Matrix

DATE
PAGE

A-1	H-8	O-15	V-22
B-2	I-9	P-16	W-23
C-3	J-10	Q-17	X-24
D-4	K-11	R-18	Y-25
E-5	L-12	S-19	Z-26
F-6	M-13	T-20	space — 0
G-7	N-14	U-21	

Code Message :- $\begin{bmatrix} 41 & 45 & 73 \\ 54 & 46 & 64 \\ 0 & 5 & 10 \end{bmatrix}$ (only sender & reader know this key)

Key Msg :- $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ firstly we have to find K^{-1} & then mult. it with code Msg

$|K| = -2$, $adj(A) = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ & 2 & 3 & & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$K^{-1} = \frac{1}{|K|} adj(K) = \frac{-1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$\begin{bmatrix} 41 & 45 & 73 \\ 54 & 46 & 64 \\ 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 23 & 5 & 12 \\ 3 & 15 & 13 \\ 5 & 0 & 0 \end{bmatrix}$

WELCOME..

DATE
PAGE

① **Row Matrix** : Any matrix with single row is called row matrix. eg:- $[1, 2, 3]$

② **Column Matrix** :- which having single column.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

③ **Square Matrix**: A Matrix with same no. of row & columns.

④ **Rectangular Matrix**:- No. of row not equal to no. of columns.

⑤ **Null Matrix** :- If each and every element of matrix is zero.

⑥ **Digonal Matrix** :- If all the non-digonal element are zero, then its called digonal Matrix.

⑦ **Scaler Matrix** :- It is a diagonal Matrix with same diagonal elements.

⑧ **Identity Matrix** :- If all the diagonal elements are one.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⑨ **upper triangular**:- If $A = [a_{ij}]$ and $a_{ij} = 0$ $\forall$ $i > j$ then matrix is said to be upper Δ.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

a_{21}, a_{31}, a_{32}

DATE
PAGE

(10) **Lower triangular Matrix :-** If $A=[a_{ij}]$ & $a_{ij}=0 \ \forall \ i<j$ then this is lower Δ.

$$\begin{bmatrix} 6 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}$$

$a_{12} \rightarrow$, $0 \rightarrow a_{13}$, $0 \rightarrow a_{23}$

(11) **Orthogonal Matrix :-** If $AA^T = A^TA = I$

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad \& \quad A^T = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1+4+4 & -2+4-2 & 2+2-4 \\ -2+4-2 & 4+4+1 & -4+2+2 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(9) Symmetric = $A^T = A$

(13) Skew symmetric = $A^T \neq A$, $A^T = -A$ (diagonal elements are zero)

* If A is orthogonal Matrix then $A^{-1} = A^T$

(14) **Hermition Matrix :-** $(\bar{A})^T = A^\theta$ or $A^\theta = A$

↳ (Transpose of conjugate of A)

$$A = \begin{bmatrix} 1 & 1+i & 4-5i \\ 1-i & 3 & 3i \\ 4+5i & -3i & -2 \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 1 & 1-i & 4+5i \\ 1+i & 3 & -3i \\ 4-5i & 3i & -2 \end{bmatrix}$$

$$(\bar{A})^T = A^\theta = \begin{bmatrix} 1 & 1+i & 4-5i \\ 1-i & 3 & 3i \\ 4+5i & -3i & -2 \end{bmatrix}$$

DATE
PAGE

(15) Skew-Hermition Matrix: $(\bar{A})^T = -A$

Note: (1) Diagonal elements of Hermition matrix are always real no.
(2) Diagonal elements of skew Hermition matrix are either zero or purely imaginary.

Date:- 6/9/22

(16) Unitary Matrix: $AA^\theta = I = A^\theta A$ $\{A^\theta = (\bar{A})^T\}$

eg:- $A = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$

$A^\theta = (\bar{A})^T =$

$\bar{A} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$, $(\bar{A})^T = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$

$AA^\theta = \frac{1}{2}\times\frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} (1-i^2+1-i^2) & 1^2+i^2-2i+1+i^2+2i \\ 1^2+i^2+2i+1+1^2-2i & 1^2-i^2+1+1^2-i^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$= \frac{1}{4}\times 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ = Indentity

Row & column opperation: (Elementry opp.)

(1) We can add or subtract any no. of rows or column
$R_i \to R_i \pm R_j \pm R_k$

(2) We can interchange two rows or columns
$R_i \leftrightarrow R_j$

DATE
PAGE

③ we can multiply any row or column with any non-zero real no. λ. eg:- $R_i \to \lambda R_i$, where $\lambda \neq 0$

④ we can use the linear combination (LC) of any no. of row or column

$R_i \to \lambda_1 R_1 \pm \lambda_2 R_2 \pm \lambda_3 R_3$, $\lambda_1, \lambda_2, \lambda_3 \neq 0$

⑤ Do Not –

① we can not multiply or divide two rows or column

② we can not multiply or divide any row/column with zero.

③ we cannot interchanging row with column.

④ we should not operate on that row or column which is already used by any other in the single step.

$R_2 \to R_2 - R_4$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

Date :- 07/09/22

- Row Echelon form :- zero's in row's must be in increasing order. only count the zero before non-zero no.

① $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x \\ 2 \\ 4 \end{matrix}$ Rank = 2

② $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} 1 \\ 3 \\ 4 \\ 5 \checkmark \end{matrix}$ Rank = 3 → non-zero rows

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} 1 \\ 2 \\ 2 \\ 4 \\ 4 \end{matrix}$ Rank = 3

x (zero must be in inc. order)

✓ (because all dement zero then we can repeat)

Rank = 3 non-zero Rows

DATE
PAGE

Row echelon form

Any matrix A is in Row echelon form if :-

1. No. of zero increases along with no the row's
2. We consider only that zero which are before first non-zero element of row.
3. Const. no. of zero's are allowed only in case of zero-row's

Rank's :→ No. of non-zero row's in row echelon form of Matrix is said to be rank of that Matrix.

Denoted by :- $\rho(A)$ → Rank of A

Note :-
Firstly we have to convert it in row echlon form.

Ques:- Find Rank $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$ & $R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix}$$

Now we don't use R_1

$R_3 \to 2R_3 + 7R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & 11 & 41 \end{bmatrix} \begin{matrix} x \\ 1 \\ 2 \end{matrix}$$

Rank = 3

Ans

use element वाले उसके नीचे वाला rows बनाना है।

Ques: Let $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

Find Rank.

$R_2 \to 3R_2 - 2R_1$

$R_3 \to 3R_3 - 5R_1$

$R_4 \to 3R_4 - 8R_1$

DATE
PAGE

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 0 & 4 & 12 & -19 \\ 0 & 4 & 12 & -19 \\ 0 & 4 & 12 & -19 \end{bmatrix}$$

use

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_3$

$$\begin{matrix} \times \\ 1 \\ 4 \\ 4 \end{matrix}\begin{bmatrix} 6 & 1 & 3 & 8 \\ 0 & 4 & 12 & -19 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 2 (Non-zero Echelon Row)

$\rho(A) = 2$

Ans 3:-

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 3 & -1 & 1 & -1 & 2 \\ 1 & 2 & -1 & 1 & 1 \\ 6 & 2 & 1 & 1 & 5 \end{bmatrix}$$

use

$R_2 \to 2R_2 - 3R_1$

$R_3 \to 2R_3 - R_1$

$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 0 & -5 & -1 & -5 & -2 \\ 0 & 3 & -3 & 1 & 0 \\ 0 & -1 & -2 & -2 & -1 \end{bmatrix}$$

use

$R_3 \to 5R_3 + 3R_2$

$R_4 \to 5R_4 - R_2$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 0 & -5 & -1 & -5 & -2 \\ 0 & 0 & -18 & -10 & -6 \\ 0 & 0 & -9 & -5 & -3 \end{bmatrix}$$

use

$R_4 \to 2R_4 - R_3$

$$\begin{matrix} \times \\ 1 \\ 2 \\ 5 \end{matrix}\begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 0 & -5 & -1 & -5 & -2 \\ 0 & 0 & -18 & -10 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

(3 - Non-zero)

DATE
PAGE

Minor:- This is determinant of submatrix. Higher order Minor should be 3×3

eg:- $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}_{3\times4}$

$M_1 = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \\ -2 & 3 & 2 \end{bmatrix}$ $|M| = 1(8-3) - 2(6+2) + 0$
$= 5 - 16 = -11 \neq 0$

Rank = 3

*1. **Matrix A has rank 'r' if** :-
(i) There exist atleast one non-zero minor of order 'r'.
(ii) Every Minor of higher order (higher than r, if exist) Must be zero.

Ques:- $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}_{3\times4}$

(go higher to lower)

$M_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{bmatrix} = 0$

Higher minor is 3×3

$M_{2'} \begin{bmatrix} 2 & 3 & 2 \\ 3 & 5 & 1 \\ 3 & 4 & 5 \end{bmatrix} = 0$

$M_3 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 1 & 4 & 5 \end{bmatrix} = 0$

$M_4 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 1 & 3 & 5 \end{bmatrix} = 0$

$M_5 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}_{2\times2} = 3-4$
$= -1 \rightarrow$ Non-zero

Rank = 2

DATE 08/09/2022
PAGE

* Normal Matrix form :-

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = [I_3, 0]$$ (Rank is 3)

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ (Rank is 2) ($\because I_2$)

Note :- Rank is equal to order of I_r

eg: $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, $[I_r, 0]$

$[I_r]$, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Ques-
$$\begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 3 & -1 & 1 & -1 & 2 \\ 1 & 2 & -1 & 1 & 1 \\ 6 & 2 & 1 & 1 & 5 \end{bmatrix}$$ $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 3 & -1 & 1 & -1 & 2 \\ 2 & 1 & 1 & 1 & 2 \\ 6 & 2 & 1 & 1 & 5 \end{bmatrix}$$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & -7 & 4 & -4 & -1 \\ 0 & -3 & 3 & -1 & 0 \\ 0 & -10 & 7 & -5 & -1 \end{bmatrix}$$ $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + C_1$, $C_4 \to C_4 - C_1$, $C_5 \to C_5 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -7 & 4 & -4 & -1 \\ 0 & -3 & 3 & -1 & 0 \\ 0 & -10 & 7 & -5 & -1 \end{bmatrix}$$ $R_4 \to R_4 - (R_3 + R_2)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -7 & 4 & -4 & -1 \\ 0 & -3 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_3 \to 7R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -7 & 4 & -4 & -1 \\ 0 & 0 & 9 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 7C_3 + 4C_2$
$C_4 \to 7C_4 - 4C_2$
$C_5 \to 7C_5 - C_2$
$C_2 = -\frac{1}{7} C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & -1 \\ 0 & 0 & 9 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - 4C_2$
$C_4 \to C_4 + 4C_2$
$C_5 \to C_5 + C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 9 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to \frac{1}{9} C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 5C_3$
$C_5 \to C_5 - 3C_3$

$$\sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \rho(A) = 3$$

Date :- 13/9/22

Ques :-

A → square Matrix

$|A| \neq 0$ Matrix is non singular

$|A| = 0$ " " singular.

DATE
PAGE

Note:→ If A is square Matrix :-

① **A is non-singular** ⇔ [Rank of A = order of A] [A^{-1} = OP]

eg:→ $\begin{bmatrix} - & - \\ - & - \end{bmatrix}_{4\times4}$, Rank of [A=4]

② **A is singular** ⇔ [Rank < order of A]

★

Ques:1 Find non-singular matrix P & Q so that PAQ is in normal form of A.

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}_{3\times4}$$

→ If we use row operation then we change only in A & $I_{3\times3}$.
→ If we use column operation then we change only in A & $I_{4\times4}$

Ans:- $A = I_{3\times3}\, A_{3\times4}\, I_{4\times4}$

(Same no. of rows) (Same no. of columns)

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -9 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} // \\ - \end{bmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & -2 & -1 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \longleftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & -6 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} A \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$$

$R_3 \to R_3 - 6R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & 0 & 28 & 56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ -6 & 1 & 9 \end{bmatrix} A \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$$

$C_3 \to C_3 - 5C_2$, $C_4 \to C_4 - 10C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 28 & 56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ -6 & 1 & 9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 28 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ -6 & 1 & 9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to (-1)R_2$, $R_3 \to \frac{1}{28} R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ -6/28 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{[I_3 \; 0] = PAQ}$$

DATE
PAGE

Q:- $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}_{3\times 3}$

$\rho(A) = 3$ (non-singular)
Rank = order.

$A = I_{3\times 3} A I_{3\times 3}$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 3R_2 - 2R_1$

$$\begin{bmatrix} 3 & -3 & 4 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + C_1$, $C_3 \to 3C_3 - 4C_1$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} & & \\ & // & \\ & & \end{bmatrix} A \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_1 \to \frac{1}{3}R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_3 \to 3R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & -3 & 3 \end{bmatrix} A \begin{bmatrix} & & \\ & // & \\ & & \end{bmatrix}$$

$C_3 \to C_3 + 4C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & -3 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

DATE
PAGE

$R_2 \to -\frac{1}{3}R_2$, $R_3 \to -\frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ -\frac{2}{3} & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

(P) (O)

$$[I, O] = P A O$$

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✳ Linear equations system:

1) **Non-Homogeneous:** If $AX = B$, if $B \neq 0$, system of Lt. is non-homogeneous.

2) **Homogeneous:** If $AX = B$, if $B = 0$, system of Lt. is homogeneous.

✳ 1) For NON-homogeneous:-

eg:- $2x + 3y + 4z = 10$
$2x + 5y + 9z = 2$
$5x + 7y + z = 10$

$AX = B$

$$\begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 9 \\ 5 & 7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 10 \end{bmatrix}$$

Steps:

1) Put this in $AX = B$
2) Augmented Matrix; $[A:B]$

$$\begin{bmatrix} 2 & 3 & 4 & : & 10 \\ 2 & 5 & 9 & : & 2 \\ 5 & 7 & 1 & : & 10 \end{bmatrix}$$

3). Find $\rho(A:B)$ & $\rho(A)$

4) (a) $\rho(A) \neq \rho(A:B)$, system is inconsistent (No sol^n)

(b) $\rho(A) = \rho(A:B)$, " " consistent (sol^n)

(i) :- $\rho(A) = \rho(A:B)$ = no. of unknown
then unique sol^n exist

(ii) :- $\rho(A) = \rho(A:B) <$ no. of unknowns
Infinite no. of sol^n.

Q.1 :- $x - 4y + 7z = 8$
$3x + 8y - 2z = 6$
$7x - 8y + 26z = 31$

$AX = B$

$$\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 31 \end{bmatrix}$$

$[A:B]$

$$\begin{bmatrix} 1 & -4 & 7 & ; & 8 \\ 3 & 8 & -2 & ; & 6 \\ 7 & -8 & 26 & ; & 31 \end{bmatrix}$$ Row echelon form.

