Resistive hose modes in Tokamaks and stability of sinusoidal flows in the viscous Hasegawa-Mima equation

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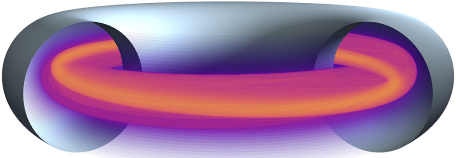


Runaway electron generation may be understood via Coulomb collision physics.

The presence of a strong electric field can accelerate a substantial number of electrons beyond the point where they can be slowed by collisions.

$$e|\boldsymbol{E}| \gtrsim mv\nu(v) = \frac{e^4 n \ln\lambda}{4\pi\epsilon_0^2 m v^2}$$

Classically, for a given electric field strength, any electrons with $v^2 \gtrsim \frac{e^3 n \ln\lambda}{4\pi\epsilon_0 m_e}$ would be expected to runaway. Considering relativistic effects sets a lower limit on the electric field strength[1].

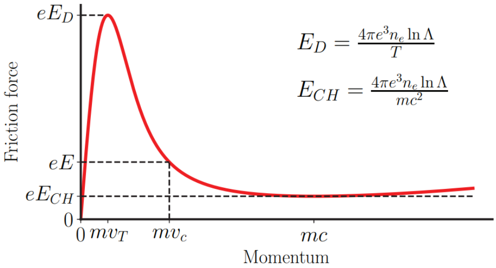


Figure: Friction force resulting from collisions as a function of electron momentum[2].

[1] J. W. Connor and R. J. Hastie, Nuclear Fusion **15**, 415–423 (1975).
[2] B. N. Breizman et al., Nuclear Fusion **59**, 10.1088/1741-4326/ab1822 (2019).

REs are produced during a tokamak thermal quench.

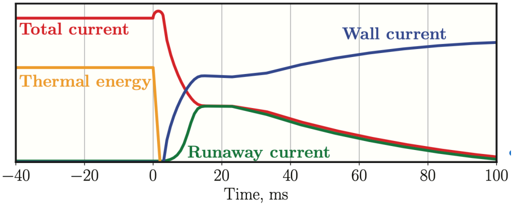


Figure 9. Anticipated evolution of the plasma thermal energy (orange), total plasma current (red), RE current (green) and the wall current (blue) in ITER. At 20 ms the density of the background plasma increases to facilitate mitigation of the RE current.

Rapid cooling of the plasma increases resistivity $\implies E_{||} = \eta J_{||}$ is large.

Electrons are accelerated into the runaway region, RE current plateau is reached.

RE current is eventually terminated - possibly damaging PFCs. This figure from Breizman illustrates the process schematically[a].

[a] B. N. Breizman et al., Nuclear Fusion **59**, 10.1088/1741-4326/ab1822 (2019).

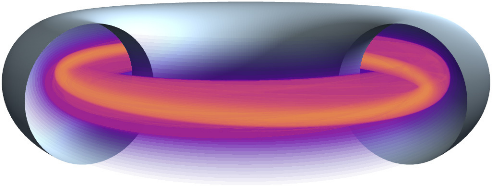
Stability of the beam+plasma system during the current plateau has been investigated.

- Lovelace[3], and Spong[4], et al. deal with stability of the a beam in a perfectly conducting background plasma.
- Avinash[5], Helander[6], and Liu[7] consider resistive instabilities in the presence of runaways.
- Liu et al. find that the runaways introduce a real frequency to the $(2, 1)$ tearing mode.

[3] R. V. Lovelace, Physics of Fluids **19**, 723–737 (1976).
[4] D. A. Spong et al., Plasma Physics **19**, 817–838 (1977).
[5] Avinash and P. Kaw, Nuclear Fusion **28**, 10.1088/0029-5515/28/1/009 (1988).
[6] P. Helander et al., Physics of Plasmas **14**, 10.1063/1.2817016 (2007).
[7] C. Liu et al., Physics of Plasmas **27**, 10.1063/5.0018559 (2020).

RE beams generated during disruptions in a tokamak can be unstable to resistive hose modes.

- Hose modes is overstable, with $|\omega| \gg \gamma$.
- Resistivity scaling of linear growth rates for a tearing unstable equilibrium shows the fastest growing mode transitions from $m = 2$ to $m = 1$ at large resistivity.
- Dominance at high resistivity suggests the hose mode may be important in post-thermal quench tokamak scenarios.



Energetic particle beams in a background plasma are susceptible to long-wavelength 'resistive hose' instabilities[8].

The literature considers self-pinched beams of relativistic electrons moving through a resistive neutralizing background plasma. The fundamental instability picture is

- perturbed particle orbits generate perturbed RE current.
- ohmic currents induced in the bulk plasma decay.
- The imperfect cancellation of the perturbed RE current by the ohmic plasma currents allows an overstable oscillation.

