The Equilibrium of a Rapidly Rotating Magnetic Mirror

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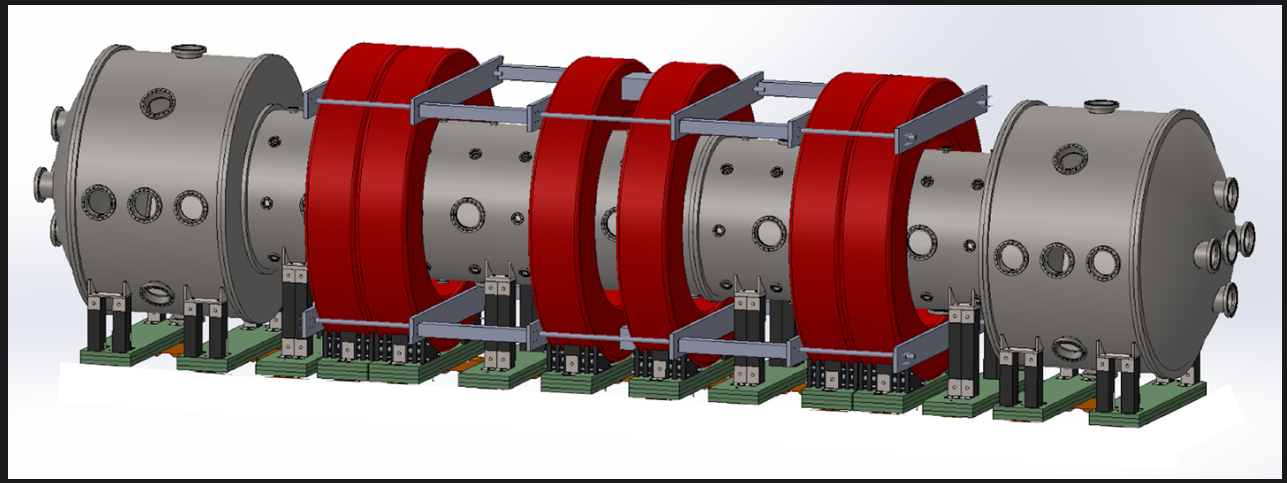
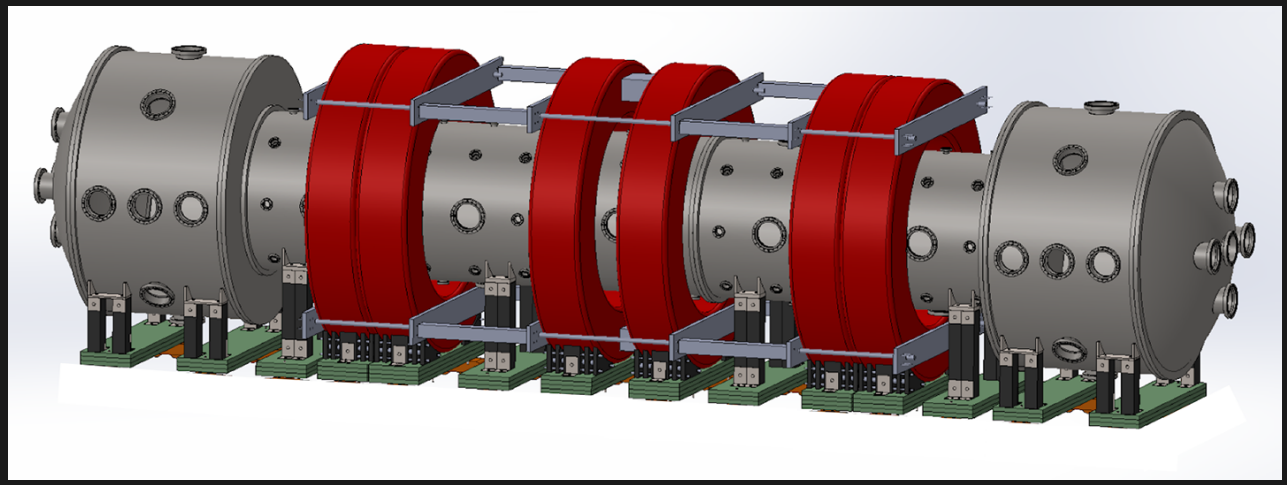
Centrifugal Mirrors: What Are They