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$

$$\sim \begin{bmatrix} 1 & -4 & 7 & ; & 8 \\ 0 & 20 & -23 & ; & -18 \\ 0 & 20 & -23 & ; & -25 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -4 & 7 & ; & 8 \\ 0 & 20 & -23 & ; & -18 \\ 0 & 0 & 0 & ; & -7 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 3 \end{matrix}$$

$\rho(A) = 2$ (∵ Non-zero rows)
$\rho(A:B) = 3$
$\rho(A) \neq \rho(A:B)$ system is non consistent

Q.2 :- $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$

DATE
PAGE

Ans: $AX = B$

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$[A:B]$

$$\begin{bmatrix} 1 & -1 & 2 & : & 7 \\ 3 & 4 & -5 & : & -5 \\ 2 & -1 & 3 & : & 12 \end{bmatrix}$$ using row echlean form

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 2 & : & 7 \\ 0 & 7 & -11 & : & -26 \\ 0 & 1 & -1 & : & -2 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & : & 7 \\ 0 & 1 & -1 & : & -2 \\ 0 & 7 & -11 & : & -26 \end{bmatrix} \quad R_3 \to R_3 - 7R_2 \sim \begin{bmatrix} 1 & -1 & 2 & : & 7 \\ 0 & 1 & -1 & : & -2 \\ 0 & 0 & -4 & : & -12 \end{bmatrix} \begin{matrix} \leftarrow x \\ \leftarrow 1 \\ \leftarrow 2 \end{matrix}$$

$P(A:B) = P(A) = 3$, consistant in nature
unique soln exist.

For checking:- (x, y & z value)

$AX = B$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -12 \end{bmatrix}$$

$x - y + 2z = 7$

$y - z = -2$

$-4z = -12 \Rightarrow \boxed{z = 3}$

$y - 3 = -2 \Rightarrow y = -2 + 3 = 1$

$\boxed{y = 1}$

$x - y + 2(3) = 7$

$x = 7 - 5 = 2$, $\boxed{x = 2}$

$\rho(A) = \rho(A:B) = 2$ < no. of unknowns

$AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 0 \end{bmatrix}$$

$x + 2y + 3z = 14$ — (1)

$-y - 2z = -8 \Rightarrow$ $y = 8 - 2z$ put in (1)

z is independent;

$x + 2(8 - 2z) + 3z = 14$

$x = -2 + z$

$z = k$, k = real no.

$x = -2 + z$

$y = 8 - 2z$

Ques:-1 $2x + 3y + 4z = 20$

$x + 2y + 3z = 14$

$x + 4y + 7z = 30$

$AX = B$

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 14 \\ 30 \end{bmatrix}$$

$[A:B]$

$$\begin{bmatrix} 2 & 3 & 4 & : & 20 \\ 1 & 2 & 3 & : & 14 \\ 1 & 4 & 7 & : & 30 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 2 & 3 & 4 & : & 20 \\ 1 & 4 & 7 & : & 30 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$,

$R_3 \to R_3 - R_1$,

$$\sim \begin{bmatrix} 1 & 2 & 3 & , & 14 \\ 0 & -1 & -2 & , & -8 \\ 0 & 2 & 4 & , & 16 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -1 & -2 & : & -8 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \begin{matrix} x \\ 1 \\ 0 \end{matrix}$$

$\rho(A:B) = 2 = \rho(A) <$ no. of unknowns

consistant (Infinit sol^n)

Note: Total no. of independent variables = Total no. of variable $- \rho(A)$

Independent variables $= 3 - 2 = 1$.

Ques:-2. Reduce to Normal form:-

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

use

$R_2 \leftrightarrow R_1$

DATE
PAGE

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

use

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

use

$C_2 \to C_2 + C_1$
$C_3 \to C_3 + 2C_1$
$C_4 \to C_4 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

use

$R_3 \to R_3 - 4R_2$
$R_4 \to R_4 - 9R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -7 \\ 0 & 0 & 33 & 38 \\ 0 & 0 & -66 & -44 \end{bmatrix}$$

$C_3 \to C_3 - 6C_2$, $C_4 \to C_4 + 7C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 38 \\ 0 & 0 & -66 & -44 \end{bmatrix}$$

use

$R_4 \to R_4 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 38 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$R_3 \to -\frac{1}{33} R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{38}{33} \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$\rho(A) = 3$

Ques: (wrong)

$x - y + 2z = 3$
$x + 2y + 3z = 5$
$3x - 4y - 5z = 13$

DATE
PAGE

$AX = B$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 13 \end{bmatrix}$$

$[A:B]$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 3 & -4 & -5 & 13 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & -11 & 4 \end{array}\right]$$

$R_3 \to 3R_3 + R_2$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -32 & 14 \end{array}\right] \begin{matrix} \to x \\ \to 1 \\ \to 2 \end{matrix}$$

$\rho(A) = 3$
$\rho(A:B) = 3$; consistent
unique sol^n.

$AX = B$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -32 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 14 \end{bmatrix}$$

$x - y + 2z = 3$ ①
$3y + z = 2$ ②
$-32z = 14$ ③ $\Rightarrow z = -\frac{7}{16}$

$\Rightarrow 3y - \frac{7}{16} = 2 \Rightarrow 3y = \frac{39}{16} \Rightarrow y = \frac{13}{16}$

$\Rightarrow x - \frac{13}{16} - \frac{7}{8} = 3 \Rightarrow x = \frac{61}{16}$

Q.3:

$x - y + 2z = 3$
$3y + z = 2$
$-32z = 14 \Rightarrow z = -\frac{7}{16}$

Q.3.

$2x + 6y = -11$
$6x + 20y - 6z = -3$
$6y - 18z = -1$

$C = [A:B]$

DATE
PAGE

$$\begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix} \begin{matrix} \to x \\ \to 1 \\ \to 3 \end{matrix}$$

$\rho(A) = 2$, $\rho(A:B) = 3$

$\rho(A) \neq \rho(A:B)$ so inconsistent in nature

Ques: $x + 2y + z = 2$
$3x + y - 2z = 1$
$4x - 3y - z = 3$
$2x + 4y + 2z = 4$

$AX = B$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix}_{4\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

Augmented Matrix $\sim [A:B]$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 3 & 1 & -2 & : & 1 \\ 4 & -3 & -1 & : & 3 \\ 2 & 4 & 2 & : & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -5 & -5 & : & -5 \\ 0 & -11 & -5 & : & -5 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_2 \to \frac{-1}{5} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 1 & : & 1 \\ 0 & -11 & -5 & : & -5 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

use

$R_3 \to R_3 + 11R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 1 & : & 1 \\ 0 & 0 & 6 & : & 6 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$R_3 \to \frac{1}{6} R_3$ ✓

$\rho(A) = 3 = \rho(A:B) \Rightarrow$ consistant (unique sol^n)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$x + 2y + z = 2 \Rightarrow x + 0 + 1 = 2 \Rightarrow \boxed{x = 1}$

$y + z = 1 \Rightarrow \boxed{y = 0}$

$\boxed{z = 1}$

Date: 15/09/22

V. Imp.

Ques:- Find λ & μ such that system has

(a) No sol^n (b) unique sol^n (c) Infinite sol^n

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

use

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & (\lambda-1) & : & (\mu-6) \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & (\lambda-3) & : & (\mu-10) \end{bmatrix}$$

(a) No. sol^n:-

$\lambda - 3 = 0, \quad \mu - 10 \neq 0$

$\rho(A) \neq \rho(A:B)$

$\lambda = 3, \quad \mu \neq 10$

$\rho(A) = \rho(A:B) = 2$ < no. of unknowns

(a) ① $\lambda - 3 = 0$, $\mu - 10 = 0$ ⇒ infinite no of sol^n

② $\lambda - 3 \neq 0$, $\mu - 10 \neq 0$ ⇒ unique sol^n $\rho(A:B) = \rho(A) = 3$

③ $\lambda - 3 = 0$, $\mu - 10 \neq 0$ ⇒ $\rho(A:B) = 3$, $\rho(A) = 2$, No. sol^n

④ $\lambda - 3 \neq 0$, $\mu - 10 = 0$ ⇒ $\rho(A) = \rho(A:B) = 3$ unique sol^n

DATE
PAGE

(b) Unique sol^n:-

$\lambda - 3 \neq 0 \Rightarrow \boxed{\lambda = 3}$, μ can have any value.

$\rho(A) = \rho(A:B) =$ no. of unknowns.

$\because \quad \lambda - 3 \neq 0, \quad \mu - 10 \neq 0 \quad -(1)$

$\lambda - 3 \neq 0, \quad \mu - 10 = 0 \quad -(2)$

(c) Infinite no. of sol^n:

$\lambda - 3 = 0 \quad \mu - 10 = 0$

$\boxed{\lambda = 3} \quad \boxed{\mu = 10}$

$\rho(A) = \rho(A:B) = 2 <$ no. of unknown.

#(2):- Homogeneous system of Linear eq^n :→

Steps: 1) $\rho(A) =$ no. of unknowns $\Rightarrow$ unique sol^n $\Rightarrow$ zero sol^n $\Rightarrow$ Trivial sol^n

2) $\rho(A) <$ no. of unknown $\Rightarrow$ infinite sol^n exist (NO. trivial or non zero sol^n)

eg:→ $x + y + z = 0$

$x + 2y + 3z = 0$

$x + 3y + 4z = 0$

$AX = B, \quad (B = 0)$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \to x \\ \to 1 \\ \to 2 \end{matrix}$$

$\rho(A) = 3 =$ no. of unknown

$\Rightarrow$ unique sol^n zero sol^n

$\boxed{x = 0} \quad \boxed{y = 0} \quad \boxed{z = 0}$ Ans

DATE
PAGE

Ques: $2x+y+w=0$
$4x+2y+z+3w=0$
$6x+3y+4z+7w=0$

$AX=B$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to x \\ \to z \end{matrix}$$

$\rho(A) = 2 <$ no. of unknown (4)
infinte no. of sol^n.

No. of independent variable $= 4-2=2$

$AX=B$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x+y+w=0$
$z+w=0$
$\boxed{z=-w}$ → ind. variable.
$\boxed{y=-w-2x}$, (x & w are ind. variables.)

Let $w=k_1$, $x=k_2$ } Real no.

$\boxed{y=-k_1-2k_2}$
$\boxed{z=-k_1}$

Date: 27/9/22

Q: Find the value of 'k' such that

DATE
PAGE

$x+y+z=0$
$x+2y+kz=0$
$x+3y+4z=0$

has (a) trivial sol^n
(b) Non-trivial sol^n

sol^n $AX=0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

Row echelon

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & (k-1) \\ 0 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & (k-1) \\ 0 & 0 & (5-2k) \end{bmatrix}$$

(a) if $5-2k \neq 0$.
$\Rightarrow \rho(A) = 3 =$ no. of unknowns
$5-2k \neq 0$
$\Rightarrow k \neq 5/2$, trivial sol^n exist

$5-2k=0$ & $5-2k \neq 0$

(b) $5-2k=0 \Rightarrow k=5/2$
$\rho(A) = 2 <$ no. of unknowns (3)
non-trivial sol^n

Short cut:

$AX=0$
If A is square matrix then
(a) for trivial sol^n $\iff |A| \neq 0$
(b) " Non-trivial sol^n $\iff |A| = 0$

* Linear dependence and independence of vectors:

eg: $2\hat{i}+3\hat{j}-5\hat{k}$; $[2 \ 3 \ -5]$

Linear combination :→

(Vector) $V_1 = [1 \ 2 \ -1]$ $V_2 = [0 \ 0 \ 1]$, $V_3 = [1 \ -2 \ 0]$
$\alpha_1, \alpha_2, \alpha_3 \in$ Real no. (scaler)

(Trivial = L.I = $|A| \neq 0$, $\rho(A)$ = no. of unknowns. (for square matrix)
Non-" = L.D = $|A| = 0$, $\rho(A) <$ no. of unknowns)

DATE
PAGE

$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = V_4$ (new vector)
[V_4 is L.C. of V_1, V_2 & V_3]

eg:→ $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$

$1[1\ 2\ -1] + 2[0\ 0\ 1] + 3[1\ -2\ 0]$

$= [1\ 2\ -1] + [0\ 0\ 2] + [3\ -6\ 0]$

$= [(1+0+3)\quad (2+0-6)\quad (-1+2+0)]$

$= [4\ \ -4\ \ 1] = V_4$ Ans V_4 is Linear comb. of $\{V_1, V_2, V_3\}$

→ Let $V_1 = [1, 2]$, $V_2 = [2\ 4]$

$\alpha_1 V_1 + \alpha_2 V_2 = 0 = [0, 0]$

(i) If $\alpha_1 = 0 = \alpha_2$

$0V_1 + 0V_2 = 0 \Rightarrow 0[1\ 2] + 0[2\ 4] = [0, 0]$

(ii) $\alpha_1 = 2$, $\alpha_2 = -1$

$2[1, 2] + (-1)[2, 4] = [0, 0]$

$[2\ 4] + [-2\ \ -4] = [(2-2)\ \ (4-4)] = [0\ \ 0]$

Here $\{V_1, V_2\}$ is Linear dependent (because α_1 & α_2 are non-zero)

2) :- $V_1 = [2\ 0]$, $V_2 = [0\ 5]$

$\alpha_1 V_1 + \alpha_2 V_2 = 0$

(i) $\alpha_1 = -5$, $\alpha_2 = 2$

$-5[2, 0] + 2[0\ 5] = [-10\ 0] + [0\ \ 10] = [-10\ 10]$

$\{V_1, V_2\}$ is Linear independent.

