boldsymbol{U''/\beta}$,
  - deriving WME/WKE from first principles: *hand-waving leads to errors*.
- Cross-scale interactions within the Hasegawa–Mima model:
  - Zonal flows tend to merge when there are driftons with $k_\theta \lesssim q$.
  - Zonal-flow merging gradually reduces the DW radial scale.
  - High-k_θ DWs are efficiently dissipated during this process.
- Tertiary instabilities and the Dimits shift:
  - Dissipation localizes the tertiary modes near the ZF-velocity extrema.
  - The growth rate of these modes can be made negative by $U'' \Rightarrow$ Dimits shift.
  - An analytic theory is developed within the Terry–Horton model.
  - Two-fluid models have additional features. (not in this talk)
- Drift-wave solitons in subcritical turbulence (not in this talk)

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