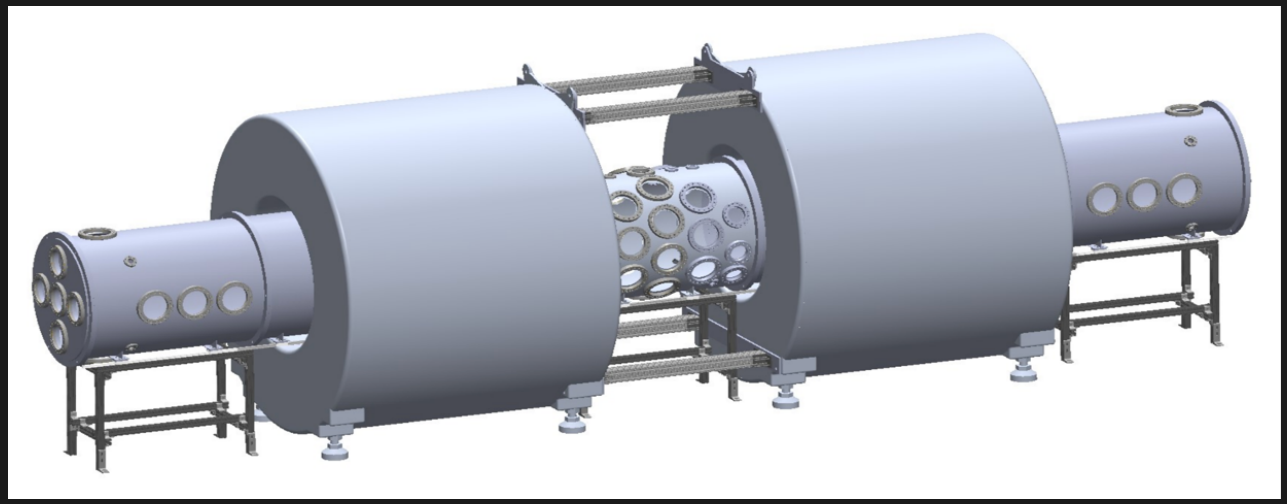
- Novel device (preceded by PSP-2 & MCX)
- Simple geometry, reducing build & maintainance costs

Centrifugal Mirrors: How do They Help?



- Ions are pushed away from the ends of the plasma, confining particles
- Elecrons follow (quasineutrality), confinining heat
- Flow shear stabilizes macro- and micro- instabilities

