Semiclassical Theory of the Koopman-van Hove Equation

Ilon Joseph

Lawrence Livermore National Laboratory

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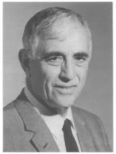
Motivation: Can wavefunctions live directly on phase space?

- **Can we represent classical mechanics using the same framework as quantum mechanics?**
  B. O. Koopman & J. von Neumann 1932

- **How do canonical symplectic transformations act on complex (wave) function spaces?**
  L. van Hove 1951

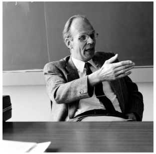
- **What is the mathematical structure behind quantization?**
  Pre-quantization before geometric quantization!
  B. Kostant 1972

Bernard Koopman



John von Neumann

Leon van Hove

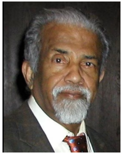


Bertram Kostant

Motivation: Can wavefunctions live directly on phase space?

- **Is there a classical or semiclassical wavefunction? and, if so, . . .**
- **Can we couple quantum & classical systems self-consistently?**
  E. C. G. Sudarshan 1976 $\rightarrow$ D. Bondar, F. Gay-Balmaz, C. Tronci PRSA 2019
- **How can we use quantum computers to simulate classical dynamics?**
  I. Joseph PRR 2020, et al POP 2023
- **Complete quantum algorithm for simulating advection-diffusion and Koopman-von Neumann dynamics!**
  I. Novikau & I. Joseph 2024 arxiv

EC George Sudarshan



Ceasare Tronci

Ivan Novikau

I. Joseph, Semiclassical Theory and the Koopman-van Hove (KvH) Equation

https://arxiv.org/abs/2306.01865 J. Phys. A: Math. Theor. 56 (2023) 484001 https://doi.org/10.1088/1751-8121/ad0533

- **Can wavefunctions live directly on phase space?**
  - **KvH equation** combines conservation of probability with nontrivial phase evolution in classical phase space
  - **Wigner-Weyl transform** applies to density matrices, not wavefunctions
- **Semiclassical derivation of KvH equation**
  - **Hamilton-Jacobi + Amplitude Transport equations = KvH** on configuration space
  - **Natural Injection:** KvH on configuration space $\rightarrow$ KvH on phase space
  - **Natural Projection:** *semiclassical KvH* on phase space $\rightarrow$ KvH on configuration space
- **KvH eigensystem for harmonic oscillator**
  - For Integrable systems, KvH spectrum = cartesian product of classical & semiclassical spectra
  - JWKB analysis eliminates classical spectrum and leads to **quantization** of the semiclassical spectrum
- **Semiclassical theory has much to say about decoherence!**

Classical dynamics has a natural linear representation on function spaces

- **Consider a set of ODEs for $z(\tau) \in \mathbb{R}^{2d}$**

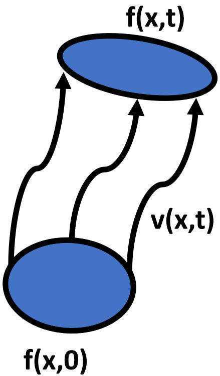
$$d\mathbf{z}/d\tau = \mathbf{V}(\tau, z)$$

- **Advection of a passive scalar $\theta(\tau, z)$**

$$d\theta/d\tau = (\partial_\tau + \mathbf{V} \cdot \boldsymbol{\nabla})\,\theta = 0$$

- **Along trajectory $z_\tau = \mathcal{T}_\tau(z_0)$, we have**

$$\theta_\tau = \mathsf{T}_\tau \theta_0 \qquad \theta_\tau(z_\tau) = \theta_0 \circ \mathcal{T}_\tau^{-1} \circ z_\tau = \theta_0(z_0)$$



Classical dynamics has a natural linear representation in terms of conservation of probability

- **Advection of a volume-form: Liouville's equation for probability density function (PDF)**
  - Assume $\mathbf{f}(\tau, z) = f(\tau, z)\mathbf{d}z^1 \wedge \mathbf{d}z^2 \cdots \wedge \mathbf{d}z^{2d}$

$$d\mathbf{f}/d\tau \;:=\; (\partial_\tau + \mathcal{L}_{\mathbf{V}})\mathbf{f} \;=\; 0 \quad \longleftrightarrow \quad df/d\tau + f\nabla \cdot \mathbf{V} \;=\; (\partial_\tau + \boldsymbol{\nabla}\cdot\mathbf{V})\, f \;=\; 0$$