Some relevant key features:

- Unstable modes are essentially transverse to the beam axis.
- The perturbed motion of the background plasma is negligible (other than neutralizing the beam).

[8] M. N. Rosenbluth, Physics of Fluids **3**, 932–936 (1960); S. Weinberg, Journal of Mathematical Physics **8**, 614–641 (1967).

The fluid model of runaway electrons evolves a beam-like distribution with density n_r - including volumetric sources.

Continuity equation for runaway electron population:

$$\frac{\partial n_r}{\partial t} + \boldsymbol{\nabla} \cdot (n_r \boldsymbol{u_r}) = S_D(\mathcal{E}_{||}) + S_A(E_{||}) + D_r \nabla^2 n_r, \tag{1}$$

where n_r is the number density of runaways, S_D, S_A are sources, D_r is a numerical diffusion coefficient and

$$\mathcal{E}_{||} \equiv \frac{E_{||}}{E_D}, \quad \boldsymbol{u_r} = -c_r \hat{\mathbf{b}} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}, \quad c_r = \text{const.} \gg v_{th,e}, v_A$$

$$E_D = \frac{n_e e^3 \ln \Lambda}{4\pi \epsilon_0 T_e}.$$

The source terms model the Dreicer and avalanche mechanisms.

The model for the Dreicer source S_D is that presented by Connor and Hastie[9]

$$S_D = n_e^2 \frac{e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_e^2 v_{th}^3} \mathcal{E}_{||}^{-\frac{3}{16}(1+Z_{eff})} \exp\left\{ -\frac{1}{4\mathcal{E}_{||}} - \sqrt{\frac{1+Z_{eff}}{\mathcal{E}_{||}}} \right\}$$
$$\times \exp\left\{ -\frac{T_e}{mc^2} \left(\frac{1}{8\mathcal{E}_{||}^2} + \frac{2}{3}\sqrt{\frac{1+Z_{eff}}{\mathcal{E}_{||}^3}} \right) \right\}$$

The avalanche source S_A is given by Rosenbluth and Putvinski[10]:

$$S_A = \frac{n_r}{\tau} \sqrt{\frac{\pi a}{3(Z+5)}} \left(\frac{E_{||}}{E_c} - 1 \right) \left(1 - \frac{E_c}{E_{||}} + \frac{4\pi(Z+1)^2}{3a(Z+5)(E_{||}^2/E_c^2 + 4/a^2 - 1)} \right)^{-1/2}$$

$$E_c = \frac{n_e e^3 \ln\Lambda}{4\pi\epsilon_0^2 mc^2}, \quad a(\varepsilon) = (1 + 1.46\sqrt{\varepsilon} + 1.72\varepsilon)^{-1}, \quad \tau = \frac{mc\ln\Lambda}{eE_c}$$

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[9] J. W. Connor and R. J. Hastie, Nuclear Fusion **15**, 415–423 (1975)

[10] M. N. Rosenbluth and S. V. Putvinski, Nuclear Fusion **37**, 1355–1362 (1997)

The runaway electrons couple to the MHD evolution via a modified Ohm's law.

$$\mathbf{E} = -\boldsymbol{V} \times \boldsymbol{B} + \eta \left(\boldsymbol{J} + e n_r \boldsymbol{u}_r \right) \tag{2}$$

This Ohm's law is valid for

$$\frac{n_r}{n_e} \ll 1, \qquad m_e n_r \frac{\mathrm{d}\boldsymbol{u}_r}{\mathrm{d}t} \approx 0$$

- This model is similar the model employed in Bandaru[11] and Matsuyama[12], and the model in M3D-C1[13].

[11] V. Bandaru et al., Physical Review E **99**, 1–11 (2019).

[12] A. Matsuyama et al., Nuclear Fusion **57**, 10.1088/1741-4326/aa6867 (2017).

[13] C. Zhao et al., Nuclear Fusion **60**, 10.1088/1741-4326/ab96f4 (2020).

Here we linearize the equations around a 0-β equilibrium supported by runaway electron current.

$$\partial_t n_r + \boldsymbol{\nabla} \cdot (n_r \boldsymbol{U}_r + N_r \boldsymbol{u}_r) = 0,$$
$$\rho \partial_t \boldsymbol{v} = \boldsymbol{j} \times \boldsymbol{B} + \boldsymbol{J} \times \boldsymbol{b}, \quad \partial_t \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{e}$$
$$\boldsymbol{\nabla} \times \boldsymbol{b} = \mu_0 \boldsymbol{j},$$
$$\boldsymbol{e} = -\boldsymbol{v} \times \boldsymbol{B} + \eta \left(\boldsymbol{j} - \boldsymbol{j}_r\right)$$
$$\boldsymbol{j}_r = -e \left(n_r \boldsymbol{U}_r + N_r \boldsymbol{u}_{\boldsymbol{r}}\right)$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$$

Uniform constant plasma density, ρ, and uniform, constant resistivity η. No equilibrium flow: $\boldsymbol{V} = 0$. $c_r > 0$ is the parallel speed of runaway electrons along magnetic field lines. Source terms and RE drift effects are neglected in this linear analysis.