(Trivial = L.I.
Non-" = L.D.)

eg: $V_1 = [1\ 2]$, $V_2 = [2\ 4]$

$\alpha_1 [1, 2] + \alpha_2 [2\ 4] = [0\ 0]$

$[\alpha_1\ \ 2\alpha_1] + [2\alpha_2 \ \ 4\alpha_2] = [0\ 0]$

$[(\alpha_1 + 2\alpha_2)\ \ (2\alpha_1 + 4\alpha_2)] = [0\ 0]$

$\alpha_1 + 2\alpha_2 = 0$
$2\alpha_1 + 4\alpha_2 = 0$ } Homogeneous system

DATE
PAGE

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 - 2R_1, \quad \rho(A) < \text{no of variables}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \rho(A) < \text{no. of variable}$$
$$1 < 2 \quad (\text{L.D.})$$

eg: $V_1 = [2 \quad 0]$, $V_2 = [0 \quad 5]$

$$\alpha_1[2 \quad 0] + \alpha_2[0 \quad 5] = [0 \quad 0]$$
$$[2\alpha_1 \quad 0] + [0 \quad 5\alpha_2] = [0 \quad 0]$$
$$[(2\alpha_1 + 0) \quad (0 + 5\alpha_2)] = [0, 0]$$

$2\alpha_1 + 0\alpha_2 = 0$
$0\alpha_1 + 5\alpha_2 = 0$

$2\alpha_1 = 0 \Rightarrow \boxed{\alpha_1 = 0}$

$5\alpha_2 = 0 \quad \boxed{\alpha_2 = 0}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \to \text{trivial sol}^n \ (\text{L.I.})$$

Note: ① **Linear independent:**

Let $V_1, V_2, V_3 \dots V_n$ are vectors
$\alpha_1, \alpha_2, \alpha_3 \dots \alpha_3$ are scalars
if $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \dots + \alpha_n V_n = 0$
implies that all α_i's are zero, then set $\{V_1, V_2, V_3 \dots V_n\}$ is Linear independent (LI)

② **Linear dependent:**

If $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \dots \alpha_n V_n = 0$
where at least on $\alpha_i \neq 0$ then set
$\{V_1, V_2, V_3 \dots V_n\}$ is Linear dependent.

Theory in case of L.D. we express one as linear combinations

DATE 28/9/22
PAGE

Type of question:

① Given set of vector is LI or L.D. → short trick.

② Find the value of some 'k' so that the given set is L.I. or L.D. → short cut

③ Express same vector in the form of other vector.

Ques 1: Check the set of vectors $[1\ 2\ 3]$, $[3\ -2\ -1]$, $[1\ -6\ -5]$ is LI or LD?

soln:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & -1 \\ 1 & -6 & -5 \end{bmatrix}$$

↓ Sqr Matrix

→ Reduce to row echlen form

Method 1: Find rank of A, [square or rectangular]

ⓐ if $\rho(A)$ = no. of vector ⇒ L.I. $\{V_1, V_2, V_3\}$

ⓑ " $\rho(A) <$ " " " ⇒ then L.D.

Method 2: If A is square matrix then [only for square.]

$|A| = 0 \iff$ L.D.

$|A| \neq 0 \iff$ L.I.

$$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -10 \\ 0 & -8 & -8 \end{bmatrix}$$

(-8 is the pivot)

$$R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -10 \\ 0 & 0 & 2 \end{bmatrix}$$

$\rho(A) = 3$ = no. of vectors (3)

system is L.I.

→ the set $\{V_1, V_2, V_3\}$ is L.I.

Ques: Are the vectors $X_1 = (1\ 3\ 4\ 2)$, $X_2 = (3\ -5\ 2\ 2)$, $X_3 = (2\ -1\ 3\ 2)$ L.D.? (ii) If so express one of these as L. combination of other.

soln

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 3 & -5 & 2 & 2 \\ 0 & -1 & 3 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

Trivial → L.I.
Non-Trivial → L.D.

DATE
PAGE

$$\sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -14 & -10 & -4 \\ 0 & -7 & -5 & -2 \end{bmatrix} \qquad R_3 \to R_3 - \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -14 & -10 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 <$ no. of vectors
$\Rightarrow$ system is Linearly dependent

(ii):- $\alpha_1 [1 \quad 3 \quad 4 \quad 2] + \alpha_2 [3 \quad -5 \quad 2 \quad 2] + \alpha_3 [2 \quad -1 \quad 3 \quad 2] = [0 \quad 0 \quad 0 \quad 0] \quad -(1)$

$\Rightarrow [\alpha_1 + 3\alpha_2 + 2\alpha_3 \quad 3\alpha_1 - 5\alpha_2 - \alpha_3 \quad 4\alpha_1 + 2\alpha_2 + 3\alpha_3 \quad 2\alpha_1 + 2\alpha_2 + 2\alpha_3] = [0 \quad 0 \quad 0 \quad 0]$

$\Rightarrow \alpha_1 + 3\alpha_2 + 2\alpha_3 = 0$
$3\alpha_1 - 5\alpha_2 - \alpha_3 = 0$
$4\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$
$2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0$

Short cut:- We can take the Transpose of A & mult. by α_1, α_2 & α_3

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_3$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & -4 & -2 \end{bmatrix} \qquad R_2 \to \frac{-1}{7} R_2$$

$R_3 \to \frac{-1}{5} R_3$; $R_4 \to \frac{-1}{2} R_4$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \qquad \text{(2) use}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 <$ no. of vectors (3) ∴ infinite sol^n.

no. of independent variables $= 3 - \rho(A) = 3 - 2 = 1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 0$ — (1)

$2\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_3 = -2\alpha_2$ put in (1) (α_2 → ind. variable)

$\alpha_1 + 3\alpha_2 + 2(-2\alpha_2) = 0$

$\alpha_1 = \alpha_2$

$\alpha_3 = -2\alpha_2$

Let $\alpha = k$

$\alpha_2 = k$, $\alpha_1 = k$, $\alpha_3 = -2k$ put in eq^n (1)

X_1 X_2 X_3

$k[1\ 3\ 4\ 2] + k[3\ -5\ 2\ 2] + -2k[2\ -1\ 3\ 2]$

$\Rightarrow k[1\ 3\ 4\ 2] = -k([3\ -5\ 2\ 2] - 2[2\ -1\ 3\ 2])$

$\Rightarrow [1\ 3\ 4\ 2] = -[3\ -5\ 2\ 2] + 2[2\ -1\ 3\ 2]$

$\boxed{X_1 = -X_2 + 2X_3}$ Ans

★ Q. Check $x = [2\ -1\ 0]$, $y = [1\ 2\ 1]$, $z = [0\ 2\ -1]$ are L.I. or L.D. & express $[3\ 2\ 1]$ as a linear combination of x, y, z.

→ $\alpha_1[2\ -1\ 0] + \alpha_2[1\ 2\ 1] + \alpha_3[0\ 2\ -1] = [3\ 2\ 1]$

$= [(2\alpha_1 + \alpha_2 + 0)\ \ (-\alpha_1 + 2\alpha_2 + 2\alpha_3)\ \ (0 + \alpha_2 - \alpha_3)] = [3\ 2\ 1]$

$2\alpha_1 + \alpha_2 + 0 = 3$

$-\alpha_1 + 2\alpha_2 + 2\alpha_3 = 2$ ⟹ Non-homogeneous.

$0 + \alpha_2 - \alpha_3 = 1$

$x_1 = \frac{8}{9}$, $x_2 = \frac{11}{9}$, $x_3 = \frac{2}{9}$

DATE
PAGE

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - 2R_1$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_3 \to R_3 - 2R_1 \end{matrix} \quad \sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 5 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

DATE
PAGE

Q:- check wheather the vectors (3, 2, 4) (1, 0, 2) & (1, -1, -1) are linearly independent or not? (L.I.)

Q②:- check the L.I. or L.D. of vectors
[1 2 4] [2 -1 3] [0 1 2] [-3 7 2]
and find the relation if it exist.

Ans:- $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 = [0\ 0\ 0]$ — (1)

$\alpha_1[1\ 2\ 4] + \alpha_2[2\ -1\ 3] + \alpha_3[0\ 1\ 2] + \alpha_4[-3\ 7\ 2] = [0\ 0\ 0]$

$[(\alpha_1 + 2\alpha_2 - 3\alpha_4) \quad (2\alpha_1 - \alpha_2 + \alpha_3 + 7\alpha_4) \quad (4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4)] = [0\ 0\ 0]$

$+0\alpha_3$

$\alpha_1 + 2\alpha_2 - 3\alpha_4 = 0$

$2\alpha_1 - \alpha_2 + \alpha_3 + 7\alpha_4 = 0$

$4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4 = 0$

$-40 + \frac{39}{2} + 2 + 2$

$\alpha_1 = -$
$\alpha_2 = 13/2$
$\alpha_3 = -1$
$\alpha_4 = 1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad R_2 \to R_2 - 2R_1, \quad R_3 \to R_4 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

use

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\rho(A) = 3 <$ no. of variable (4)

ind. variables $= 4 - 3 = 1$.

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\alpha_1 + 2\alpha_2 - 3\alpha_4 = 0$

$-5\alpha_2 + \alpha_3 + 13\alpha_4 = 0$

$\alpha_3 + \alpha_4 = 0 \Rightarrow \boxed{\alpha_3 = -\alpha_4}$

$-5\alpha_2 = -13\alpha_4 + \alpha_4$

$-5\alpha_2 = -12\alpha_4 \Rightarrow \boxed{\alpha_2 = \frac{12}{5}\alpha_4}$ → ind. variable.

$\alpha_1 + 2\left(\frac{12}{5}\alpha_4\right) - 3\alpha_4 = 0$

$\alpha_1 + \frac{24}{5}\alpha_4 - 3\alpha_4 = 0$

$\alpha_1 + \frac{9}{5}\alpha_4 = 0 \Rightarrow \boxed{\alpha_1 = -\frac{9}{5}\alpha_4}$

$\alpha_4 = k$

$\boxed{\alpha_1 = -\frac{9}{5}k, \ \alpha_2 = \frac{12}{5}k, \ \alpha_3 = -k, \ \alpha_4 = k}$ put in (1)

$-\frac{9}{5}k\begin{bmatrix}1 & 2 & 4\end{bmatrix} + \frac{12}{5}k\begin{bmatrix}2 & -1 & 3\end{bmatrix} - k\begin{bmatrix}0 & 1 & 2\end{bmatrix}$

$+k\begin{bmatrix}-3 & 7 & 2\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$

$\Rightarrow -\frac{9}{5}k\begin{bmatrix}1 & 2 & 4\end{bmatrix} = -k\left[\frac{12}{5}(2\ -1\ 3) - 1(0\ 1\ 2) + 1(-3\ 7\ 2)\right]$

$\Rightarrow \frac{9}{5}\begin{bmatrix}1 & 2 & 4\end{bmatrix} = \frac{12}{5}(2\ -1\ 3)^{x_2} - 1(0\ 1\ 2)^{x_3} + 1(-3\ 7\ 2)^{x_4}$

$\Rightarrow \begin{bmatrix}\frac{9}{5} & \frac{18}{5} & \frac{36}{5}\end{bmatrix} = \begin{bmatrix}\frac{24}{5} & -\frac{12}{5} & \frac{36}{5}\end{bmatrix} - \begin{bmatrix}0 & 1 & 2\end{bmatrix} + \begin{bmatrix}-3 & 7 & 2\end{bmatrix}$

OR

$\boxed{\frac{9}{5}x_1 = \frac{12}{5}x_2 - x_3 + x_4}$

Ans 1:

I st Method

$\begin{bmatrix}3 & 2 & 4\\ 1 & 0 & 2\\ 1 & -1 & -1\end{bmatrix}$, $|A| = 3(0+2) - 2(-1-2) + 4(-1+0)$

$= 6 + 6 - 4 = 8 \neq 0$

L.I.

II nd Method

$R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix}1 & -1 & -1\\ 1 & 0 & 2\\ 3 & 2 & 4\end{bmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

Date :
Page No. :

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 5 & 7 \end{bmatrix} \quad R_3 \to R_3 - 5R_2$$

7-15

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -8 \end{bmatrix}$$

$\rho(A) = 3 =$ no. of variable
so L.I. Ans

Q:- Find the value of k so that the set of vector $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ & $\begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix}$ are L.D.

$$A = \begin{bmatrix} 1 & 1 & k \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

$|A| = 0$ as system is L.D.

$\Rightarrow 1(2-0) - 1(-1+0) + k(2-6) = 0$

$2 + 1 + 2k - 6k = 0 \Rightarrow \boxed{k = 3/4}$

Date: 29-9-22

* Eigen values & Eigen vectors:

A - square Matrix $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ ($1 \to (1-\lambda)$, $2 \to (2-\lambda)$)

Step 1:- $[A - \lambda I]$ → identity of same order as of given matrix A

$$[A - \lambda I] = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2-0 \\ 3-0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 2 \\ 3 & (2-\lambda) \end{bmatrix} \checkmark$$

Step 2:- characteristic polynomial $= |A - \lambda I|$

$= (1-\lambda)(2-\lambda) - 6$

$= 2 - \lambda - 2\lambda + \lambda^2 - 6$

$\Rightarrow \lambda^2 - 3\lambda - 4$ ✓

Date :
Page No.

① $[A-\lambda I]$
② $|A-\lambda I|$
③ $|A-\lambda I| = 0$
④ Find roots.

Step 3. $|A-\lambda I| = 0$ → characteristic eqⁿ

$\lambda^2 - 3\lambda - 4 = 0$

Step 4: Find roots of this eqⁿ : These root are called Eigen value.

$\lambda^2 - 4\lambda + \lambda - 4 = 0 \Rightarrow \lambda(\lambda-4) + 1(\lambda-4) = 0$

$(\lambda-4)(\lambda+1) = 0$

$\boxed{\lambda = 4, -1}$ → Eigen values.

eg:- $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

$[A-\lambda I] = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (2-\lambda) & -1 \\ 1 & -\lambda \end{bmatrix}$

$|A-\lambda I| = -\lambda(2-\lambda) - (-1) = -2\lambda + \lambda^2 + 1$

$\lambda^2 + 2\lambda + \lambda + 1 \Rightarrow \lambda(\lambda+1) + 1(\lambda+1)$ (1,1)

Q:- $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Sum of diagonal element called trace of A.

$[A-\lambda I] = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$

$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$.