- **Along trajectory $z_\tau = \mathcal{T}_\tau(z_0)$, we now multiply by the Jacobian**

$$\mathbf{f}_\tau = \mathsf{T}_\tau \mathbf{f}_0 \qquad f_\tau(z_\tau) = \mathcal{J}_\tau^0 f_0(z_0) \qquad \mathcal{J}_\tau^0 = \det\left(\frac{\partial z_0}{\partial z_\tau}\right)$$

The Koopman-von Neumann (KvN) equation evolves probability amplitudes

- **KvN evolves the square root of a volume form**
  - Assume $|\boldsymbol{\psi}|^2 = \mathbf{f}$ so that $\boldsymbol{\psi} = \mathbf{f}^{1/2}e^{i\theta} = \psi(\tau, z)\sqrt{\mathbf{d}z^1 \wedge \mathbf{d}z^2 \cdots \wedge \mathbf{d}z^{2d}}$ and $\psi = f^{1/2}e^{i\theta} \in \mathbb{C}$

$$d\boldsymbol{\psi}/d\tau := (\partial_\tau + \mathcal{L}_{\mathbf{V}})\boldsymbol{\psi} = 0 \quad \longleftrightarrow$$

$$\boxed{i\hbar\left(d\psi/d\tau + \tfrac{1}{2}\psi\nabla\cdot\mathbf{V}\right) = i\hbar\left(\partial_\tau + \tfrac{1}{2}\{\mathbf{V}\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}\cdot\mathbf{V}\}\right)\psi = 0}$$

- **Along trajectory $z_\tau = \mathcal{T}_\tau(z_0)$, we now multiply by sqrt Jacobian**

$$\boldsymbol{\psi}_\tau = \mathsf{T}_\tau\boldsymbol{\psi}_0 \qquad \psi_\tau(z_\tau) = (\mathcal{J}_\tau^0)^{1/2}\psi_0(z_0) \qquad \mathcal{J}_\tau^0 = \det\left(\frac{\partial z_0}{\partial z_\tau}\right)$$

The Koopman-van Hove (KvH) equation evolves wavefunctions that have phase dynamics

- **KvH also evolves the complex phase $\psi = \mathbf{f}^{1/2} e^{i\theta}$**
  - Phase has nontrivial dynamics
    $$\hbar d\theta/d\tau = \hbar \left(\partial_\tau + \mathbf{V} \cdot \boldsymbol{\nabla}\right) \theta = L(\tau, \mathbf{z})$$
  - Feynman's prescription for path integral chooses $L(\tau, \mathbf{z})$ **= classical Lagrangian**
    $$\hbar d\theta/d\tau = L(\tau, \mathbf{z}) = \mathbf{p} \cdot \nabla_{\mathbf{p}} H(\tau, \mathbf{z}) - H(\tau, \mathbf{z})$$
  - Combining phase evolution with Liouville equation for f yields KvH
    $$d\boldsymbol{\psi}/d\tau := (\partial_\tau + \mathcal{L}_{\mathbf{V}})\boldsymbol{\psi} = iL\boldsymbol{\psi}/\hbar \quad \longleftrightarrow$$
    $$\boxed{i\hbar \left(d\psi/d\tau + \tfrac{1}{2}\psi(\nabla \cdot \mathbf{V})\right) = i\hbar \left(\partial_\tau + \tfrac{1}{2}\{\mathbf{V} \cdot \boldsymbol{\nabla} + \boldsymbol{\nabla} \cdot \mathbf{V}\}\right) \psi = -L\psi}$$
- **Does this actually agree with semiclassical theory?** $\psi_\tau(z_\tau) = (\mathcal{J}_\tau^0)^{1/2} \psi_0(z_0) e^{i \int_0^\tau L d\tau/\hbar}$

Important Questions!