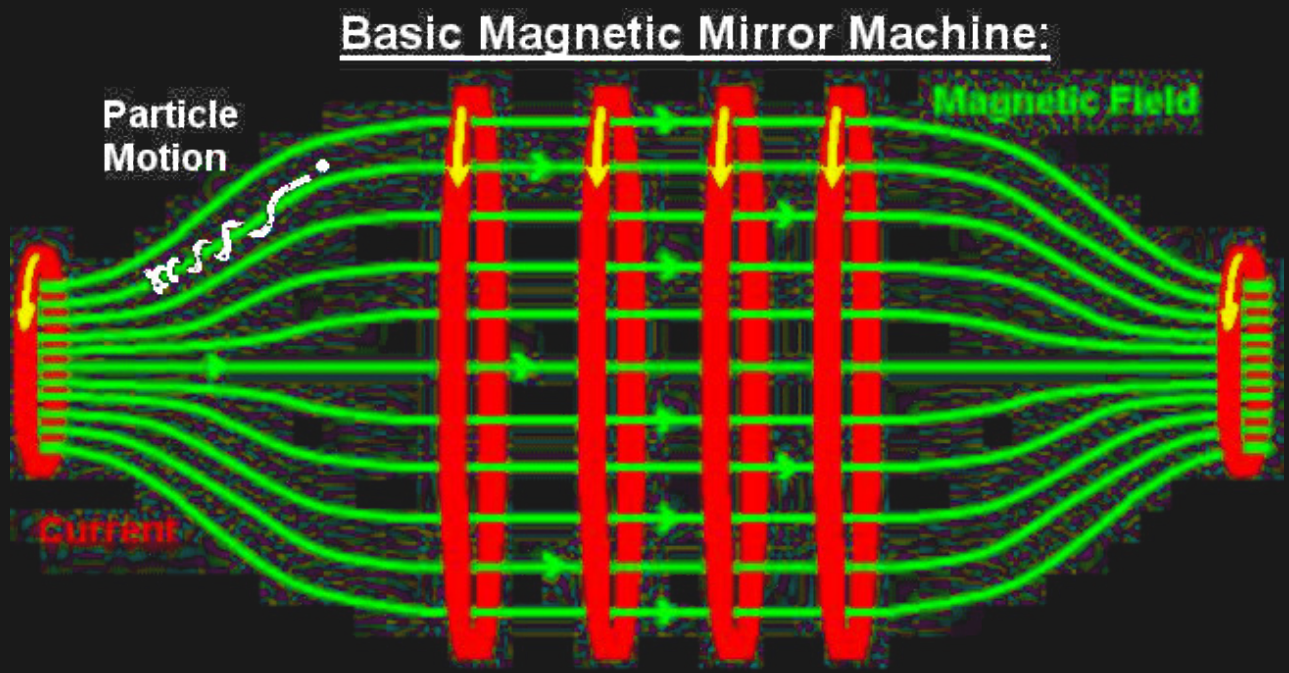
The CMFX Experiment



Improvements over MCX

- Superconducting MRI Magnets 3 Tesla, long pulse capable
- High-performance Aluminum Vacuum Vessel from UHV Atlas, for neutral control
- RF Pre-heat for robust density control
- Improved HV Capacitor system, up to $100\mathrm{kV}$

Theory and Modelling for CM



0D-Modelling

- Assume Pastukhov-like losses along field lines
- Include classical perpendicular losses
- Include simple neutral and radiation model

Small-Gyroradius Expansion

Assuming a steady-state, small-gyroradius, axisymmetric plasma, the lowest-order momentum equation is:

$$\boldsymbol{E} + \boldsymbol{u}_s \times \boldsymbol{B} = \boldsymbol{0} \tag{1}$$

This implies that

$$\boldsymbol{E} = -\frac{d\Phi}{d\psi}\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\varphi, \qquad \text{and} \qquad \boldsymbol{u}_s = \boldsymbol{u} = \omega(\psi)R^2\boldsymbol{\nabla}\phi, \tag{2}$$

To next order in the gyroradius expansion

$$w_{\|}\boldsymbol{b}\cdot\nabla|_{\varepsilon} F_s = C[F_s], \qquad \text{where} \qquad \boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u} \tag{3}$$

with the unique, confined, solution of a co-rotating Maxwellian:

$$F_s = N_s(\psi)\left(\frac{m_s}{2\pi T_s(\psi)}\right)^{3/2} \exp\left[-\frac{m_s}{2T_s}\left(w^2 - \omega^2 R^2\right) - \frac{Z_s e}{T_s}\varphi\right], \tag{4}$$

N.B. $N_s(\psi)$ is a pseudo-density that is a parameter of the Maxwellian. Not the particle density.

Small-Gyroradius Expansion: Density and Quaisneutrality

The density of this Maxwellian is given by

$$n_s = N_s(\psi) \exp\left[\frac{m_s\omega^2 R^2}{2T_s} - \frac{Z_s e}{T_s}\varphi\right] \tag{5}$$

We solve for the potential by equating electron and ion charge densities:

$$N_e \exp\left[\frac{e\varphi}{T_e} + \frac{m_e\omega^2 R^2}{2T_e}\right] = Z_i N_i \exp\left[-\frac{Z_i e\varphi}{T_i} + \frac{m_i\omega^2 R^2}{T_i}\right], \tag{6}$$