** Short cut: For (3×3)

$\lambda^3 - [\text{sum of the diagonal elements of } A]\lambda^2 + [\text{sum of minors of diagonal of } A]\lambda - \det A = 0$

$\lambda^2 - [\text{sum of dia.} - \text{First root}]\lambda + \frac{\det A}{\text{First root}} = 0$

$M_{11} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} = 5-1 = 4$, $M_{12} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = 1-9 = -8$, $M_{33} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = 4$

Sum of Minors $= 4+(-8)+4 = 0$

DATE
PAGE

$\Rightarrow \lambda^3 - 7\lambda^2 + 0 - (-36) = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$

to Find roots: (Hit & trial method)

$\lambda = -2$, so $(\lambda+2) = 0$

$-8 - 28 + 36 = 0$

do:-

$(\lambda+2) \overline{)\ \lambda^3 - 7\lambda^2 + 36\ (\ \lambda^2 + 9\lambda + 18}$

$\lambda^3 + 2\lambda^2$

$-9\lambda^2 + 36$

$-9\lambda^2 - 18\lambda$

$+18\lambda + 36$

$18\lambda + 36$

0

$9\lambda^2 - 9\lambda + 18 = 0$

$\lambda(\lambda-6) - 3(\lambda-6) = 0$

$\lambda = -2, 6, 3$

$|A| = 1(4) - 1(-2) + 3(-14)$

$= 4 + 2 - 42 = -36$

$\lambda = 2$	$\lambda = 1$
$8 - 7\times4 + 36$	$1 - 7 + 36 \neq 0$
$8 - 28 + 36 \neq 0$	

	λ_3	λ^2	λ	const
-2	1	-7	0	36
		-2	18	-36
	1	-9	18	0
		λ^2	λ	const

$\lambda^2 - 9\lambda + 18 = 0$

$\lambda^2 - 6\lambda + 3\lambda + 18 = 0$

$\lambda(\lambda-6) - 3(\lambda-6) = 0$

$\lambda = -2, 6, 3$

Ques- $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \Rightarrow [A-\lambda I] = \begin{bmatrix} (-2-\lambda) & 2 & -3 \\ 2 & (1-\lambda) & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$

$(-3, -3, 5)$

$[A-\lambda I] = \begin{vmatrix} (-2-\lambda) & 2 & -3 \\ 2 & (1-\lambda) & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$

$-2-\lambda+1-\lambda-\lambda$

-1

$= \lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45$

$= \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$(\lambda+3) = 0$

$\lambda+3 \overline{)\ \lambda^3 + \lambda^2 - 21\lambda - 45 = 0\ (\ \lambda^2 - 2\lambda - 15}$

$\lambda^3 + 3\lambda^2$

$-2\lambda^2 - 21\lambda - 45$

$-2\lambda^2 - 6\lambda$

$-15\lambda - 45$

$M_{11} = \begin{bmatrix} 1 & -6 \\ -2 & 0 \end{bmatrix} = -12$

$M_{22} = \begin{bmatrix} -2 & -3 \\ -1 & 0 \end{bmatrix} = -3$

$M_{33} = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} = -2-4 = -6$

DATE
PAGE

$\lambda^2 - 2\lambda - 15 = 0$

$\lambda^2 - 5\lambda + 3\lambda - 15 = 0$

$\lambda(\lambda+3) - 5(\lambda+3) = 0$

$(\lambda-5)(\lambda+3) = 0$

$\boxed{\lambda = 5, -3}$ & $\boxed{\lambda = -3}$

so $\boxed{\lambda = 5, -3, -3}$ ✓

Date :- 4-10-22

No Topic

Eigen vectors:

Non-zero column matrix.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}_{3\times3}$$

In Eigen value

$$[A - \lambda I] = \begin{bmatrix} (1-\lambda) & 1 & 3 \\ 1 & (5-\lambda) & 1 \\ 3 & 1 & (1-\lambda) \end{bmatrix}$$

$\boxed{\lambda = -2, 3, 6}$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$ be an Eigen vector corresp. to $\lambda = 3$

$\boxed{AX = \lambda X}$

start from here in ques →

$AX - \lambda X = 0$

$\boxed{X[A - \lambda I] = 0}$

In this we have to check for all the values of λ

For $\lambda = 3$

so $[A - 3I]X = 0$

$$[A - 3I] = \begin{bmatrix} (1-3) & 1 & 3 \\ 1 & (5-3) & 1 \\ 3 & 1 & (1-3) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho[A - 3I] = 2$

DATE
PAGE

No. of Indep. Eigen vector corresp. to ($\lambda=3$) = order of Matrix (no. of variables) − $\rho(A-3I)$

$= 3-2=1$

Now, $[A-3I]X=0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x+2y+z=0 \\ 0+5y+5z=0 \\ 0+0+0=0 \end{matrix}$$

$\Rightarrow \boxed{y=-z}$

$x+2(-z)+z=0$

$\boxed{x=z}$, z is indep.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$ → Eigen vector corresp to $\lambda=3$

(ii) For $\boxed{\lambda=-2}$

$[A+2I]X=0$

$$[A+2I] = \begin{bmatrix} (1+2) & 1 & 3 \\ 1 & (5+2) & 1 \\ 3 & 1 & (1+2) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad \sim \begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A+2I)=2$

No. of Indep. Eigen vector corresp $\boxed{\lambda=-2}$ = order of Matrix − $\rho[A+2I] = 3-2=1$

$[A+2I]X=0$

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x+7y+z=0$

$-20y=0 \Rightarrow \boxed{y=0}$ → Fixed

$x+z=0 \Rightarrow \boxed{x=-z}$ → indep.

$$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-z\\0\\z\end{bmatrix}=z\begin{bmatrix}-1\\0\\1\end{bmatrix}$$ → Eigen value to $\lambda=9$

(iii) For $\boxed{\lambda=6}$

$X[A-6I]=0$

$$[A-6I]=\begin{bmatrix}(1-6) & 1 & 3\\ 1 & (5-6) & 1\\ 3 & 1 & (1-6)\end{bmatrix}=\begin{bmatrix}-5 & 1 & 3\\ 1 & -1 & 1\\ 3 & 1 & -5\end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim\begin{bmatrix}1 & -1 & 1\\ -5 & 1 & 3\\ 3 & 1 & -5\end{bmatrix} \quad \begin{matrix}R_2\to R_2+5R_1\\ R_3\to R_3-3R_1\end{matrix} \quad \sim\begin{bmatrix}1 & -1 & 1\\ 0 & -4 & 8\\ 0 & 4 & -8\end{bmatrix}$$

$R_3\to R_3+R_2 \quad \sim\begin{bmatrix}1 & -1 & 1\\ 0 & -4 & 8\\ 0 & 0 & 0\end{bmatrix}$

$\rho[A-6I]=2$

Indep. vectors $=3-2=1$

$(A-6I)X=0$

$$\begin{bmatrix}1 & -1 & 1\\ 0 & -4 & 8\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}x-y+z=0\\ -4y+8z=0\\ y=2z\end{matrix}$$

$\boxed{y=2z}$ → indep vector.

$x-2z+2z=0 \Rightarrow \boxed{x=0}$

$x-2z+z=0 \Rightarrow x-z=0 \Rightarrow \boxed{x=z}$

$$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}z\\2z\\z\end{bmatrix}=z\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Ans

DATE
PAGE

(2) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ Find eigen vectors of this Matrix

$\boxed{\lambda = 5, -3, -3}$

(i) Eigen vector corresp. to $\lambda = -3$

$X[A+3I] = 0$

$[A+3I] = \begin{bmatrix} (-2+3) & 2 & -3 \\ 2 & (1+3) & -6 \\ -1 & -2 & (0+3) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\rho(A+3I) = 1$

No. of indep. eigen vectors $= 3 - 1 = \boxed{2}$

$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x + 2y - 3z = 0$

$\boxed{x = 3z - 2y}$

take two indep. vectors (y & z)

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z - 2y \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} z$ Ans

(ii) Eigen vector corresp. to $\boxed{\lambda = 5}$

$X[A-5I] = 0$

$[A-5I] = \begin{bmatrix} (-2-5) & 2 & -3 \\ 2 & (1-5) & -6 \\ -1 & (2+5) & (0-5) \end{bmatrix} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$

$R_3 \leftrightarrow R_1$

$\begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 7R_1$ $= \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & +16 & 32 \end{bmatrix}$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A-3I) = 2$ L.I eigen

No. of Independent vectors $= 3-2 = 1$

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x-2y-5z=0$

$-8y-16z=0$

$y = \frac{-16}{8}z = -2z$ L.I

$\boxed{y=-2z}$

$-x+2(-2z)-5z=0$

$-x+4z-5z=0 \Rightarrow -x-z=0 \Rightarrow \boxed{x=-z}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$ is eigen vector corresp. to $\boxed{\lambda=5}$

Date:- 6-10-22

Ques 1:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & -1 & 4 \end{bmatrix}$ Find eigen values & vectors.

$$[A-\lambda I] = \begin{bmatrix} (1-\lambda) & 0 & 0 \\ 0 & (2-\lambda) & 1 \\ -1 & -1 & (4-\lambda) \end{bmatrix}$$

Charac polynomial.

$|A-\lambda I| = \lambda^3 - (\text{sum of dig. element})\lambda^2 + [\text{sum of minor}]\lambda - \det A = 0$

OR

$|A-\lambda I| = (1-\lambda)[(2-\lambda)(4-\lambda)+1] + 0 0$

$= (1-\lambda)[8-6\lambda+\lambda^2+1]$

$= (1-\lambda)(\lambda^2-6\lambda+9)$.

Char eqn: $|A-\lambda I| = 0$

$(1-\lambda)(\lambda^2-6\lambda+9) = 0$

$(1-\lambda)(\lambda-3)^2 = 0$

$\boxed{\lambda = 1, 3, 3}$ → eigen value.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$ be an eigen vector corresp, $\lambda = 1$

$[A - 1I]X = 0$

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & (2-1) & 1 \\ -1 & -1 & (4-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} -1 & -1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho[A - 1I] = 2$

L.I. eigen vector (Ind) $= 3 - 2 = 1$

Now

$$\begin{bmatrix} -1 & -1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x - y + 3z = 0$

$-y + z = 0$

$\Rightarrow \boxed{y = z}$ → indp. vector

$-x + z + 3z = 0$

$-x + 4z = 0$

$\boxed{x = 4z}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$ Ans

$\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$ = eigen vector corresp to $\boxed{\lambda = 1}$

(ii) Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$ be an eigen vector corres. to $\boxed{\lambda = 3}$

$[A - 3I]X = 0 \Rightarrow$

$$\begin{bmatrix} (1-3) & 0 & 0 \\ 0 & (2-3) & 1 \\ -1 & -1 & (4-3) \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

$R_3 \to R_1$

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ -2 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_1 \quad \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$ $\sim \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\rho(A-3I) = 2$

no of L.I. eigen vector $= 3-2 = 1$

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x-y+z=0$
$-y+z=0$
$\boxed{y=z}$

$-x-y+z=0 \Rightarrow \boxed{x=0}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ eigen vector corresp to $\boxed{\lambda=3}$

Diagonilazable:- Any square matrix, have same no. of ~~eig~~ L.I eigen vector as of the order of given matrix

→ means there exist a non-singular matrix P i.e. $\boxed{P^{-1}AP = D}$

eg:- $P = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix}$ (columns: $\downarrow$ -3, $\downarrow$ -3, $\downarrow$ 5)

$$P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Date : 11-10-22

★ **Properties of Eigen values:**

1) ~~Some~~ Sum of the diagonal elements of A = trace A.
2) Sum of the eigen values = trace of A.
3) Product of eigen values = det A.

eg: $\begin{bmatrix} 2 & - & - \\ - & 2 & - \\ - & - & 3 \end{bmatrix}_{3\times3}$

(a) (1, 2, 3) (b) (2, 4, 1) (c) (5, 6, 0)

DATE
PAGE

Let λ is an eigen value of A

1.)	A^T	λ
2.)	A^{-1}	$\frac{1}{\lambda}$, $\lambda \neq 0$
3.)	A^m	λ^m
4.)	Adj(A)	$\frac{\lvert A \rvert}{\lambda}$
5)	$[A-kI]$	$\lambda - k$

If 0 is eigen value of some matrix then,

Ques: If A has eigen value λ then $(kA)^T$ = ?

$(kA)^T = (\lambda k)$

6.) If zero is an eigen value of some Matrix A, then A is singular matrix, means A^{-1} doesn't exist.

* For triangular Matrix/diagonal Matrix, eigen values are just the diagonal elements of given matrix. eg: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\Rightarrow \lambda = 1, 2, 1$

For $A^2 \Rightarrow \lambda_1^2, \lambda_2^2, \lambda_3^2$

$\lambda = 1, 4, 1$

(1) For Hermition Matrix $(\bar{A})^T = A$ or $A^\theta = A$	Real
(2) For skew-Hermition matrix $A^\theta = -A$	Purly imaginary or zero.
(3) Unitary Matrix $A^\theta A = I = AA^\theta$	complex no. with ± 1 (Modulus of 1.)
(4) Orthogonal Matrix $AA^T = I = A^T A$	± 1
(5) Symmetric matrix $A^T = A$	Real
(6) Skew-symmetric $A^T = -A$	Imaginary or zero
(7) Idempotent Matrix $A^2 = A$	0 or 1

eg:- eigen value of square matrix is $\lambda = 0$, then

Ans:- $\det = 0$ = product of eigen vector

$|A| = 0$ = singular

DATE
PAGE

Ques Find inverse by using Matrix inversion Method,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Ans $A = IA$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \qquad R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} A \qquad \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} A \qquad R_1 \to 3R_1 + 2R_3$$

$-3-2 \qquad 3-4$

$-3+2$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} +2 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} A$$

$R_3 \to \frac{-1}{3} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ 0 & 1 & 0 \\ -1/3 & -1/3 & +2/3 \end{bmatrix} A$$

DATE
PAGE

Q:- $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$

$M_1 = \begin{bmatrix} 2 & 4 & 3 \\ -3 & -2 & -1 \\ 6 & -1 & 7 \end{bmatrix}$ $\quad 2(-14-1)-4|-3|$

$= +3(3+12)$

$= -30 - 4(66) + 45$

$= -30 + 60 + 45 = 75$

$\boxed{\rho(A) = 3}$

★ Cayley Hamilton theorm:- Every square matrix satisfies its characteristic equation

$\lambda^3 - x\lambda^2 + \lambda - |A| = 0 \Rightarrow A^3 - xA^2 + A - |A| = 0$

Ques:- Verify Cayley Hamilton theorm for the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ also find A^{-1} & $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

Char eqn $= [A - \lambda I]$

$|A - \lambda I| = 0$

$\lambda^3 - (\text{sum of dia. element of } A)\lambda^2 + (\text{sum of minor (diagonal elements)})\lambda - \det A = 0$

$= \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

Now $A^3 - 5A^2 + 7A - 3 = 0 \quad -(1)$

$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$A^3 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$

substitute in (1)

$\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 14-25+14-3 & 13-20+7+0 & 13-20+7+0 \\ 0 & 1-5+7-3 & 0 \\ 13-20+7 & 13-20+7+0 & 14-25+14-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence verified

$$A^3 - 5A^2 + 7A - 3I = 0$$ Ans

(ii) Find A^{-1} :-

$$A^3 - 5A^2 + 7A - 3I = 0$$

multiply by A^{-1}

$$A^{-1}(A^3 - 5A^2 + 7A - 3I) = A^{-1}0 \Rightarrow A^2 - 5A + 7I - 3A^{-1} = 0$$

$$\Rightarrow A^2 - 5A + 7I = 3A^{-1}$$

$$A^{-1} = \frac{1}{3}\left[\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{3}\begin{bmatrix} (5-10+7) & (4-5) & (4-5) \\ 0-0+0 & (1-5+1) & (0+0+0) \\ 4-5 & 4-5 & 5-10+7 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

(iii) Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ — (1)

$A^3 = 5A^2 - 7A + 3I$ multi with A^5, — (2) (calculate with highest power.)