The perturbe RE velocity is complicated by including the perpendicular drift.

Simultaneous solution of Ohm's law and $\boldsymbol{u}_r = \boldsymbol{E} \times \boldsymbol{B}/B^2$ gives

$$\boldsymbol{u}_{r,\perp}(1+\alpha^2) = \boldsymbol{v}_\perp - c_r \frac{\boldsymbol{b}_\perp}{B} - \eta\alpha \boldsymbol{j}_\perp + \left(\eta \frac{\boldsymbol{j}}{B} + \alpha \boldsymbol{v} - \alpha c_r \frac{\boldsymbol{b}}{B}\right) \times \frac{\boldsymbol{B}}{B} \tag{3}$$

$$\alpha \equiv \frac{\eta e N_r}{B} \tag{4}$$

Note that $\alpha \ll 1$ for experimentally relevant η, N_r and B. At present, we neglect terms from the $\boldsymbol{E} \times \boldsymbol{B}$ drift

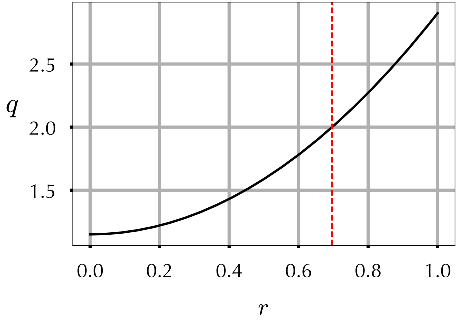
$$\boldsymbol{u}_r = -c_r \frac{\boldsymbol{b}_\perp}{B} \tag{5}$$

We also do not consider sources.

Cylindrical equilibrium taken from Liu et. al[14] was used to benchmark implementation.

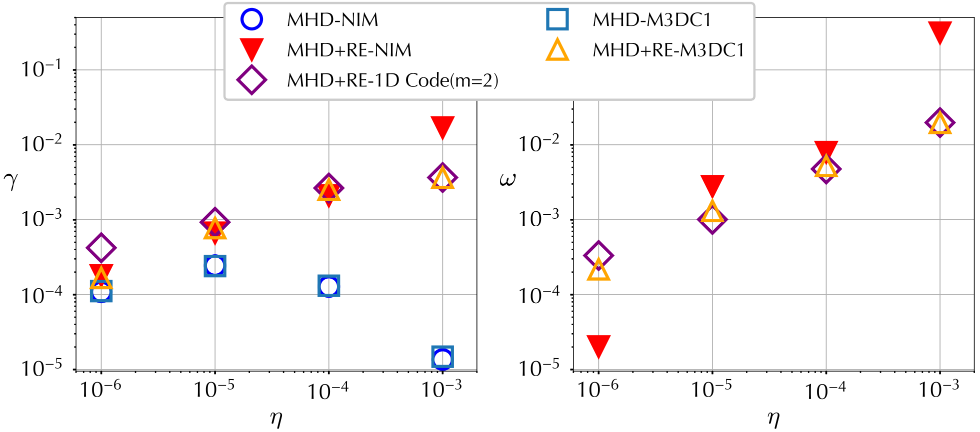
$$L = 20\pi, \quad a = 1,$$
$$B_z(r = 0) = 1, \quad \mu_0 \rho = 1,$$
$$\implies V_A = 1,$$
$$c_r = 20V_A, \ \beta = 0,$$
$$q(r) = 1.15(1 + (1.234r)^2)$$

- All the equilibrium current is carried by runaways
- In resistive MHD the most unstable mode is the $m = 2$ tearing mode



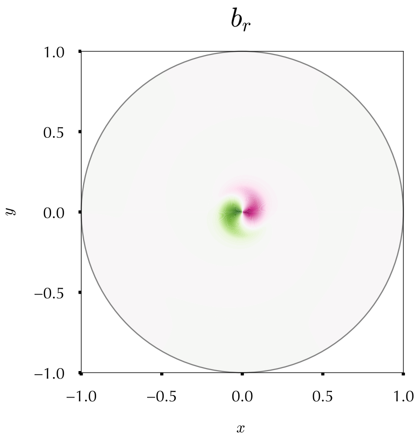
[14] C. Liu et al., Physics of Plasmas **27**, 10.1063/5.0018559 (2020).

Both NIMROD and eigenvalue code reproduces published result on tearing with REs for small η.



NIMROD sees a different fastest growing mode at higher resistivity.

- Eigenfunction for the most unstable $n = 1$ mode at $\eta = 10^{-3}$ is localized near $r = 0$.
- These modes are also oscillatory, but have a $\omega_r \gg \gamma$.