This can be solved (employing $m_e/m_i \ll 1$ and assuming $Z = 1$ & $T_e = T_i$) to give:

$$\varphi = \frac{m_i}{4T_i}\omega^2 R^2, \tag{7}$$

and hence (on a mild redefinition of N_s):

$$n_s = N_s \exp\left[\frac{m_s\omega^2}{4T_s}\left(R^2 - R_{\max}^2\right)\right]. \tag{8}$$

Small-Gyroradius Expansion: The Magnetic Equilibrium

With these formulae for φ and the variation of n_s along a field line in hand, we are left with the problem of determining the magnetic equilibrium, and hence the shape of the field lines.
This is done by solving the usual equations of force balance, with the addition of the centrifugal force:

$$-n_s m_s \omega^2 R \nabla R = -\nabla \left(p_i + p_e + \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \boldsymbol{B} \cdot \nabla \boldsymbol{B} \qquad (9)$$

N.B. We choose not to solve a Grad-Shafranov equation, although one could easily be written for ψ.

Ordering

We will solve the equilibrium equation asymptotically, under the following assumptions:

- The plasma is rapidly rotating $M = u/c_s \gg 1$.
- The plasma thermal energy is small compared to the rotational energy $T_i \ll m_i u^2$
- The line-average rotational energy is comparable to the magnetic energy $m_i \bar{n}_i \omega^2 R \sim B^2 \big/ 2\mu_0$

Thus, the solution will be low β and have an **Alfvén Mach Number** M_A of order unity. We also assume reflection symmetry in the vertical plane $z = 0$.

Layer Solution: 1

Dropping terms small in β, we have to solve

$$m_i N_i \exp\left[\frac{m_s\omega^2}{2T_s}\left(R^2 - R_{\max}^2\right)\right]\omega^2 R\nabla R = -\nabla\frac{B^2}{2\mu_0} + \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\boldsymbol{B}, \tag{10}$$

we expect the rapid rotation to localise the density into a disc-like layer near the midplane (i.e at $z = 0$). Making the assumption that the density localises and that gradients in z dominate over gradients in R, we have to solve for a field that balances centrifugal forces and magnetic tension in the radial direction:

$$m_i N_i \exp\left[\frac{m_s\omega^2}{2T_s}\left(R^2 - R_{\max}^2\right)\right]\omega^2 R = \frac{1}{\mu_0}B_z\frac{\partial B_R}{\partial z} \tag{11}$$

Layer Solution: 2

Introducing the field line shape as $R = R(\psi, z)$ we can write this as an equation for R:

$$m_i N_i \exp\left[\frac{m_s\omega^2}{2T_s}\left(R^2 - R_{\max}^2\right)\right]\omega^2 R = \frac{1}{\mu_0} B_z \frac{\partial}{\partial z}\left(B_z \left.\frac{\partial}{\partial z}\right|_\psi R\right). \tag{12}$$

To reduce the complexity of the system, we note that

$$\nabla \cdot \boldsymbol{B} = \frac{\partial B_z}{\partial z} + \frac{\partial B_R}{\partial R} \approx \frac{\partial B_z}{\partial z} = 0, \tag{13}$$

and so B_z is constant (with respect to z) inside the layer. Then, we observe that

$$\begin{aligned} B_z \left.\frac{\partial}{\partial z}\right|_R &= B_z \left.\frac{\partial}{\partial z}\right|_\psi - B_z \left.\frac{\partial R}{\partial z}\right|_\psi \left.\frac{\partial}{\partial R}\right|_z \\ &= B_z \left.\frac{\partial}{\partial z}\right|_\psi - B_R \left.\frac{\partial}{\partial R}\right|_z \approx B_z \left.\frac{\partial}{\partial z}\right|_\psi \end{aligned} \tag{14}$$

Layer Solution: 3

Hence we have an equation purely along the field line:

$$m_i N_i \exp\left[\frac{m_s\omega^2}{2T_s}\left(R^2 - R_{\max}^2\right)\right]\omega^2 R = \frac{1}{\mu_0}\left(B_z^2\,\frac{\partial^2}{\partial z^2}\bigg|_{\psi} R\right). \tag{15}$$