$A^8 = 5A^7 - 7A^6 + 3A^5$ put in (1)

$(5A^7 - 7A^6 + 3A^5) - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^4 - 5A^3 + 8A^2 - 2A + I$, highest power 4; — (3)

Now multi (3) with A

$A^4 = 5A^3 - 7A^2 + 3A$ put in (3)

$= (5A^3 - 7A^2 + 3A) - 5A^3 + 8A^2 - 2A + I$

$= A^2 + A + I$

Exp $= A^2 + A + I$

$$\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

DATE
PAGE

Shortcut:

① A is square matrix 3×3, rank = 2
this is singular, (Rank < 3)

② $\begin{bmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{bmatrix}$ Find x so that rank < 3, ↔ Singular |A|=0
so use |A| = 0 , use $C_1+C_2+C_3$

$\begin{bmatrix} 3x-8 & 3 & 3 \\ 3x-8 & 3x-8 & 3 \\ 3x-8 & 3 & 3x-8 \end{bmatrix} = 3x-8 \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & 3x-11 & 0 \\ 0 & 0 & 3x-11 \end{bmatrix} =$

③ For Normal form :- $\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}_{3\times4}$ Let $\rho = 3$ so $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(a) $[I_3, 0]$ (b) $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

④ only for 4×4 $V_1 = [2\ 3\ 5]$, $V_2 = [1\ 1\ 1]$, $V_3 = [2\ -1\ 0]$
$V_4 = [5\ 6\ -9]$

(a) $V_4 = 5V_1 + 7V_2 - V_3$ (b) $V_1 = V_2 - V_3 + V_4$
check by putting values.

⑤ Eigen vector & values :-
$\lambda^3 - (\text{diagonal sum})\lambda^2 + (\text{sum of minors})\lambda - \det A = 0$

⑥ For roots :- remaing root
$\lambda^2 - [\text{sum of diagonal} - \text{First root})\lambda + \frac{\det A}{\text{1st root}} = 0$

⑦ Eigen vectors :-
check for $AX = \lambda X$.
eg:- (a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ For $\lambda = 3$
so $A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Unit 2 :-

Complex Number

DATE 27/10/22
PAGE

$a+ib$; a & b are real no.

$i^2 = -1$

$a+ib$, real part $= a$

imaginary part $= b$

(1) If real part $= 0$, then this is purely imaginary.

(2) If img. part $= 0$, " " " " real, (eg:- $2 = 2+i0$)

Conjugate ($\bar{z}$) :-

$z = 2+3i$ then $\bar{z} = 2-3i$

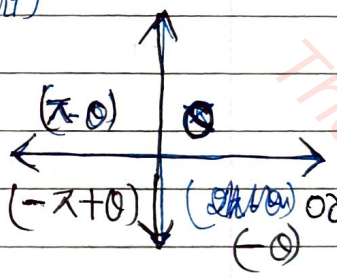
Modulus = $|z|$

if $z = a+bi$

then $|z| = \sqrt{(a)^2+(b)^2} = \sqrt{(\text{Real pt})^2+(\text{img part})^2}$

Argument (θ) :

$$\theta = \tan^{-1}\left(\frac{\text{Img. part}}{\text{Real part}}\right)$$



Q. $-1-\sqrt{3}i$

$\bar{z} = -1+\sqrt{3}i$

$|z| = \sqrt{(-1)^2+(-\sqrt{3})^2}$

$= 2$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)$

$\theta = -\frac{2\pi}{3}$

$-\pi \le \theta \le \pi$

(2) $-1+i\sqrt{3}$

$\bar{z} = -1-i\sqrt{3}$

$|z| = \sqrt{(-1)^2+(\sqrt{3})^2}$

$= 2$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$

$\theta = \frac{2\pi}{3}$

(3) $1-i\sqrt{3}$ (IV quad)

$\bar{z} = 1+i\sqrt{3}$

$|z| = 4$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$

$\theta = -\frac{\pi}{3}$

DATE 01-11-22
PAGE

Note: → We can not determine which complex no. is greater or smaller.

$a+ib = c+id$ OR $[a\ b]\ [c\ d]$

$a=c$ & $b=d$

⇒ zero is always a complex number.

$0+i0$

⇒ Every non-zero complex no. can be expressed as $r(\cos\theta+i\sin\theta)$

$$a+ib = r(\cos\theta+i\sin\theta) = r\,cis\,\theta$$

$r = \sqrt{a^2+b^2}$ modulus

$\theta = \tan^{-1}\frac{b}{a}$ (considering the quadrant)

* → cis form of:

$1 = \cos 0 + i\sin 0 = 1+0 = cis\,0$ $[\because -\pi \le \theta \le \pi]$

$-1 = \cos\pi + i\sin\pi = cis\,\pi$

$i = \cos(\pi/2) + i\sin(\pi/2) = cis(\pi/2)$

$-i = \cos(\frac{-\pi}{2}) + i\sin(\frac{-\pi}{2}) = cis(\frac{-\pi}{2})$

Ques: $-2-2i$ convert to cis form ?

$-2-2i = r[\cos\theta + i\sin\theta]$

$-2-2i = r\cos\theta + i\,r\sin\theta$

$\therefore -2 = r\cos\theta$ & $-2 = r\sin\theta$ ①

Now squaring & adding

$(-2)^2+(-2)^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$8 = r^2(\cos^2\theta+\sin^2\theta)$ (↳ 1)

$r^2 = 8$

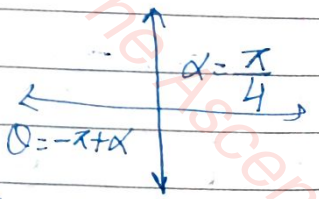
$r = 2\sqrt{2}$ put in ①

↳ distance form origin

$-2 = 2\sqrt{2}\cos\theta$ & $-2 = 2\sqrt{2}\sin\theta$

$\cos\theta = \frac{-1}{\sqrt{2}}$ & $\sin\theta = \frac{-1}{\sqrt{2}}$ → III rd quadrant

$\theta = -\pi + \frac{\pi}{4} = \boxed{\frac{-3\pi}{4}}$



DATE
PAGE

so $-2-2i = 2\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$

$= \boxed{2\sqrt{2}\ \text{cis}\left(\frac{-3\pi}{4}\right)}$ Ans

De-Moiver's Thm :-

Statement :- ① If n is an integer then $(\cos\theta + i\sin\theta)^n = [\cos(n\theta) + i\sin(n\theta)]$

② If n is a fraction then one of the value of $[\cos\theta + i\sin\theta]^n$ will be $[\cos n\theta + i\sin n\theta]$

$(1)^{1/3}$ = It have only 3 roots = $1, \omega, \omega^2$

$(1)^3$ = " " " 1 roots

Case 1: Let n be an positive integer :-

for $n=1$

LHS $(\cos\theta + i\sin\theta)^n = [\cos\theta + i\sin\theta]$

RHS $[\cos(1\theta) + i\sin(1\theta)] = \cos\theta + i\sin\theta$

$\Rightarrow$ LHS = RHS so result is true for $\boxed{n=1}$

Assume that result is true for $n=k$

$\Rightarrow (\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$ ——— ①

To prove that the result is true for $n = k+1$

Consider $[\cos\theta + i\sin\theta]^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)^1$

from eqⁿ ①

$[\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= (\cos k\theta \cdot \cos\theta + i^2 \sin k\theta \cdot \sin\theta) + i(\sin k\theta \cdot \cos\theta + \cos k\theta \cdot \sin\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

$\Rightarrow$ Result is true for $n = k+1$

$\Rightarrow$ Result is true for all +ve integers

$\cos A \cos B \mp \sin A \sin B = \cos(A \pm B)$

$\sin A \cos B \mp \cos A \sin B = \sin(A \pm B)$

DATE 02-11-21
PAGE

Case II :- when n is -ve integer $[n = -d]$, where d is some +ve integer.

Consider $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-d}$

$= \frac{1}{(\cos\theta + i\sin\theta)^d} = \frac{1}{(\cos d\theta + i\sin d\theta)}$ (From I)

Now

$= \frac{1}{(\cos d\theta + i\sin d\theta)} \times \frac{\cos d\theta - i\sin d\theta}{(\cos d\theta - i\sin d\theta)}$

$= \frac{\cos d\theta - i\sin d\theta}{\cos^2 d\theta - i^2\sin^2\theta} = \frac{\cos(-d\theta) + i\sin(-d\theta)}{\cos^2\theta + \sin^2\theta}$

$= \cos n\theta + i\sin(n\theta)$ $(\because n = -d)$

$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ when n is -ve integer.

Case III: when n is equals to zero $[n = 0]$

L.H.S $(\cos\theta + i\sin\theta)^0$

R.H.S $[\cos 0\theta + i\sin 0\theta] = 1$

So result is 1 for $n = 0$

∴ From case 1, 2, 3, we conclude that Result is true for all integer.

Part (b) :- Let $\left[n = \frac{p}{q}\right]$ Here q is +ve integer & p can be any integer (-ve or +ve)

Consider :- $\left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^q$

$= \left[\cos\frac{\theta}{q} \times q + i\sin\frac{\theta}{q} \times q\right]$ {using part 1}

$(\sin\theta+i\cos\theta)^n = (i)^n[\cos\theta - i\sin\theta)^n$

$(\sin\theta-i\cos\theta)^n = (-i)^n[\cos\theta+i\sin\theta)^n$

{ where n's value must be given }

DATE
PAGE

$\Rightarrow \left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^q = (\cos\theta+i\sin\theta)$

$\Rightarrow$ Take q^{th} root on both sides,

$\Rightarrow$ one of the value of $[\cos\theta+i\sin\theta]^{\frac{1}{q}}$ is $\left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^{q\times\frac{1}{q}}$

* Raise the power 'p'.

one of the value of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}$ is $\left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^p$

" " " " " " is $\left[\cos\left(\frac{p}{q}.\theta\right)+i\sin\left(\frac{p}{q}.\theta\right)\right]$

$\Rightarrow (\cos\theta+i\sin\theta)^n$ has one value as $(\cos\theta+i\sin\theta)$ when n is in fraction.

* (1) $(\cos\theta - i\sin\theta)^n = [\cos(n\theta) - i\sin(n\theta)]$

(2) $\frac{1}{(\cos\theta+i\sin\theta)} = (\cos\theta - i\sin\theta)$

(3) $\frac{1}{\cos(10\theta)+i\sin(10\theta)} = \cos(10\theta) - i\sin(10\theta) = (\cos\theta+i\sin\theta)^{-10}$

Simplify :-

$$\frac{(\cos3\theta+i\sin3\theta)^5(\cos2\theta-i\sin2\theta)^3}{(\cos4\theta+i\sin4\theta)^{-9}(\cos5\theta+i\sin5\theta)^9}$$

By using de Moivre's theorm

$$= \frac{(\cos\theta+i\sin\theta)^{5\times3}(\cos\theta+i\sin\theta)^{-6}}{[\cos\theta+i\sin\theta]^{-36}\cdot[\cos\theta+i\sin\theta]^{45}}$$

$$= \frac{(\text{cis}\theta)^{15}\cdot(\text{cis}\theta)^{-6}}{(\text{cis}\theta)^{-36}(\text{cis}\theta)^{45}} = \frac{(\text{cis}\theta)^{15-6}}{(\text{cis}\theta)^{-36+45}} = \frac{(\text{cis}\theta)^9}{(\cos\theta)^9} = 1$$

Note :- $[\sin\theta+i\cos\theta]^7$ then $(i)^7[\cos\theta+i\sin\theta]$

$i^7\left[\frac{1}{i}\sin\theta+\cos\theta\right]^7 = -i[-i\sin\theta+\cos\theta]^7$

⑨ $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$

Ques: Prove that $[1+\sin\theta+i\cos\theta]^n + [1+\sin\theta-i\cos\theta]^n = 2^{n+1}\cos^n\left[\frac{\pi}{4}-\frac{\theta}{2}\right]\cos\left[\frac{n\pi}{4}-\frac{n\theta}{2}\right]$

Consider:- $1+\sin\theta+i\cos\theta = 1+\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\theta\left[\frac{\pi}{2}-\theta\right]$

$= 2\cos^2\frac{1}{2}\left(\frac{\pi}{2}-\theta\right) - i\,2\sin\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\cos\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)$

$1+\cos 2\theta = 2\cos^2\theta$
$1-\cos 2\theta = 2\sin^2\theta$
$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\cos\theta\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right) + i\sin\left[\frac{\pi}{4}-\frac{\theta}{2}\right]\right]\right]$

$\therefore (\cos\theta+i\sin\theta)^n = 2^n\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\text{cis}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n$

$= 2^n\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n\right]$ — (1)

Note:-> change $i \to -i$

$(1+\sin\theta-i\cos\theta)^n = 2^n\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\cos n\left(\frac{\pi}{4}-\frac{\theta}{2}\right) - i\sin n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$ — (2)

Adding (1) & (2) we get.

$(1+\sin\theta+i\cos\theta)^n + (1+\sin\theta-i\cos\theta)^n = 2^n\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\left[\cos\left(n\frac{\pi}{4}-\frac{n\theta}{2}\right) + i\sin\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)\right]\right.$

$+ 2^n\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\left(\cos\left(n\frac{\pi}{4}-\frac{n\theta}{2}\right) - i\sin n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)\right]$

$= 2^n\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cdot\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)(1+1) = 2^{n+1}\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)$

Ques: PT $\dfrac{(1+\sin\theta+i\cos\theta)^n}{(1+\sin\theta-i\cos\theta)^n} = \cos\left(\frac{n\pi}{2}-\theta n\right) + i\sin\left(\frac{n\pi}{2}-\theta n\right)$

$1 = i^2\cos^2\theta - i^2\sin^2\theta$

Ques 2:-

Q.3:-

Ans:1

Que

DATE
PAGE

Ques 2:- $\dfrac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 3\theta + i\sin 3\theta)^{-4}}{(\cos 3\theta + i\sin 3\theta)^2 (\cos 4\theta + i\sin 4\theta)^{-3}}$

$= \dfrac{(\text{cis}\theta)^6 (\text{cis}\theta)^{-12}}{(\text{cis}\theta)^6 (\text{cis}\theta)^{-12}} = 1.$

Q.3: $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \dfrac{(\cos\theta + i\sin\theta)^4}{i^4 \left(\sin\theta \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]\right)^4}$

$= \dfrac{(\text{cis}\theta)^4}{i^4 \left[\frac{1}{i}\sin\theta + \cos\theta\right]^4} = \dfrac{(\text{cis}\theta)^4}{[-i\sin\theta + \cos\theta]^4} = \dfrac{(\text{cis}\theta)^4}{[\text{cis}(-\theta)]^4}$ $(\because i^4 = 1)$

$= \dfrac{[\text{cis}\theta]^4}{[\text{cis}\theta]^{-4}} = [\text{cis}\theta]^8 = \boxed{\text{cis}(8\theta)}$ Ans

Ans 1: $i = \sin^2\theta - i^2\cos^2\theta$

LHS $\left[\dfrac{\sin^2 - i^2\cos^2\theta + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right]^n$ $\quad \because a^2 - b^2 = (a+b)(a-b)$

$= \left[\dfrac{(\sin\theta + i\cos\theta)(\sin\theta - i\cos\theta) + [\sin\theta + i\cos\theta]}{1 + \sin\theta - i\cos\theta}\right]^n$

$= \left[\dfrac{\sin\theta + i\cos\theta\,(\sin\theta - i\cos\theta + 1)}{(1 + \sin\theta - i\cos\theta)}\right]^n$

$= \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$ Ans

Ques:- $\dfrac{(\cos\theta + i\sin\theta)^3 (\cos\theta - i\sin\theta)^5}{(\cos 2\theta + i\sin 2\theta)^4}$

Ans:- $\frac{(\cos\theta)^3 (\text{cis}\theta)^{-5}}{(\text{cis})^8} = \frac{\text{cis}\theta^{(-2)}}{\text{cis}\theta^{(8)}} = \text{cis}(-10\theta)$ Ans

Ques 5:- Prove that $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cdot \cos\frac{n\pi}{4}$

$1+i = r[\cos\theta + i\sin\theta]$.