Neglecting velocity perturbations does not affect the hose mode.

Set $\boldsymbol{v} \sim 0$, and change variables to $\Lambda_r = \mu_0 e c_r N_r / B$, $\lambda_r = \mu_0 e c_r n_r / B$, then since $\partial_t \boldsymbol{B} = 0$:

$$\partial_t \lambda_r - \frac{c_r}{B} \left(\boldsymbol{B} \cdot \nabla \lambda_r + \boldsymbol{\nabla} \cdot (\Lambda_r \boldsymbol{b}_\perp) \right) = 0, \tag{6}$$

$$\partial_t \boldsymbol{b} = \eta \boldsymbol{\nabla} \times (\lambda_r \boldsymbol{B} + \Lambda_r \boldsymbol{b}_\perp) - \eta \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}. \tag{7}$$

We can find normal modes of this simplified system in cylindrical geometry with the 1D ODE eigenvalue code.

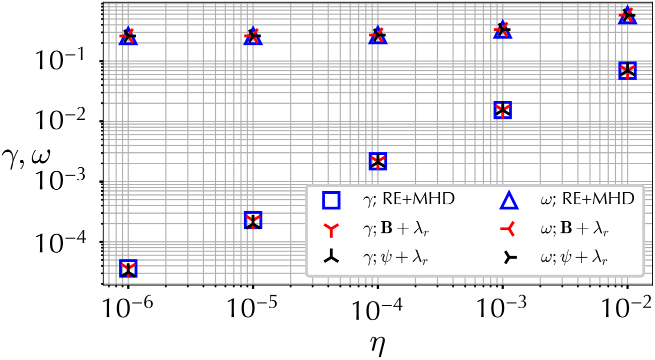
A further reduced model reveals that the instability is primarily driven by the gradient in the equilibrium runaway current.

Since the axial component $|b_z| \ll |b_r|, |b_\theta|$, simplify the system via $\boldsymbol{b} = \nabla\psi \times \hat{\boldsymbol{z}}$. Additionally, we neglect terms of order $r/R \sim B_\theta/B$, and assume $B_z \sim B_0 = \mathrm{const.}$ (large-aspect ratio).

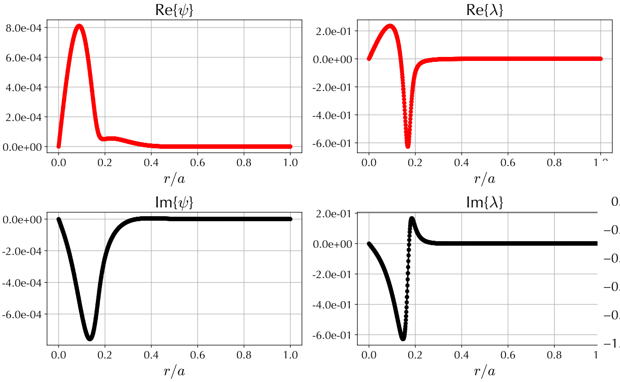
$$
\begin{aligned}
iF(r) &\equiv \frac{\boldsymbol{B}\cdot\nabla}{B},\\
(\omega + c_r F(r))\lambda_r &= -c_r \frac{m\Lambda_r'}{r}\frac{\psi}{B},\\
\omega\psi - i\eta\left(\frac{1}{r}(r\psi')' - \frac{m^2}{r^2}\psi\right) &= i\eta B_z \lambda_r.
\end{aligned}
$$

The reduced models are sufficient to describe the instability.

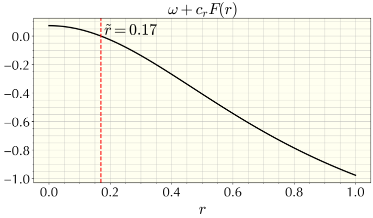
- EV calculations of growth rates and frequencies of the beam mode with $m = 1$ agree between all three models.
- The ability to neglect $\boldsymbol{v}$ means that it is not an MHD kink mode, but true a beam-plasma mode.



Radial profiles of reduced model eigenfunctions give some insight.



- The radial structure is localized within a region of width $\tilde{r}$ around the origin.
- The location of $\tilde{r}$ seems to be determined by a resonance of $\mathrm{Re}\{\omega\} + c_r F(\tilde{r}) = 0$.