Simplifying by assuming that R changes only by a small amount inside the layer, we write

$$R \approx R_{\max}(\psi) - \delta R \tag{16}$$

we can solve to find that

$$\delta R = \frac{R_{\max}}{2M^2}\ln\left[\cosh\left(8M^2\lambda\frac{z}{R_{\max}}\right)\right], \tag{17}$$

with $M = \omega R_{\max}/c_s$ and

$$\lambda = \left(\frac{4}{M^2}\frac{N_i m_i \omega^2 R_{\max}^2}{B_z^2/2\mu_0}\right)^{1/2} \tag{18}$$

Layer Solution: 4

Conesquences of this solution:
We can now calculate the density profile:

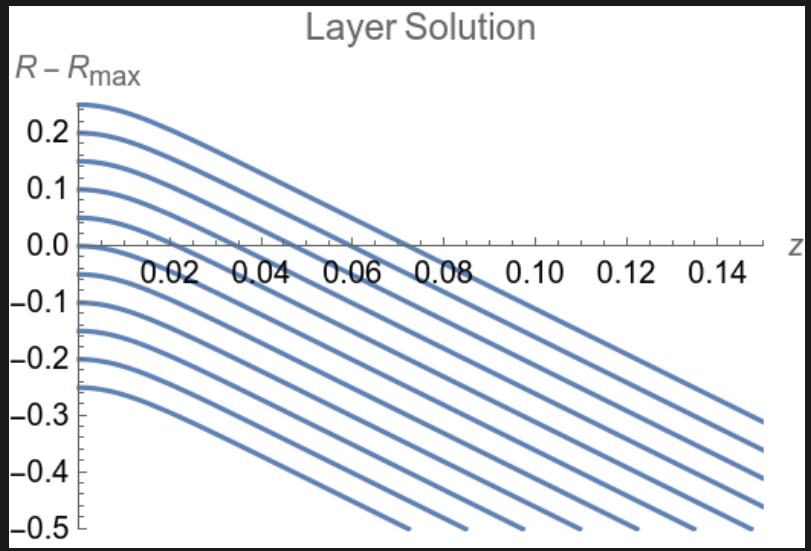
$$n_i = N_i \,\mathrm{sech}^2\left(8M^2\lambda\frac{z}{R_{\max}}\right) \qquad (19)$$

and calculate the field-line-averaged density $\bar{n}_i$ in terms of N_i to finally eliminate N_i:

$$N_i = \frac{\bar{n}_i}{R_{\max}} M^2 \bar{M}_A^2, \qquad (20)$$

where the average Alfven Mach number is

$$\bar{M}_A^2 = \frac{\bar{n}_i m_i \omega^2 R_{\max}}{B_z^2/2\mu_0} \qquad (21)$$



Exterior Solution

Now we need to solve outside the layer. Thankfully, our solution for n_i is such that it becomes a delta function (consistent with our assumptions). The current layer due to the plasma is

$$J_\phi = [B_R]_{0^-}^{0^+} = 8\lambda B_z \tag{22}$$

Greens Function

For a current layer at $z = 0$:

$$G(R, z, R') = \frac{1}{2\pi}\sqrt{(R + R')^2 + z^2}\left[\left(1 - k^2\right)K(k) - E(k)\right]J_\phi \tag{23}$$