- **Is there a semiclassical derivation of the KvH and/or KvN equations?**
- **If so, does the physics agree with semiclassical theory?**
- **How accurate is KvN or KvH?**
- **What about quantization conditions?**
- **What about decoherence?**

Semiclassical analysis for generalized Schrödinger equation follows JWKB theory

- **Consider asymptotic JWKB analysis for generalized Schrödinger equation**

$$i\hbar\partial_\tau\psi = \hat{H}(\tau, \hat{x}, \hat{p})\psi$$

  - Position $\hat{x} = x \in \mathbb{R}^d$ and momentum $\hat{p} = -i\hbar\partial_x$ operators satisfy CCR: $[\hat{x}, \hat{p}] = i\hbar$

- **"Local" JWKB ansatz for wavefunction:**

$$\psi = Re^{i\theta} = e^{i\left(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2\right)} = e^{i\theta_0} R_1 R_2 \cdots$$

  - Asymptotic expansion: $1 \gg \epsilon$ implies $d\theta_0 \gg d\log R_1 \gg d\log R_2 \cdots$
  - Lowest order phase defines **classical action** $S = \hbar\theta_0$
  - Lowest order approximation to $\hat{p}\log\psi$ defines **classical momentum** $p := \partial_x S = \hbar\partial_x\theta_0$

- **Variational principle yields asymptotic JWKB result:** $H := \hat{H}(\tau, x, p)$

$$\mathcal{A}l[\psi, \psi^\dagger; \tau, x] = \int \psi^\dagger \left(i\hbar\partial_\tau - \hat{H}\right)\psi dx d\tau \simeq -\int \psi^\dagger \left(\partial_\tau S + H\right)\psi dx d\tau + \ldots$$

Semiclassical theory yields Hamilton-Jacobi equation & Hamilton's equations

- **At 0th order, we obtain the Hamilton-Jacobi equation (HJE) for the phase** $\psi = R\exp(i\bar{\theta}/\hbar)$

$$\hbar\partial_\tau\theta = \partial_\tau S = -H(\tau, x, p) \qquad\qquad p(\tau, x) := \partial_x S$$

- **Group velocity** derived from stationary phase approximation $dS = p \cdot dx - H d\tau$

$$\dot{x} := dx/d\tau = -\partial_p\partial_\tau S = \partial_p H\big|_x$$

- **Generalized forces** for evolution of momentum $p = \partial_x S$ derived using chain rule

$$\dot{p} := \frac{d}{d\tau}p(\tau, x) = \partial_\tau p + \dot{x}\cdot\partial_x p = -\left.\partial_x H\right|_{\tau,p(x)} + \partial_p H\big|_x \cdot \partial_x p = -\left.\partial_x H\right|_p$$

- **Infinitesimal HJE:** classical action is driven by the classical Lagrangian L

$$\frac{d}{d\tau}S(\tau, x) = \partial_\tau S + \dot{x}\cdot\partial_x S = -H + \partial_p H\big|_x \cdot p =: L$$

Semiclassical theory yields a nonlinear KvH equation on configuration space

- **At 1st order, we obtain a conservation law for probability** $f = |\psi|^2 = R^2$

$$df/d\tau = -f(\partial_x \cdot \dot{x}) \qquad \longleftrightarrow \qquad (\partial_\tau + \partial_x \cdot \dot{x})\, f = 0$$

- **This implies the amplitude transport equation (ATE) for the amplitude** $R = f^{1/2}$

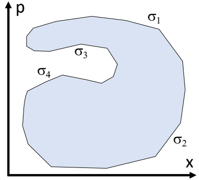
$$dR/d\tau = -\tfrac{1}{2} R(\partial_x \cdot \dot{x}) \qquad \longleftrightarrow \qquad \left(\partial_\tau + \tfrac{1}{2}\left\{\partial_x \cdot, \dot{x}\right\}\right) R = 0$$

- **Combining HJE and ATE generates the KvH equation ... but on configuration space!**

$$\hbar d\psi/d\tau = -\tfrac{1}{2}\hbar\psi(\partial_x \cdot \dot{x}) + iL\psi \qquad \longleftrightarrow \qquad \hbar\left(\partial_\tau + \tfrac{1}{2}\left\{\partial_x \cdot, \dot{x}\right\}\right)\psi = iL\psi$$

  - Configuration space KvH is **nonlinear in phase dynamics** due to dependance of $\dot{x}(\tau, x, p)$ on $p = \partial_x S$