Uniform beam case is analytically tractable in the ψ, λ system

$\Lambda = \Lambda_0(1 - H(1-\rho_b))$, where $H(\rho)$ is the Heaviside function. $\rho = r/a$. Then the ψ equation is

$$\psi'' + \frac{1}{\rho}\psi' + \left(\kappa^2 - \frac{m^2}{\rho^2}\right)\psi = -\frac{B_z}{B}\frac{\alpha m a \Lambda_0}{\rho}\frac{\delta(\rho-\rho_b)}{\sigma a^2 \omega + \alpha a F}\psi.$$

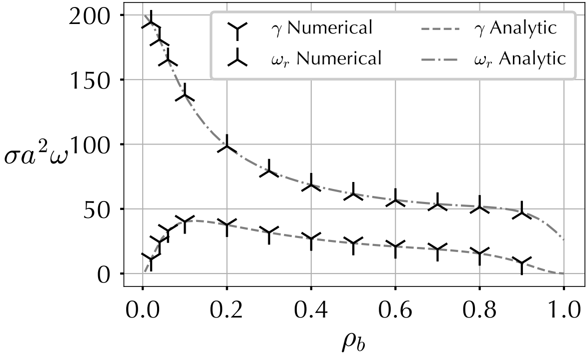
$$\alpha \equiv \sigma a c_r\ ,\ \kappa^2 \equiv i\sigma a^2\omega - a^2 k^2.$$

$$\psi = c_0 J_m(\kappa\rho), \quad 0 \leq \rho \leq \rho_b,$$
$$\psi = c_1 J_m(\kappa\rho) + c_2 Y_m(\kappa\rho), \quad \rho_b \leq \rho \leq 1.$$

The dispersion relation is

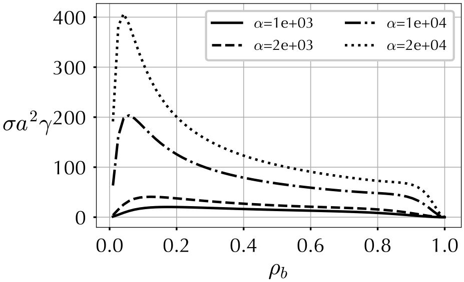
$$\sigma a^2\omega + \alpha a F = \frac{m\pi\alpha a\Lambda_0}{2}\left(Y_m(\kappa)J_m(\kappa\rho_b) - Y_m(\kappa\rho_b)J_m(\kappa)\right)\frac{J_m(\kappa\rho_b)}{J_m(\kappa)},$$

Solutions to dispersion relation match numerical growth rates and frequencies.

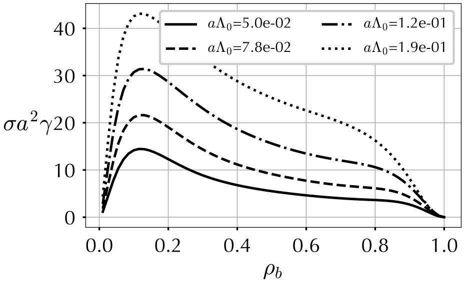


Analytic dispersion as a function of ρ_b at $\alpha = 2 \times 10^2, \Lambda_0 = 0.174$, $m = 1, ak = -0.1$

Maximum growth rate depends on α and Λ_0.



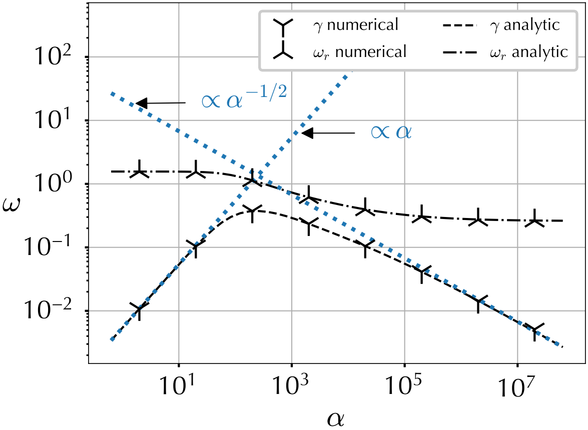
Analytic dispersion as a function of ρ_b at several α values and $\Lambda_0 = 0.174$



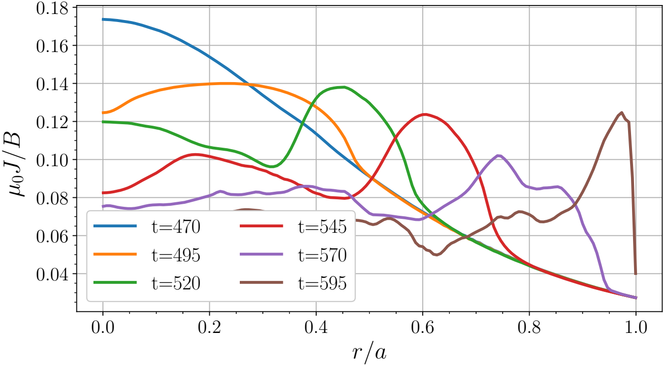
Analytic dispersion as a function of ρ_b at several $a\Lambda_0$ values and $\alpha = 2 \times 10^3$

Dispersion relation gives insight into asymptotic scaling of growth rates at large α.