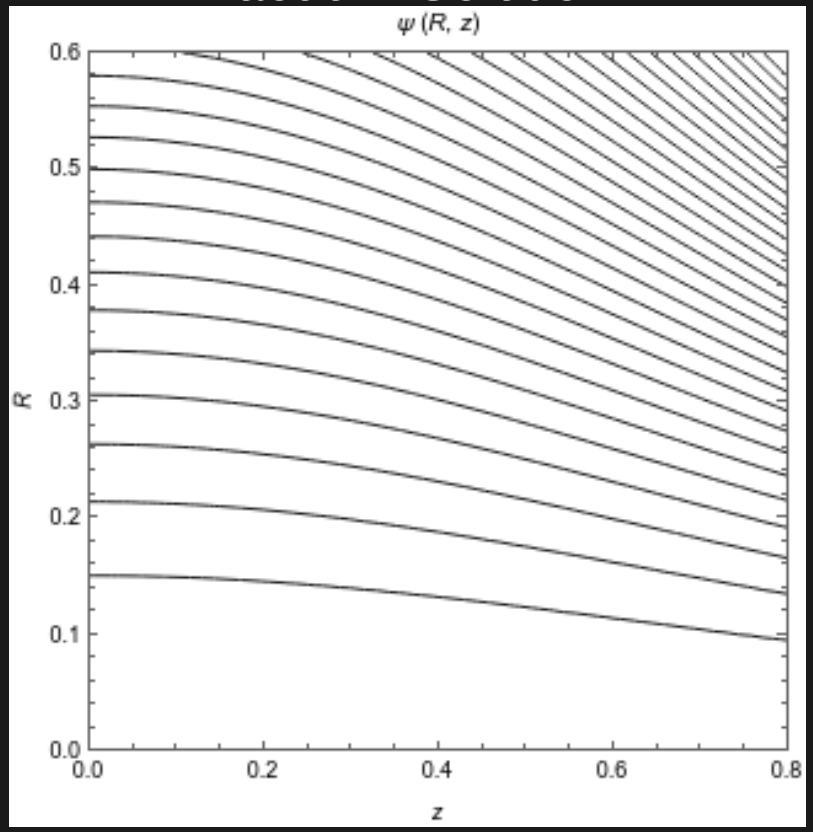
Giving

$$\psi(R, z) = \psi^{\text{coil}} + \int_a^b G(R, z, R')J_\phi(R')dR' \tag{24}$$

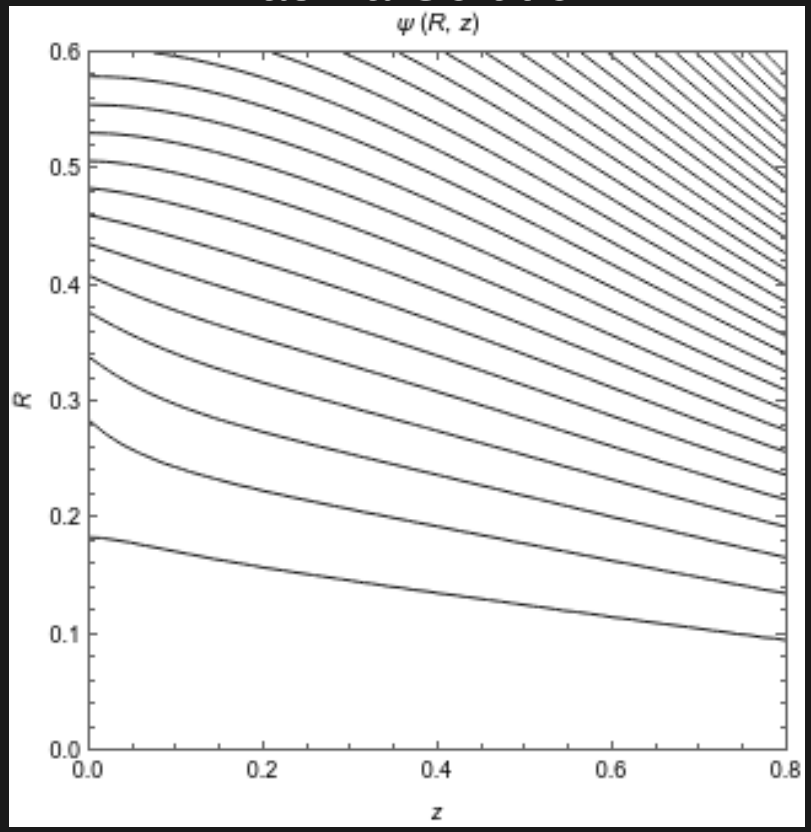
Integral Equation for ψ. Iterate the solution to find consistent ψ & B_z

Exterior Solution: 2

Vacuum Solution



Plasma Solution



Introduction

The Equations of Collisionless Equilibrium

Rapidly Rotating Solution

Summary

References

- Detailed calculation available from the author!
- 0D-Modelling available on github – http://github.com/IanAbel/MCTrans