There is a natural injection of configuration space into phase space



- **Solving for the trajectories requires solving for both position and momentum given initial data $x(\tau = 0)$, $p(\tau = 0)$**

$$\dot{x} = \partial_p H(\tau, x, p) \qquad \dot{p} = -\partial_x H(\tau, x, p)$$

- **A general solution of the HJE, $S(\tau, x, J)$, depends on d constants of the motion J**

$$\dot{J} := dJ/d\tau = 0$$

$$d/d\tau = \partial_\tau + \dot{z} \cdot \partial_z = \partial_\tau + \dot{x} \cdot \partial_x + \dot{J} \cdot \partial_J = \partial_\tau + \dot{x} \cdot \partial_x$$

- **For general solutions, the phase and amplitude are naturally functions of phase space $z = \{x, J\}$ and so is the wavefunction $\psi(\tau, z)$**

$$d\psi/d\tau = \left(\partial_\tau + \tfrac{1}{2}\left\{\dot{z}\cdot, \partial_z\right\}\right)\psi = \left(\partial_\tau + \tfrac{1}{2}\left\{\dot{x}\cdot, \partial_x\right\}\right)\psi = iL\psi/\hbar$$

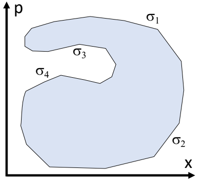
General asymptotic solution is a superposition of terms in the local asymptotic form

- **Local asymptotic solution has issues near classical turning points, caustics, etc.**
  - Multiple trajectories can have the same final position, when starting from different initial data $x(\tau = 0), \; p(\tau = 0)$

- **Linearity of Schrödinger's equation $\longrightarrow$ wavefunction = superposition of solutions**

- Thus, we can postulate a superposition of local asymptotic solutions on different branches σ

$$\psi(\tau, x) = \sum_\sigma R_\sigma e^{iS_\sigma/\hbar}$$



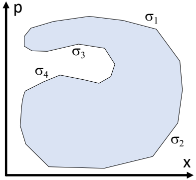
- Configuration space HJE is weakly nonlinear: only affects phase $\partial_\tau S = -H(\tau, x, \partial_x S)$

- **If we include the ability to form superpositions in phase space, then we arrive at a natural projection from phase space to configuration space**

$$\psi(\tau, x) = \sum_\sigma \int R_\sigma e^{iS_\sigma/\hbar} d^d p = \sum_\sigma \int R_\sigma e^{iS_\sigma/\hbar} \left(\det \partial p/\partial J\right) d^d J$$

A modified semiclassical KvH equation naturally projects onto configuration space

- **Natural projection from phase space $\longrightarrow$ configuration space**



$$\psi(\tau, x) = \int \psi^{\mathrm{SC}}(\tau, x, p) d^d p = \sum_\sigma \int \psi_\sigma^{\mathrm{SC}}(\tau, x, p) \left|\det \partial p / \partial J\right| d^d J$$

- **This requires ψ^{SC} to be a volume form in p and the square root of a volume form in x**

$$\boldsymbol{\psi}^{\mathrm{SC}}(\tau, \mathbf{z}) = \psi^{\mathrm{SC}} \mathbf{d}p^1 \wedge \mathbf{d}p^2 \cdots \wedge \mathbf{d}p^d \wedge \sqrt{\mathbf{d}x^1 \wedge \mathbf{d}x^2 \cdots \wedge \mathbf{d}x^d}$$

  - Modified amplitude transport for $R^{\mathrm{SC}} = |\psi^{\mathrm{SC}}|$

$$dR^{\mathrm{SC}}/d\tau = -R^{\mathrm{SC}} \left(\partial_p \cdot \dot{p} + \tfrac{1}{2} \partial_x \cdot \dot{x}\right) \qquad \longleftrightarrow \qquad \left(\partial_\tau + \partial_p \cdot \dot{p} + \tfrac{1}{2} \left\{\partial_x \cdot, x\right\}\right) R^{\mathrm{SC}} = 0$$