- Here ω is plotted, not $\sigma a^2 \omega$.
- frequency and growth rate for different α values at $\rho_b = 0.5$ and $a\Lambda_0 = 0.174$,$m = 1, ak = -0.1$

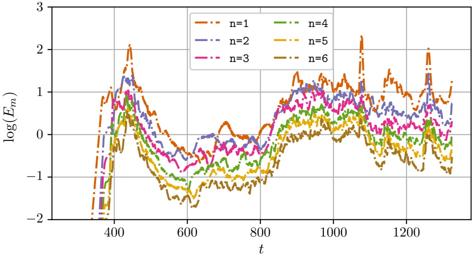


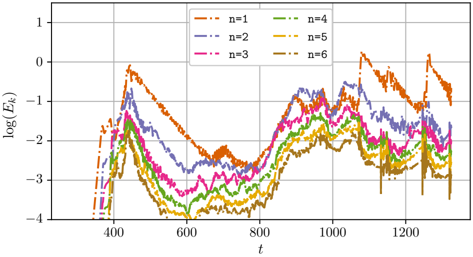
Nonlinear evolution of the hose instability results in a spreading of the parallel current



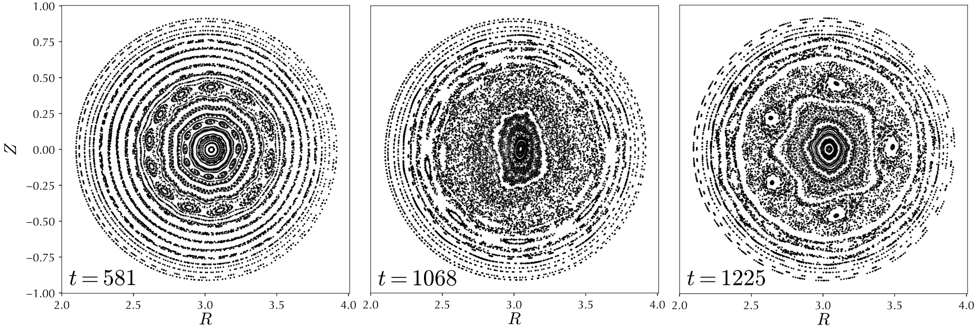
Cylindrical nonlinear calculation with $\boldsymbol{v} = 0$, $\eta = 10^{-2}$. Flux surface-averaged parallel current profiles are shown at several times as the $n = 1$ mode starts to saturate.

Toroidal nonlinear case shows some intermittency later in time.

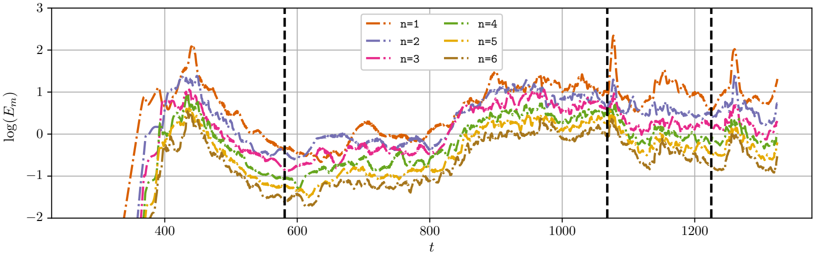




- Bursts of $n = 1$ activity ~ 600 after initial saturation.
- Fluctuating magnetic energy (left) is much larger than kinetic. $\implies$ $\boldsymbol{v}$ plays little role
- $R/a = 3$

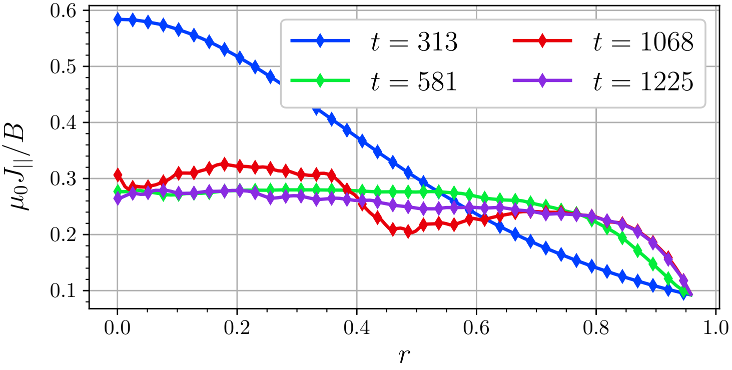


$t = 581$
$t = 1068$
$t = 1225$
Z
R
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
2.0
2.5
3.0
3.5
4.0



n=1
n=2
n=3
n=4
n=5
n=6
$\log(E_m)$
t
3
2
1
0
−1
−2
400
600
800
1000
1200

Flattening of parallel current profile does not extend to the wall in toroidal case.



Conclusions

- Runaway electron beams in a post-thermal quench tokamak can be unstable to resistive hose modes.
- These modes can exhibit strong scaling with the resistivity background plasma, and exist even when $\boldsymbol{v} = 0$.
- The reduced model for the large-aspect ratio cylinder with uniform Λ provides a semi-analytic dispersion relation.
- Nonlinear calculations show current profile flattening, and changes in magnetic topology in the core.