$r = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

so

$1 = r\cos\theta$ & $1 = r\sin\theta$

$1 = \sqrt{2}\cos\theta$ & $1 = \sqrt{2}\sin\theta$

$\cos\theta = \frac{1}{\sqrt{2}}$ & $\sin\theta = \left(\frac{1}{\sqrt{2}}\right)$

$\cos\theta = \frac{\pi}{4}$, $\sin\theta\left(\frac{\pi}{4}\right)$

$(1+i)^n = \left(\sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]\right)^n$ — ①

$(1-i)^n = \left[\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]^n$ — ② $= (\sqrt{2})^n \text{cis}\left(\frac{-\pi}{4}\right)^n$

① + ②

$(1+i)^n + (1-i)^n = (\sqrt{2})^n \text{cis}\left(\frac{\pi}{4}\right)^n + (\sqrt{2})^n \text{cis}\left(\frac{-\pi}{4}\right)^n$

$= (\sqrt{2})^n \left[\text{cis}\left(\frac{\pi}{4}\right)^n + \text{cis}\left(\frac{-\pi}{4}\right)^n\right]$

$= 2^{n/2} \cdot \left[\cos\left(\frac{\pi}{4}\right)\right]^n [1+1]$

$= 2^{n/2} \cdot 2 \left[\cos\left(n\frac{\pi}{4}\right)\right]$

$= 2^{\frac{n}{2}+2} \left[\cos n\frac{\pi}{4}\right]$ hence proved.

Q.6:- $\tan\alpha - i$ convert to standard form.

$\tan\alpha - i = r[\cos\theta + i\sin\theta]$

$\tan\alpha = r\cos\theta$ — ①

$-1 = r\sin\theta$ ——②

Squaring & adding

$1^2 + \tan^2\alpha = r^2$ (1)

$\sec^2\alpha = r^2$

$[r = \sec\alpha]$ put in ① & ②

$\cos\theta = \frac{\tan\alpha}{\sec\alpha} = \frac{\sin\alpha}{\cos\alpha} \times \cos\alpha$

$\cos\theta = \sin\alpha$ $\left[\because \sin\alpha = \cos\left(\frac{\pi}{2} - \theta\right)\right]$

$= \cos\left(\alpha - \frac{\pi}{2}\right)$ (4th quadrant)

Note:

Date :- 03-11-22

* ① $\text{cis}\theta_1 \cdot \text{cis}\theta_2 \cdot \text{cis}\theta_3$

$\Rightarrow (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) = \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$

② $\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} = \frac{\text{cis}\theta_1}{\text{cis}\theta_2} = \text{cis}(\theta_1 - \theta_2)$

Ques 1:→ If $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$, show that

$x_1 x_2 x_3 \cdots x_n = \cos\left[\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right] + i\sin\left[\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right]$

Hence show that $\lim_{n\to\infty} x_1 x_2 x_3 \cdots x_n = i$

$x_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \text{cis}\left(\frac{\pi}{3}\right)$

$x_2 = \cos\frac{\pi}{(3)^2} + i\sin\frac{\pi}{3^2} = \text{cis}\left(\frac{\pi}{3^2}\right)$

$x_3 = \cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3} = \text{cis}\left(\frac{\pi}{3^3}\right)$

$\vdots$

$x^n = \cos\frac{\pi}{3^n} + i\sin\frac{\pi}{3^n}$

$x_1 \cdot x_2 \cdot x_3 \cdots x_n = \text{cis}\frac{\pi}{3} \cdot \text{cis}\left(\frac{\pi}{3^2}\right) \cdot \text{cis}\left(\frac{\pi}{3^3}\right) \cdots \text{cis}\left(\frac{\pi}{3^n}\right)$

$= \text{cis}\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots \frac{\pi}{3^n}\right)$

$= \text{cis}\left[\frac{\pi}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \frac{1}{3^{n-1}}\right)\right]$

↳ in G.p.

Sum $= \frac{a(1-r^n)}{1-r}$; r (common ratio) $= \frac{1}{3}$

$= \text{cis}\left[\frac{\pi}{3}\left[\frac{\left(1-\frac{1}{3^n}\right)}{\left(1-\frac{1}{3}\right)}\right]\right] = \text{cis}\left[\frac{\pi}{3}\left(\frac{1-\frac{1}{3^n}}{2/3}\right)\right]$

$= \text{cis}\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)$

$x_1 \cdot x_2 \cdot x_3 \cdots x_n = \cos\frac{\pi}{2}\left(1-\frac{1}{3^n}\right) + i\sin\frac{\pi}{2}\left(1-\frac{1}{3^n}\right) = \text{RHS}$

II nd part:- If $\lim_{n\to\infty} x_1 \cdot x_2 \cdot x_3 \cdots x_n = \lim_{n\to\infty}\left[\text{cis}\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)\right]$

$= \lim_{x\to\infty}\left[\cos\frac{\pi}{2}\left(1-\frac{1}{3^n}\right) + i\sin\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)\right]$

$= \lim_{x\to\infty}\left[\cos\frac{\pi}{2}(1-0) + i\sin\left(\frac{\pi}{2}\right)(1-0)\right]$

$= \boxed{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}$ Ans

$= 0 + i1 = i$ hence proved.

$\because \frac{1}{3^n} = \frac{1}{3^\infty}$

$\frac{1}{\infty} = 0$

Ques: If 'α' denotes $\cos 2\alpha + i\sin 2\alpha$, find with similar expression for b, c, d, prove that, (i) $\sqrt{abcd} + \frac{1}{\sqrt{abcd}} = 2\cos(\alpha+\beta+\gamma+\delta)$

(ii) $\sqrt{\frac{ab}{cd}} + \sqrt{\frac{cd}{ab}} = 2\cos(\alpha+\beta-\gamma-\delta)$.

DATE
PAGE

Ans:

$a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$

$c = \cos 2\gamma + i\sin 2\gamma$, $d = \cos 2\delta + i\sin 2\delta$

(i) $abcd = \text{cis}\,2\alpha \cdot \text{cis}\,2\beta \cdot \text{cis}\,2\gamma \cdot \text{cis}\,(2\delta)$

$= \text{cis}\,(2\alpha + 2\beta + 2\gamma + 2\delta) = \text{cis}\,2[\alpha+\beta+\gamma+\delta]$

(ii) $\sqrt{abcd}$ → one of the value of this

$= [\text{cis}\,2(\alpha+\beta+\gamma+\delta)]^{\frac{1}{2}} = \text{cis}\cdot\frac{1}{2}\times 2[\alpha+\beta+\gamma+\delta]$

$= \cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)$ — (1)

→ $\frac{1}{abcd} = (abcd)^{-1} = [\text{cis}\,2(\alpha+\beta+\gamma+\delta)]^{-1}$

$= \cos 2(\alpha+\beta+\gamma+\delta) - i\sin 2(\alpha+\beta+\gamma+\delta)$ —

(By de-Moivre's theorem)

One of the value of :-

$\frac{1}{\sqrt{abcd}} = (abcd)^{-\frac{1}{2}} = [\cos 2(\alpha+\beta+\gamma+\delta) - i\sin 2(\alpha+\beta+\gamma+\delta)]^{\frac{1}{2}}$

$= \cos\frac{2}{2}\cdot\frac{1}{2}(\alpha+\beta+\gamma+\delta) - i\sin\frac{2}{2}\cdot\frac{1}{2}(\alpha+\beta+\gamma+\delta)$ — (2)

adding (1) & (2)

$\sqrt{abcd} + \frac{1}{\sqrt{abcd}} = 2\cos(\alpha+\beta+\gamma)$

(iii) :- $\sqrt{\frac{ab}{cd}} + \sqrt{\frac{cd}{ab}} = 2\cos(\alpha+\beta-\gamma-\delta)$

Consider

$\frac{ab}{cd} = \frac{\text{cis}\,2\alpha\ \text{cis}\,2\beta}{\text{cis}\,2\gamma \cdot \text{cis}\,2\delta} = \frac{\text{cis}\,(2\alpha+2\beta)}{\text{cis}\,(2\gamma+2\delta)} = \text{cis}\,[2\alpha+2\beta-2\gamma-2\delta]$

$\sqrt{\frac{ab}{cd}} = \left(\frac{ab}{cd}\right)^{\frac{1}{2}} = \text{cis}\,[2\alpha+2\beta-2\gamma-2\delta]^{\frac{1}{2}} = \text{cis}\,2(\alpha+\beta+\gamma+\delta)^{\frac{1}{2}}$

$= \text{cis}\,\frac{2}{2}\cdot\frac{1}{2}(\alpha+\beta+\gamma+\delta) = \text{cis}\,(\alpha+\beta-\gamma-\delta)$

$$\sqrt{\frac{ab}{cd}} = \cos(\alpha+\beta-\gamma-\delta) + i\sin(\alpha+\beta-\gamma-\delta) \quad \text{--- (1)}$$

Now $\frac{cd}{ab} = \left(\frac{ab}{cd}\right)^{-1} = \text{cis}[2(\alpha+\beta-\gamma-\delta)]^{-1} = \text{cis}[-2(\alpha+\beta-\gamma-\delta)]$

$$\sqrt{\frac{cd}{ab}} = \left(\frac{cd}{ab}\right)^{1/2} = \text{cis}\left[-2(\alpha+\beta-\gamma-\delta)\right]^{1/2} = \text{cis}\left[-\cancel{2}\cdot\frac{1}{\cancel{2}}(\alpha+\beta-\gamma-\delta)\right]$$

$$= \text{cis}[-(\alpha+\beta-\gamma-\delta)] \quad \text{--- (2)}$$

$$= \cos(\alpha+\beta-\gamma-\delta) - i\sin(\alpha+\beta-\gamma-\delta) \quad \text{--- (3)}$$

① & ② adding

$$\sqrt{\frac{ab}{cd}} + \sqrt{\frac{cd}{ab}} = 2\cos(\alpha+\beta-\gamma-\delta)$$ hence proved.

Date:- 9-11-22

Ques 3:- If α, β be roots of $t^2 - 2t + 2 = 0$ prove that

(a) $\frac{(x+\alpha)^n - (x+\beta)^n}{(\alpha-\beta)} = \frac{\sin(n\phi)}{\sin^n\phi}$

(b) $\frac{(x+\alpha)^n + (x+\beta)^n}{\alpha+\beta} = \frac{\cos(n\phi)}{\sin^n\phi}$, where $x+1 = \cot\phi$

step, $t^2 - 2t + 2 = 0$

roots $1+i$, $1-i$

α ↗ β ↗

$x+\alpha = x+1+i$, $x+\beta = x+1-i$

$= \cot\phi + i$, $\cot\phi - i$

$= \frac{\cos\phi}{\sin\phi} + i$, $\frac{\cos\phi}{\sin\phi} - i$

$= \frac{\cos\theta + i\sin\theta}{\sin\phi}$, $\frac{\cos\phi - i\sin\phi}{\sin\phi}$

★ Roots of the eqⁿ :→ (Complex no.) :→

$(\cos\theta + i\sin\theta)^{p/q}$ = here we have q no. of values

period of $\sin\theta = \cos\theta = 2\pi$

Steps:

1) By De-Moiver's theorem

$(\cos\theta + i\sin\theta)^{p/q} = (\cos p\theta + i\sin p\theta)^{1/q}$

2.) $[\cos(2n\pi + P\theta) + i\sin(2n\pi + P\theta)]^{1/q}$ (periodicity of final)

3.) $\left[\cos\left(\frac{2n\pi + P\theta}{q}\right) + i\sin\left(\frac{2n\pi + P\theta}{q}\right)\right]$, $n = 0, 1, 2, \ldots, q-1$

if $n = 0$

then $\cos\left(\frac{P\theta}{q}\right) + i\sin\left(\frac{P\theta}{q}\right)$

$n = 1$, $\cos\left(\frac{2\pi + P\theta}{q}\right) + i\sin\left(\frac{2\pi + P\theta}{q}\right)$

$\vdots$

$n = (q-1) = \cos\left(\frac{2(q-1)\pi + P\theta}{q}\right) + i\sin\left(\frac{2(q-1)\pi + P\theta}{q}\right)$

If we put $n = q$ (values will starting repeat).