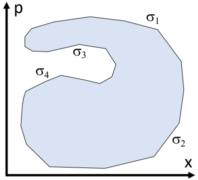
- **Modified semiclassical KvH equation on phase space**

$$\hbar d\boldsymbol{\psi}^{\mathrm{SC}}/d\tau = iL\boldsymbol{\psi}^{\mathrm{SC}} \qquad \longleftrightarrow \qquad \boxed{i\hbar \left(\partial_\tau + \partial_p \cdot \dot{p} + \tfrac{1}{2} \left\{\partial_x \cdot, \dot{x}\right\}\right) \psi^{\mathrm{SC}} = -L\psi^{\mathrm{SC}}}$$

Orthonormality & completeness are different in phase space vs. configuration space

- **Configuration space inner product**

$$\langle \phi \,|\psi \rangle = \int \phi^*(x)\psi(x)d^d x$$



- **Phase space inner product** $z = \{x, p\}$

$$\langle\langle \Phi \,|\Psi \rangle\rangle = \int \Phi^*(z)\Psi(z)d^{2d} z$$

- **Phase space eigenfunctions should still use configuration space inner product**

- **While we require $\psi(x) \in L^2$, we only require $\psi^{\mathrm{SC}}(z) \in L^1$, e.g. Dirac delta function in J**

On phase space, ψ^{SC} vanishes near turning points instead of blowing up

- **Standard KvH $\psi(\tau, z)$ is continuous in phase space $\longrightarrow$ jumps in configuration space**

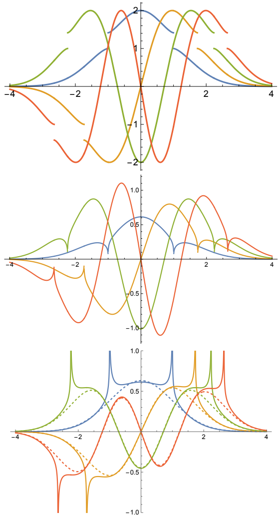
$$R(\tau, z) \sim \text{constant}$$
$$R(\tau, x) \sim \text{piecewise continuous}$$

- **Semiclassical KvH $\psi^{\mathrm{SC}}(\tau, z)$ has 1/sqrt(Van Vleck det) for the amplitude**

$$R^{\mathrm{SC}}(\tau, z) \sim (\det \partial J/\partial p)^{1/2} \sim p^{1/2}$$

- **For JWKB, $\psi(\tau, x)$, integrating over $d^d p = |\det \partial p/\partial J| d^d J$ multiplies by Van Vleck so the total has sqrt(Van Vleck det)**

$$R(\tau, x) \sim (\det \partial p/\partial J)^{1/2} \sim p^{-1/2}$$



KvH analysis and JWKB analysis can be made to agree, but requires care!

- **Semiclassical analysis has some subtleties**
  - **Jeffreys-Wentzel-Kramers-Brillouin (JWKB)** phase matching conditions = boundary conditions at classical turning points & mode coupling points
  - **Einstein-Brillouin-Keller (EBK)** quantization: $J/\hbar = n + \mu/4$ where $\mu \in \mathbb{N}$ is the Maslov index
  - Tunneling requires extending analysis to complex phase space
  - Completeness & orthonormality relations defined in configuration space

- **KvH analysis has both classical and semiclassical parts to the physics**
  - Classical spectrum should be eliminated to reproduce JWKB
  - JWKB boundary conditions should be applied to get correct phases, Maslov index, & EBK quantization conditions
  - Completeness & orthonormality relations usually considered on phase space, but should actually be defined in configuration space!

For integrable systems, the KvH spectrum is complete over phase space

- **An integrable system has action-angle coordinates $\dot{\vec{J}} = 0$ and $\dot{\vec{\theta}} = \vec{\Omega}(J) = \vec{\partial}_J H(J)$**

- **Standard KvH eigenfunctions on phase space have the form**

$$\phi_{\vec{J},\vec{m}}(\tau, z) = R_{\vec{J},\vec{m}} \exp\left(i\left[\vec{m} \cdot (\vec{\theta} - \vec{\Omega}\tau) + S/\hbar\right]\right)$$

  - **Classical spectrum** arises from amplitude transport: $\partial_\tau \psi + \vec{\Omega} \cdot \vec{\partial}_\theta \psi = 0 \quad \longrightarrow \quad \omega = \vec{m} \cdot \vec{\Omega}$
  - **Semiclassical spectrum** arises from Hamilton-Jacobi equation: $\hbar\omega = -\partial_\tau S = H(J)$