Viscous instability of zonal flows in Hasegawa-Mima

Hasegawa-Mima models 2-D electrostatic dynamics in a magnetized plasma[15]

- Uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$,electrostatic $\boldsymbol{E} = -\nabla\varphi$
- Ion drift motion in the x, y plane, $\boldsymbol{u} \approx \boldsymbol{v}_E + \boldsymbol{v}_p$
- Adiabatic electron response: $\delta n_e = -n_0 e\varphi/T_e$.
- uniform $T_i = T_e$, nonuniform $n_i = n_0(x)$
- Units: $\Omega_c \equiv eB/m_i, c_s^2 \equiv T_e/m_i, \rho_s = c_s\Omega_c$

$$(1 - \nabla^2)\partial_t\varphi - v_d\partial_y\varphi - \nabla\varphi \times \hat{\boldsymbol{z}} \cdot \nabla\nabla\varphi = 0 \tag{8}$$

Where $v_d \equiv c_s\rho_s n_0/n_0'$ is the normalized diamagnetic drift velocity

[15] K. Mima and A. Hasegawa, Physics of Fluids **21**, 81–86 (1978).

Viscosity resulting from collisions can be included

Adding collisional ion perpendicular ion viscosity $\mu\nabla_\perp^4\varphi$

$$(1-\nabla^2)\partial_t\varphi - v_d\partial_y\varphi - \nabla\varphi\times\hat{\boldsymbol{z}}\cdot\nabla\nabla\varphi + \mu\nabla_\perp^4\varphi = 0 \tag{9}$$

- μ is the viscosity normalized by c_s^2/Ω_c.
- This model is limited to small $k_\perp$, uniform $\boldsymbol{B}$, and weak density gradient.

Linearizing for perturbations around a steady zonal flow $U(x) = u_0 \sin(x/L)\hat{\boldsymbol{y}}$

Set $\varphi \to \Phi + \varphi$, where $\Phi = \int U(x)dx$, and $|\varphi| \ll |\Phi|$

$$(1-\nabla^2)\partial_t\varphi + U(1-\nabla^2)\partial_y\varphi + (U'' - \beta)\partial_y\varphi + \mu\nabla_\perp^4\varphi = 0 \tag{10}$$

Define new units based on flow: $x, y \to L\hat{x}, L\hat{y}$, $t \to L\hat{t}/u_0$, and $\phi \to L\hat{\varphi}/u_0$.

$$\left(\partial_{\hat{t}} + \sin(x)\partial_{\hat{y}}\right)\left(L^2 - \nabla^2\right)\hat{\varphi} - \left(\sin(x) - L^2\beta\right)\partial_{\hat{y}}\hat{\varphi} + \frac{1}{\mathrm{Re}}\nabla^4\hat{\varphi} = 0 \tag{11}$$

Where now $\beta \equiv v_d/u_0$ and $\mathrm{Re} = u_0 L/\mu$ is the Reynolds number associated with the zonal flow. Zhu, et al. treat this exact problem in the inviscid case[16]

[16] H. Zhu et al., Physics of Plasmas **25**, 10.1063/1.5038859 (2018).

Solution of the linear eigenvalue problem via Floquet modes

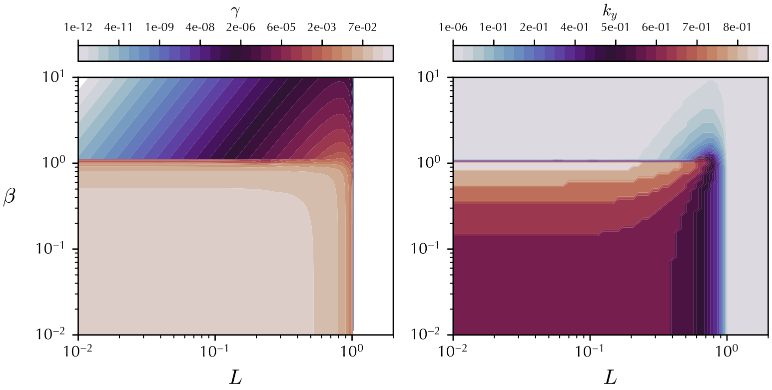
Drop the hat notation. Normal mode analysis of the linear problem:

$$\varphi(x,y,t) = e^{\lambda t + i k_y y} \sum_{m=-\infty}^{\infty} \varphi_m e^{i(m+\delta)x} \tag{12}$$

An eigenvalue solver (numpy.eigs in this case) gives eigenvalues, λ, for the truncated expansion for $m = -M, M$, given k_y, β, Re, and δ.

Viscosity affects the stability criteria for this system.