$\cos\left(\frac{2q\pi + P\theta}{q}\right) + i\sin\left(\frac{2q\pi + P\theta}{q}\right) = \cos\left(\frac{2q\pi}{q} + \frac{P\theta}{q}\right) + i\sin\left(\frac{2q\pi}{q} + \frac{P\theta}{q}\right)$

$= \cos\left(2\pi + \frac{P\theta}{q}\right) + i\sin\left(2\pi + \frac{P\theta}{q}\right)$

(Because 2π is the periodic funct. of cos θ & sin θ)

**

Ques 1:- Prove that n^{th} root of unity form a G.P series. Sum of these roots are zero & product is $(-1)^{n+1}$

Ans (i) consider $(1)^{1/n}$ (∵ nth the root of unity) $= (\text{cis } 0)^{1/n}$

$= [\text{cis}(2r\pi + 0)]^{1/n}$

By De-Moiver's theorem,

$[\text{cis}(2r\pi + 0)]^{1/n} = \text{cis}\left(\frac{2r\pi}{n} + \frac{0}{n}\right)$, $r = 0, 1, 2, \ldots n-1$

at $r = 0$, $\text{cis } 0 = 1$

at $r = 1$, $\text{cis}\left(\frac{2\pi}{n}\right) = P$

at $r = 2$, $\text{cis}\left(\frac{4\pi}{n}\right) = \left[\text{cis}\left(\frac{2\pi}{n}\right)\right]^2$ (By De-Moiver's th^m)

$= P^2$

at $r = 3$, $\text{cis}\left(\frac{6\pi}{n}\right) = \left[\text{cis}\left(\frac{2\pi}{n}\right)\right]^3 = P^3$

at $r=(n-1)$, $\text{cis}\left(\frac{2(n-1)\pi}{n}\right) = \text{cis}\left(\frac{2\pi}{n}\right)^{n-1} = p^{(n-1)}$

These roots $1, p, p^2, p^3 \dots p^{(n-1)}$ are in G.P. with common ratio p. where $p = \text{cis}\left(\frac{2\pi}{n}\right)$

(ii) Sum of roots is zero.

proof :- Sum $= 1+p+p^2+p^3 \dots\dots + p^{(n-1)}$

Sum in $S_n = \frac{a(1-r^n)}{1-r}$ n = no. of terms

$$\text{Sum} = \frac{1(1-p^n)}{(1-p)} = \frac{(1-\text{cis}(\frac{2\pi}{n})^n)}{1-\text{cis}(\frac{2\pi}{n})}$$

$$= \frac{1-(\text{cis}(2\pi))}{1-\cos(\frac{2\pi}{n})} = \frac{1-\cos 2\pi - i\sin(2\pi)}{1-\cos(\frac{2\pi}{n})}$$

$$= \frac{1-1-0}{1-p} = 0 \quad \text{hence proved}$$

(iii) Product of the roots $= (-1)^{n-1}$

proof :- $1 \cdot p \cdot p^2 \cdot p^3 \dots p^{(n-1)} = p^{0+1+2+3+\dots+(n-1)}$

Sum of n no. $\frac{n(n+1)}{2}$

$$= p^{\frac{(n-1)(n)}{2}} = p^{\frac{n(n-1)}{2}}$$

$$= \text{cis}\left(\frac{2\pi}{n}\right)^{\frac{n(n-1)}{2}}$$

here $\frac{n(n-1)}{2}$ is integer

so use De-Moivre's n^{th}

$$= \text{cis}\left(\frac{2\pi}{n} \cdot \frac{n}{2}\right)^{(n-1)} = \text{cis}(\pi)^{(n-1)}$$

$$= (-1)^{(n-1)}$$

hence proved

* Sum of m^{th} power of n^{th} root of unity :-

Note :- If we have some power of roots, then we have 2 ans either power (m) or zero

zero = if n is not the multiple of m,

Otherwise sum will be n. (m being a +ve integer)

eg :- $1, p, p^2 \dots p^{(n-1)}$

$1^m, p^m, p^{2m} \dots p^{m(n-1)}$

DATE
PAGE

sum: $1 + p^m + p^{2m} + \dots p^{m(n-1)}$

$= 1 + \text{cis}\left(\frac{2\pi}{n}\right)^m + \text{cis}\left(\frac{2\pi}{n}\right)^{2m} \dots \text{cis}\left(\frac{2\pi}{n}\right)^{m(n-1)}$

$= 1 + p^m + (p^m)^2 + (p^m)^3 \dots + (p^m)^{(n-1)}$

$= \frac{1(1-(p^m)^n)}{1-p^m}$

eg: 15^{th} power of 5^{th} root of unity.

$1, p, p^2, p^3, p^4$

$1^{15} + p^{15} + p^{2(15)} + p^{3(15)} + p^{4(15)} = 5$ { 15 is multiple of 15 then ans is 5

0 (power is not multiple of 5 then 0)

5 (5 is multi. of 15)

$= \frac{1}{1}\left(\text{cis}\frac{2\pi}{n} \times m\right)^1 \cdot \left[\text{cis}\left(\frac{2\pi m}{n}\right)\right]^2 \dots \left[\text{cis}\left(\frac{2\pi m}{n}\right)\right]^{(n-1)}$

Let $\boxed{\text{cis}\frac{2\pi}{n} \cdot m = t}$

sum:- $1 + t + t^2 + \dots t^{(n-1)}$

using sum in GP $= \frac{1(1-r^n)}{1-r}$ ✓ Common ratio $= t = \text{cis}\frac{2\pi}{n} \cdot m$

sum $= \frac{1(1-t^{(n)})}{1-t}$

* **Case 1:** when m is not multiple of n

$\frac{1(1-t^n)}{1-t}$ (n = no. of terms)

$= \frac{1 - \left[\text{cis}\left(\frac{2\pi}{n} \cdot m\right)\right]^n}{1-t} = \frac{1 - \text{cis}\frac{2\pi m}{n} \cdot n}{1-t}$

$= \frac{1-1}{1-t} = 0$ Ans

* **Case II:** When m is multiple of n.

so $m = n(d)$ where $(d = 1, 2, 3)$

d +ve integer.

$$\text{sum} = 1 + t + t^2 + t^3 + \dots + t^{(n-1)}$$

$$= 1 + \text{cis}\left(\frac{2\pi m}{n}\right) + \left[\text{cis}\left(\frac{2\pi m}{n}\right)\right]^2 + \dots + \left[\text{cis}\left(\frac{2\pi m}{n}\right)\right]^{n-1}$$

$$= 1 + \frac{\text{cis}\, 2\pi(nd)}{n} + \left[\text{cis}\left(\frac{2\pi \cdot nd}{n}\right)\right]^2 + \dots \left[\text{cis}\left(\frac{2\pi \cdot nd}{n}\right)\right]^{n-1}$$

$$= 1 + 1 + 1 \dots + 1 = n \quad \text{Ans} \quad [\because m = nd]$$

Ques:- Solve the eqⁿ $x^5 - 1 = 0$ and show that n^{th} power of roots always vanishes (0) unless n is multiple of 5, then n being an integer.

$x^5 = 1 \Rightarrow x = (1)^{1/5} = \text{cis}(0)^{1/5}$

$= \text{cis}(2\pi \times 0)^{1/5}$

By De-Moivre's thm

$(1)^{1/5} = \text{cis}\left(\frac{2\pi}{5} + \frac{0}{5}\right)$

at $r = 0$, $\text{cis}(0) = 1$

$r = 1$, $\text{cis}\left(\frac{2\pi}{5}\right) = t$

$r = 2$, $\left[\text{cis}^2\left(\frac{2\pi}{5}\right)\right] = \text{cis}\left(\frac{2\pi}{5}\right)^2 = t^2$

$r = 3$, $\text{cis}\left(\frac{2\pi}{5} \cdot 3\right) = \text{cis}\left(\frac{2\pi}{5}\right)^3 = t^3$

$r = 4 \quad = t^4$

Sum: $1 + t + t^2 + t^3 + t^4$

GP: $\frac{1(1 - t^5)}{(1 - t)}$ $\quad \frac{1 - \left(\text{cis}\frac{2\pi}{5}\right)^5}{1 - t}$ $\quad \frac{1 - \text{cis}\, 5\left(\frac{2\pi}{5}\right)}{1 - t}$

$= \frac{1 - 1}{1 - t} = 0$

Q: $(-1)^{1/4}$

$(-1)^{1/4} = (\text{cis}\,\pi)^{1/4} = \left(\text{cis}\,[2r\pi+\pi]\right)^{1/4}$

or $\left[\text{cis}\,(2r+1)\pi\right]^{1/4}$

$= \text{cis}\,(2r+1)\cdot\frac{\pi}{4}$ put $r = 0, 1, 2, 3$

$r=0$, $\text{cis}\,(0+1)\frac{\pi}{4} = \text{cis}\,\frac{\pi}{4} = \cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)$

$= \boxed{\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}}$

$r=1$, $\text{cis}\,(2(1)+1)\cdot\frac{\pi}{4} = \text{cis}\left(3\frac{\pi}{4}\right) = \text{cis}\left(\pi-\frac{\pi}{4}\right) = \boxed{\frac{-1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}}$

$r=2$, $\text{cis}\left(5\frac{\pi}{4}\right) = \text{cis}\left(\pi+\frac{\pi}{4}\right) = \cos\left(\pi+\frac{\pi}{4}\right)+i\sin\left(\pi+\frac{\pi}{4}\right)$

$= \boxed{\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}$

$r=3$, $\text{cis}\left(7\frac{\pi}{4}\right) = \text{cis}\left(2\pi-\frac{\pi}{4}\right) = \cos\left(2\pi-\frac{\pi}{4}\right)+i\sin\left(2\pi-\frac{\pi}{4}\right)$

$= \boxed{\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}}$ Ans

Ques 32 → If α, β be roots of $t^2 - 2t + 2 = 0$ prove that

(a) $\dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta} = \dfrac{\sin(n\phi)}{\sin^n\phi}$

(b) $\dfrac{(x+\alpha)^n + (x+\beta)^n}{\alpha+\beta} = \dfrac{\cos(n\phi)}{\sin^n\phi}$, where $x+1 = \cot\phi$

Ans: $t^2 - 2t + 2 = 0$

$t = \dfrac{2 \pm \sqrt{4-8}}{2(1)}$

$$x = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

roots are $1+i$ & $1-i$ put in eq^n (1)

$$\frac{(x+1+i)^n - (x+1-i)^n}{2i} = \frac{(\cot\phi + i)^n - (\cot\phi - i)^n}{2i}$$

$$= \frac{\left(\frac{\cos\phi}{\sin\phi} + i\right)^n - \left(\frac{\cos\phi}{\sin\phi} - i\right)^n}{2i} = \frac{\left(\frac{\cos\phi + i\sin\phi}{\sin\phi}\right)^n - \left(\frac{\cos\phi - i\sin\phi}{\sin\phi}\right)^n}{2i}$$

using De moivre's theorm:-

$$\frac{\cos n\phi + i\sin n\phi}{[\sin(n\phi)]^n} - \frac{(\cos n\phi - i\sin n\phi)}{[\sin(n\phi)]^n} = \frac{0 + 2i\sin n\phi}{2i \sin^n\phi}$$

$$= \frac{\sin(n\phi)}{\sin^n\phi} \text{ hence proved.}$$

(ii) $$\frac{(x+2)^n + (x+\beta)^n}{\alpha+\beta} = \frac{(x+1+i)^n + (x+1-i)^n}{2i}$$

$(\because x+1 = \cot\phi)$ $$= \frac{(\cot\phi + i)^n + (\cot\phi - i)^n}{2i} = \left(\frac{\cos\phi}{\sin\phi} + i\right)^n + \left(\frac{\cos\phi}{\sin\phi} - i\right)^n$$

$$= \frac{\frac{(\cos\phi + i\sin\phi)^n}{\sin^n\phi} + \frac{(\cos\phi - i\sin\phi)^n}{\sin^n\phi}}{2i} = \frac{2\cos n\phi}{2i \cdot \sin^n\phi}$$

hence proved.

date :- 10-11-22

* sol^n of equations :-

Ques:1 $x^3 + 8 = 0$

$x^3 = -8$

$x = (-8)^{1/3}$

$= 2(-1)^{1/3}$

$= 2[\text{cis}\,\pi]^{1/3}$

$= 2[\text{cis}(2n\pi + \pi)]^{1/3}$

DATE
PAGE

$= 2\left[\text{cis}\frac{(2r+1)\pi}{3}\right]$, $r=0,1,2$

at $r=0$

$x = 2\left(\text{cis}\frac{\pi}{3}\right) = 2\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right] = 2\left[\frac{1}{2}+\frac{i\sqrt{3}}{2}\right]$

at $r=1$,

$x = 2\,\text{cis}\left(\frac{2\pi+\pi}{3}\right) = 2\,\text{cis}\left(\frac{3\pi}{3}\right) = 2(-1) = -2$

at $r=2$,

$x = 2\,\text{cis}\left(\frac{4\pi+\pi}{3}\right) = 2\,\text{cis}\left(\frac{5\pi}{3}\right) = 2\left[\cos\left(2\pi-\frac{\pi}{3}\right)+i\sin\left(2\pi-\frac{\pi}{3}\right)\right]$

$= 2\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = 2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right]$

Roots are :→

$-2,\ 2\left[\frac{1}{2}+\frac{i\sqrt{3}}{2}\right],\ 2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$ Ans

Ques: solve $x^4-x^3+x^2-x+1=0$ → G.P. with common ratio $\frac{-1}{x}$

OR

$1-x+x^2-x^3+x^4=0$ = sum of G.P series with C.R $=-x$ $\left\{\frac{a(1-r^n)}{1-r}\right\}$

$1-x+x^2-x^3+x^4=0$ $\frac{1\left(1-(-x)^5\right)}{1-(-x)} = \frac{1+x^5}{1+x} = 0$ $x\neq -1$

$1+x^5=0$,

$x^5 = -1$

$x = (-1)^{1/5} = [\text{cis}(\pi)]^{1/5}$

periodicity

$x = \text{cis}(2r\pi+\pi)^{1/5} = \text{cis}\left(\frac{2r\pi+\pi}{5}\right)$, $r=0,1,2,3,4$

at $r=0$

$x = \text{cis}\left(\frac{\pi}{5}\right) = \cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$

at $r=1$

$x = cis\left(\frac{2\pi+\pi}{5}\right) = cis\left(\frac{3\pi}{5}\right) = \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)$

at $r=2$

$x = cis\left(\frac{4\pi}{5}\right) = cis(\pi) = -1$ rejected

at $r=3$,

$x = cis\left(\frac{6\pi}{5}\right) = \cos\left(2\pi-\frac{3\pi}{5}\right) + i\sin\left(2\pi-\frac{3\pi}{5}\right)$

$= \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

at $r=4$,

$x = cis\left(\frac{8\pi}{5}\right) = \cos\left(2\pi-\frac{\pi}{5}\right) + i\sin\left(2\pi-\frac{\pi}{5}\right)$

$= \cos\frac{\pi}{5} - i\sin\left(\frac{\pi}{5}\right)$

Reject $x=-1$

Roots, $\cos\frac{m\pi}{5} \pm i\sin\frac{m\pi}{5}$ where $m = 1, 3$.