- **KvH eigenfunctions $\phi_{\vec{J},\vec{m}}(\tau, z)$ are complete over phase space**
  - Classical spectrum is complete over θ, while semiclassical spectrum is complete over J
  - The classical & semiclassical spectra are **continuous** due to the ability to choose **non-integer** values of $\vec{m}$

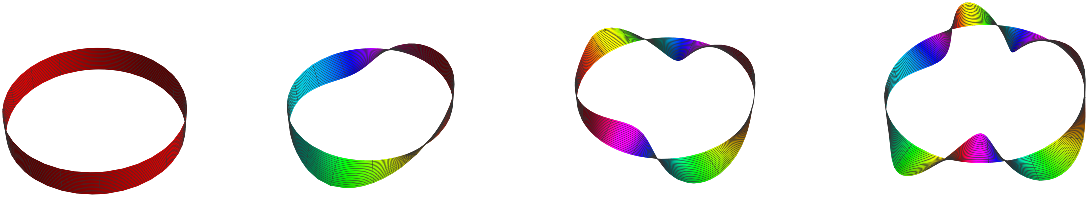
The KvH spectrum is quantized by eliminating classical spectrum!

- **Phase space KvH eigenfunctions are overcomplete in configuration space**

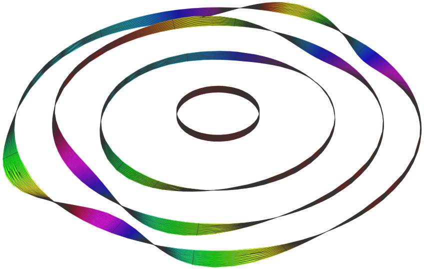
- **JWKB theory uses HJE and eliminates ATE:**
  asymptotic expansion eliminates classical spectrum by requiring $\vec{m} = 0$
  - Eliminates classical spectrum
  - Semiclassical spectrum becomes discrete
  - Eliminates completeness in phase space
  - Retains completeness in configuration space

- **Quantization conditions for J demand continuity of the phase around closed loops**
  - Standard KvH yields Bohr-Sommerfeld quantization: $J/\hbar = n \in \mathbb{N}$
  - Semiclassical KvH uses JWKB BCs for the phase across turning points which yields
  - Einstein-Brillouin-Keller (EBK) quantization: $J/\hbar = n + \mu/4$ where μ is the Maslov index

- $n = 0$ $\quad n = 1$ $\quad n = 2$ $\quad n = 3$ $\quad \ldots$

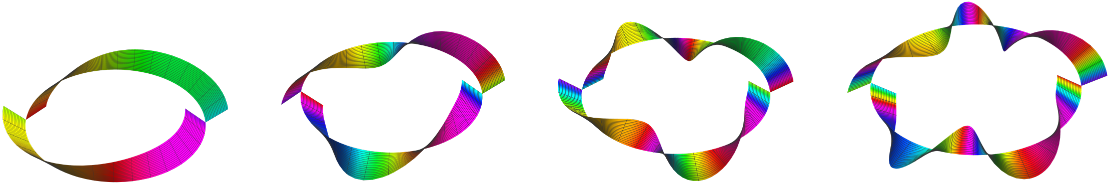


- **Superposition of semiclassical eigenfunctions including** $n = 0, \ldots, 3$



Harmonic Oscillator: Semiclassical KvH $\longrightarrow$ Einstein-Brillouin-Keller quantization

- $n = 0$ $\qquad n = 1$ $\qquad n = 2$ $\qquad n = 3$ $\qquad \ldots$



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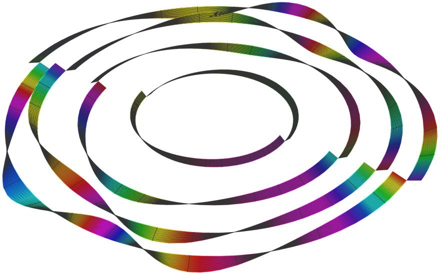
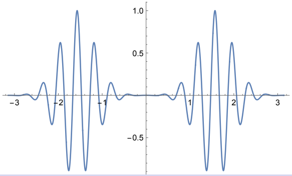