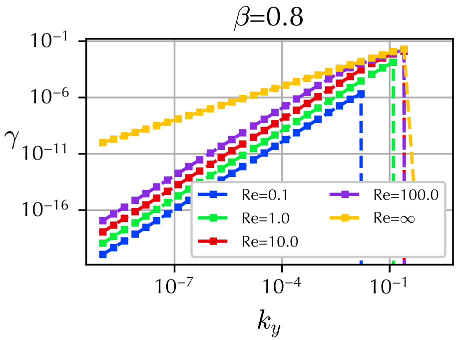
Rayleigh-Kuo criterion[17] in the inviscid case: unstable if $\beta < 1$ in our units. Also need $L < 1$. With viscosity, we have instability at $\beta > 1$. Plots for $\mathrm{Re} = 100$

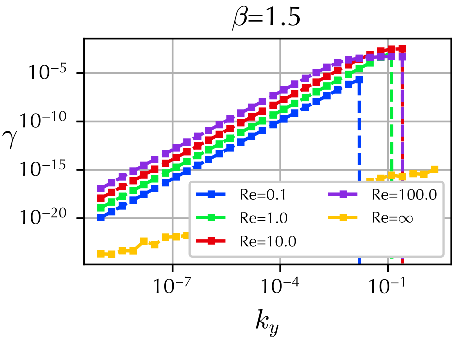


[17] H. Zhu et al., Physics of Plasmas **25**, 10.1063/1.5038859 (2018).

Growth rates as a function of k_y show different behavior as $k_y \to 0$

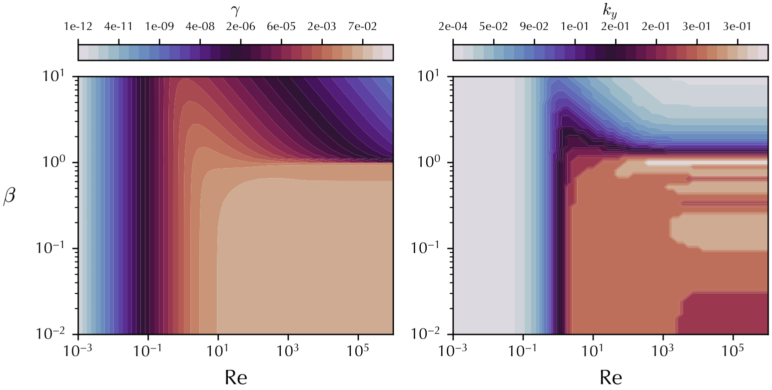
- The inviscid case has $\gamma \propto k_y$ as $k_y \to 0$.
- vertical scales are different. $\beta < 1$(left) case has larger max γ.





2D parameter scan of β and Re is insightful

- max γ and corresponding k_y.
- γ plateaus at large Re when $\beta < 1$. $\beta > 1$ modes have $k_y \ll 1$



Only $m = \pm 1$ are needed to describe the low-k_y modes

Only the mode where $\varphi_{-1} = -\varphi_1$ is unstable. This gives a 2×2 system:

$$\lambda \boldsymbol{B} \cdot \boldsymbol{\varphi} = \boldsymbol{M} \cdot \boldsymbol{\varphi}, \tag{13}$$

where

$$\boldsymbol{\varphi} \equiv \begin{bmatrix} \varphi_0 \\ \varphi_1 \end{bmatrix}, \quad \boldsymbol{B} \equiv \begin{bmatrix} L^2 + K_0^2 & 0 \\ 0 & L^2 + K_1^2 \end{bmatrix}, \tag{14}$$

$$\boldsymbol{M} \equiv \begin{bmatrix} i\beta k_y - K_0^4 / \operatorname{Re} & -k_y(L^2 + K_1^2 - 1) \\ \frac{k_y}{2}(L^2 + K_0^2 - 1) & i\beta k_y - K_1^4 / \operatorname{Re} \end{bmatrix} \tag{15}$$

Eigenvalues of this system are the solution of a quadratic. This can be written down exactly-but it remains complicated an uninformative

Expanding about $k_y = 0$ shows that $L^2 < 1$ is required for instability.

Expanding the matrix elements and solving the characteristic polynomial to order $\mathcal{O}(k_y^2)$ gives

$$\lambda \approx i\beta \frac{k_y}{L^2} + \mathrm{Re}\frac{k_y^2}{2}(1 - L^2) \tag{16}$$

- The frequency of these modes is nominally that of a drift wave.
- When $L^2 > 1$ in this limit, the mode is damped.

Conclusions and future work

- the Hasegawa-Mima equation with viscosity exhibits instability in an extended parameter regime compared to the inviscid case.
- The viscous modes have a different dependence on k_y as $k_y \to 0$.
- The spatial scales of the unstable perturbation is much larger that of the flow
- Work remains to identify the exact mechanism by which the viscosity enables addtional unstable modes.

Continuing to k_y^4 does not capture fastest growing mode, but Padé does better.

Using the first four terms to compute a Padé approximant gives a better estimate for the fastest growing mode.