Semiclassical theory provides an explanation for the Cophenhagen interpretation

- **Why does classical probability theory arise from quantum theory?**
  - Why are superpositions forbidden classically but required quantumly ?
- **Lowest order semiclassical theory only has local observables and local observables $\longrightarrow$ classical probability**
  - Lowest order semiclassical theory treats all observables as local in phase space $|z\rangle = |x, p\rangle$
  - Born's rule implies that probability is classical in the eigenbasis of the observable $\hat{O}\,|z\rangle = O(z)\,|z\rangle$

$$\left\langle \hat{O} \right\rangle = \mathrm{Tr}\left(\hat{O}\hat{\rho}\right) = \sum_{yz} \langle y|\,\hat{O}\,|z\rangle \,\langle z|\,\hat{\rho}\,|y\rangle = \sum_{z} O(z)\,\langle z|\,\hat{\rho}\,|z\rangle = \sum_{z} O(z) f(z)$$

- **Semiclassical theory can certainly describe interference effects, but this requires going beyond the lowest order theory to perform "nonlocal operations"**
  - JWKB eigenfunctions certainly display interference
  - **Copenhagen macro/micro dividing line = size of the wavepackets!**



Why is it so hard to maintain quantum coherence? Semiclassical theory provides an explanation

- **Answer**: Semiclassical phase is very large & plays an important role in coupled semiclassical-quantum systems

$$\delta S/h \gg 1$$

- **Statistical:** Many moving parts means that small perturbations to the action create order unity changes in the phase

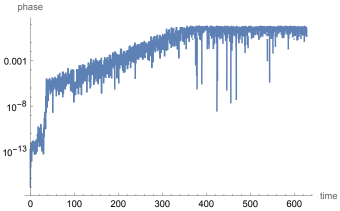
$$\delta S/h \sim \mathcal{O}(1)$$

- **Dynamical:** Chaotic trajectories generate exponential separation in the phase. If λ = Lyapunov exponent, then

$$\delta S \sim \delta S(\tau = 0)\exp(\lambda\tau)$$

- **Collisionless Damping:** decoherence likely controlled by collisionless processes

$|\delta S/h|$ for two chaotic trajectories



Conclusion: Semiclassical Theory of the Koopman-van Hove (KvH) Equation

- **The KvH equation arises from the first two orders of asymptotic JWKB theory**
  - Simply combines the infinitesimal Hamilton-Jacobi equation with the amplitude transport equation
  - **Natural Injection:** from configuration space into phase space linearizes KvH
  - **Natural Projection:** from phase space back to configuration space requires **semiclassical KvH equation**
- **Semiclassical phase space KvH has the same accuracy as configuration space JWKB**
  - Standard KvH is correct to 0th order but not 1st order: Bohr-Sommerfeld quantization
  - Semiclassical KvH is accurate through 1st order including the Maslov index & EBK quantization
  - Like JWKB, loses accuracy near turning points and caustics, etc.
- **Accurate coupling of quantum & semiclassical wavefunctions is now possible**
  - Semiclassical phase play an important role in the dynamiccs
  - Provides a physical explanation for decoherence

Outlook

- **Mathematical aspects that require further investigation**
  - Is there a symplectically invariant description for KvH? Is there a symplectically invariant asymptotic theory? Lopez& Dodin
  - Better understanding of general solutions of the Hamilton-Jacobi equation for finite times
  - Tunneling and resonance phenomena: Analytic continuation and complexification of phase space
- **Goal: Self-consistent coupling of quantum and semiclassical systems**
  - Zeroth order theory developed by Bondar, Gay-Balmaz, Tronci PRSA 2019 requires first order corrections in order to achieve semiclassical accuracy
  - Self-consistent theory accurate to first order developed by Littlejohn & Flynn in early 1990's
  - Extended by Dodin, Ruiz, & Fu over last decade to include accurate spin-orbit coupling effects

Advanced Topics

- **Can phase space & configuration space inner products agree with each other?**
  Sqrt delta function $\delta^{1/2}(z)$
- **Tunneling & Resonances:** Analytic continuation of phase space
- **Multicomponent wavefunctions = quantum-classical coupling:**
  Mode Coupling, Spin-Orbit Coupling, Quantum measurement theory