Ques:- $x^8+x^6+x^4+x^2+1=0$

$1+x^2+x^4+x^6+x^8=0$

Sum of G.P. series with C.R. $= x^2$

Sum: $\frac{1(1-(x^2)^5)}{1-x^2} = \frac{1-x^{10}}{1-x^2} \neq 0$, $x \neq \pm 1$ { Total 10 roots 2 are Rejected so 8 left. }

$1-x^{10}=0$

$x^{10}=1 \Rightarrow x = 1^{\frac{1}{10}} = 1^{\frac{-10}{10}}$

$x = cis(0)^{1/10} = cis(2\pi r+0)^{1/10} = cis\left(\frac{2\pi r}{10}+0\right)$

$\Rightarrow x = cis\left(\frac{\pi r}{5}+0\right)$ $r = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

$r=1$, $x = cis\left(\frac{\pi}{5}\right)$

$r=2$, $x = cis\left(\frac{2\pi}{5}\right)$

$r=3$:- $x = \text{cis}\left(\frac{3\pi}{5}\right)$, $r=4$, $x = \left(\frac{4\pi}{5}\right) = \text{cis}\left(\pi - \frac{\pi}{5}\right)$

$= -\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)$

at $r=5$,

$x = \text{cis}\left(\frac{5\pi}{5}\right) = \text{cis}\,\pi = -1$ (we have to reject this value)

at $r=8$, $x = \text{cis}\left(\frac{8\pi}{5}\right) = \text{cis}\left(2\pi - \frac{\pi}{5}\right)$

all root are valid except 1 & -1 Ans

Q.4:- $z^4 + 4z^2 + 16 = 0$

$16 + 4z^2 + z^4 = 0$, $CoR = 4z^2$

$x = \frac{16\left(1 - \left(\frac{z^2}{4}\right)^3\right)}{1 - \frac{z^2}{4}}$ $= \frac{16\left[64 - z^6\right]}{64} = 0$, $z \neq \pm 2$

$64 - z^6 = 0$

$z^6 = 64 \Rightarrow z = \left[64^{1/2}\right]^{1/3} = [8]^{1/3} = 2 = 2(1)^{1/6}$

$z = 2\,\text{cis}(0)^{1/6} = 2\,\text{cis}\left(\frac{2r\pi + 0}{6}\right)$, $r = 0, 1, 2, 3, 4, 5$

at $r=0$, $z = 2\,\text{cis}(0) = 2[1] = 2$

at $r=1$, $z = 2\,\text{cis}\left(\frac{2\pi}{3}\right) = 2\left[\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right]$

at $r=2$, $z = 2\,\text{cis}\left(\frac{4\pi}{3}\right) = 2\left[\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right]$

at $r=3$, $z = 2\,\text{cis}\left(\frac{6\pi}{3}\right) = 2[-1] = -2$

at $r=4$, $z = 2\,\text{cis}\left(\frac{4\pi}{3}\right) = 2\,\text{cis}\left(\pi + \frac{\pi}{3}\right) = 2\left[\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right]$

at $r=5$, $z = 2\,\text{cis}\left(\frac{5\pi}{3}\right) = 2\,\text{cis}\left(2\pi - \frac{\pi}{3}\right) = 2\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$

[at $r=6$, $z = cis\left(\frac{6\pi}{3}\right) = cis(2\pi) = cos(2\pi) + i\sin(2\pi) = 1$] ✗

roots are :- $\sqrt{3}+i$, $\sqrt{3}-i$, $-\sqrt{3}-i$, $-\sqrt{3}+i$

we have to ignore ± 2 value.

Ques 5 ★★ $x^9 - x^5 + x^4 - 1 = 0$

$x^5[x^4-1] + 1[x^4-1] = 0$

$[x^5+1][x^4-1] = 0$

$x^5 = -1$	$x^4 = 1$
$x = (-1)^{1/5}$	$x = (1)^{1/4}$
$x = cis(\pi)^{1/5}$	$x = cis(0)^{1/4}$
$x = cis(2r\pi+\pi)^{1/5}$	$x = cis(2r\pi+0)^{1/4}$

Ques: 6. P.T. roots of the eq^n $(x-1)^n = x^n$ (n being +ve integer) are $\frac{1}{2}\left(1+i\cot\frac{r\pi}{n}\right)$, where r have value $0, 1, 2, \ldots$

$(x-1)^n = x^n$

$\frac{(x-1)^n}{x^n} = 1 \Rightarrow \frac{x-1}{x} = (1)^{1/n}$

$\frac{x}{x} - \frac{1}{x} = (cis\,0)^{1/n}$

$1 - \frac{1}{x} = [cis(2r\pi+0)]^{1/n}$

$1 - \text{cis}\left(\frac{2r\pi}{n}\right)^{1/n} = \frac{1}{x}$

$1 - \text{cis}\left(\frac{2r\pi}{n}\right) = \frac{1}{x}, \quad r = 0, 1, 2, \ldots (n-1)$

$1 - \text{cis}\,\theta = \frac{1}{x}, \quad \text{where } \theta = \frac{2r\pi}{n} \quad \text{---} (1)$

$\Rightarrow x = \frac{1}{1 - \text{cis}\,\theta}$

$x = \frac{1}{1 - \cos\theta - i\sin\theta} = \frac{1}{2\sin^2(\theta/2) - i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$\left\{ \begin{array}{l} 1 - \cos 2A = 2\sin^2 A \\ \sin 2A = 2\sin A \cos A \end{array} \right.$

$= \frac{1}{2\sin\frac{\theta}{2}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]}$ → convert to $\cos\theta + i\sin\theta$

$= \frac{1}{-2i\sin\frac{\theta}{2}} \left[\frac{1}{i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}\right]$

$= \frac{1}{-2i\sin\theta/2} \left[\frac{1}{i\sin\frac{\theta}{2} + \cos(\theta/2)}\right]$

$\Rightarrow \frac{1}{-2i\sin(\theta/2)} \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]^{-1}$

$= \frac{-1}{2i}\left[\frac{\cos(\theta/2) - i\sin(\theta/2)}{\sin\theta/2}\right] = \frac{-1}{2i}\left[\cot\frac{\theta}{2} - i\right]$

$= \frac{1}{2}\left[\frac{-1}{i}\cot\frac{\theta}{2} + 1\right] = \frac{1}{2}\left[1 + i\cot\frac{\theta}{2}\right] = \frac{1}{2}\left[1 + i\cot\frac{2r\pi}{2}\right]$ (from ①)

$= \frac{1}{2}\left(1 + i\cot r\pi\right) = \text{R.H.S.}$

DATE 15-11-22
PAGE

* Exponential functions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad z = a + ib$$

$$\boxed{e^{ib} = \cos b + i \sin b}$$ — Eular formula

Note :→ period of exponential funct. is $\boxed{2\pi i}$

- Prove that $e^{iz} = e^{i(z+2\pi n)}$

$$e^{iz} = \cos(z) + i\sin(z)$$
$$= \cos(2\pi n + z) + i\sin(2\pi n + z)$$
$$= e^{i(2\pi n + z)}$$
$$= e^{iz + 2\pi n i} = \text{R.H.S}$$

Ques 1: Separate real & imaginary part of $e^{(6+5i)^2}$.

$$e^{(6+5i)^2} = e^{(36 - 25 + 60i)}$$
$$= e^{(11+60i)} = e^{11} \cdot e^{60i} = e^{11}[\cos 60 + i\sin(60)]$$
$$= [e^{11}\cos 60 + e^{11} \cdot i\sin(60)]$$

↓ real part ↓ imaginary part.

Ques 2: Prove that $[\sin(\alpha - \theta) + e^{-i\alpha}\sin\theta]^n = \sin^{(n-1)}\alpha\,[\sin(\alpha - n\theta) + e^{-i\alpha}\sin(n\theta)]$

Ans :- L.H.S

$$[\sin\alpha\cos\theta - \cos\alpha\sin\theta + (\cos\alpha - i\sin\alpha)\sin\theta]^n$$
$$= [\sin\alpha\cos\theta - \cancel{\cos\alpha\sin\theta} + \cancel{\cos\alpha\sin\theta} - i\sin\alpha\sin\theta]^n$$
$$= [\sin\alpha\cos\theta - i\sin\alpha\sin\theta]^n$$
$$= \sin^n\alpha\,[\cos\theta - i\sin\theta]^n$$

using De Moivre's theorem

$$= \sin^n\alpha\,[\cos(n\theta) - i\sin(n\theta)] \quad \text{— (1)}$$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$e^{-ib} = \cos b - i\sin b$

R.H.S $\sin^{(n-1)}\alpha\,[\sin(\alpha - n\theta) + e^{-i\alpha}\sin(n\theta)]$

$$= \sin^{(n-1)}\alpha\,[\sin\alpha\cos n\theta - \cos\alpha\sin n\theta + (\cos\alpha - i\sin\alpha)\sin n\theta]$$
$$= \sin^{(n-1)}\alpha\,[\sin\alpha\cos n\theta - \cancel{\cos\alpha\sin n\theta} + \cancel{\sin n\theta\cos\alpha} - i\sin\alpha\sin n\theta]$$
$$= \sin^{(n-1)}\alpha\,[\sin\alpha\cos n\theta - i\sin\alpha\sin n\theta]$$

① principle value ($\log_e^{\omega}$)
② General value ($\log_e^{\omega}$)

DATE
PAGE

$= \sin^{(n-1)}\alpha \cdot \sin\alpha[\cos n\theta - i\sin n\theta] = \sin^{(n-1+1)}[\cos n\theta - i\sin\theta]$

$= \sin^n[\cos n\theta - i\sin\theta]$ ——②

eq^n ① = eq^n ② Hence proved!

* Logarithmic functions :→ Let w & z both are complex no., when $\boxed{w = e^z}$

$\log_e^{\omega} = \log_e(e^z) = z$

$\boxed{z = \log_e^{\omega}}$ ✓✓

$\log_e^{\omega}$ is many valued (periodic) function.

1) General value of log :-

$\text{Log}_e^{\omega} = 2n\pi i + \log_e^{\omega}$

$\text{Log}(\alpha+i\beta)$ → General formula.

Ques. Sperate real & imaginary parts of $\text{Log}(\alpha+i\beta)$. ⇒ it will give genral log

sol^n: $\alpha + i\beta = r[\cos\theta + i\sin\theta] = re^{i\theta}$

$\alpha = r\cos\theta$ —① , $\beta = r\sin\theta$ —①

squaring & adding ① & ①

$\alpha^2 + \beta^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\boxed{r = \sqrt{\alpha^2+\beta^2}}$ —③

Divide ② & ①

$\boxed{\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)}$ —④

$\log(\alpha+i\beta) = \log(\alpha+i\beta) + 2n\pi i$ $\{\log a\cdot b = \log a + \log b\}$

$= \log(re^{i\theta}) + 2n\pi i$

$= \log r + \log e^{i\theta} + 2n\pi i$

$= \log\sqrt{\alpha^2+\beta^2} + i\theta + 2n\pi i$ $(\because ③ \text{ eq}^n)$

$= \log(\alpha^2+\beta^2)^{1/2} + i\theta + 2n\pi i$

$= \frac{1}{2}\log(\alpha^2+\beta^2) + i(\theta + 2n\pi)$ $\{\log a^b = b\log a\}$

$= \boxed{\frac{1}{2}\log(\alpha^2+\beta^2) + i\left(\tan^{-1}\frac{\beta}{\alpha} + 2n\pi\right)}$ $(\because ④)$

↓ Real part ↓ Imaginary part.

eg: $\log(4+3i)$

$= \frac{1}{2}\log(3^2+4^2) = \frac{1}{2}\log(25) = \frac{1}{2}\log(5)^2 = \log 5$ ← real part

Note: →

$$\text{Real}[\log(\alpha+i\beta)] = \frac{1}{2}\log(\alpha^2+\beta^2)$$

$$\text{Img}[\log(\alpha+i\beta)] = \tan^{-1}\left(\frac{\beta}{\alpha}\right) + 2n\pi$$

Date:- 16-11-22

Ques 1: $\text{Log}(-4) = \log(-4) + 2n\pi i$

$= \log(-1)(4) + 2n\pi i$

$= \log(-1) + \log(4) + 2n\pi i$

$= \log(\text{cis}\,\pi) + \log(2)^2 + 2n\pi i$

$= \log e^{i\pi} + 2\log 2 + 2n\pi i$

$= i\pi + 2\log 2 + 2n\pi i$

$= 2\log 2 + (2n+1)\pi i$

principle value.

Note:

$$\text{Log}(-N) = \pi i + \log N$$ ← Principal value

$$\text{Log}(-N) = \pi i + 2n\pi i + \log N$$ ← General value

Ques 2: Prove that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2+b^2}$

$a+ib = r[\cos\theta + i\sin\theta] = re^{i\theta}$

$a-ib = r[\cos\theta - i\sin\theta] = re^{-i\theta}$

$r = \sqrt{a^2+b^2}$ — (1) $\quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$ — (2)

L.H.S. $= \tan\left[i\log\left(\frac{re^{-i\theta}}{re^{i\theta}}\right)\right] = \tan\left[i\log e^{-2i\theta}\right]$

$= \tan[i(-2i\theta)] \quad \because (-i^2 = -(-1) = 1)$

$= \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

DATE
PAGE

(From ①)

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{b}{a}\right)}{1-\frac{b^2}{a^2}} = \frac{\frac{2b}{a}}{\frac{a^2-b^2}{a^2}} = \frac{2ab}{a^2-b^2}$$

= R.H.S.

$x+iy = r[\cos\theta + i\sin\theta] = r\cdot e^{i\theta}$

$x-iy = r[\cos\theta - i\sin\theta] = r\cdot e^{-i\theta}$

$\log\left(\frac{re^{i\theta}}{re^{-i\theta}}\right) = \log(e^{2i\theta}) = 2i\theta = 2i\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$

H.W.:→ Prove that $\log\left(\frac{x+iy}{x-iy}\right) = 2i\tan^{-1}\left(\frac{y}{x}\right)$.

Ques 8:- $\text{Log}\,\text{Log}(i) = 2m\pi i + \log[\text{Log}(i)]$ — ①

consider $\text{Log}(i) = 2r\pi i + \log i$

$= 2r\pi i + \log\text{cis}\left(\frac{\pi}{2}\right) \Rightarrow 2r\pi i + \log e^{i(\pi/2)}$

$= 2r\pi i + i\frac{\pi}{2}$

$= i\left(4r+1\right)\frac{\pi}{2}$

$\log(i) = \frac{\pi}{2}i(4r+1)$ put in eq ①

$\text{Log}\,\text{Log}(i) = 2m\pi i + \log\left[i\frac{(4r+1)\pi}{2}\right]$ $\{\log ab = \log a + \log b\}$

$= 2m\pi i + \log i + \log(4r+1)\frac{\pi}{2}$ $\left(\because i = \text{cis}\left(\frac{\pi}{2}\right)\right]$

$= 2m\pi i + \log e^{i\pi/2} + \log(4r+1)\frac{\pi}{2}$

$= 2m\pi i + i\frac{\pi}{2} + \log(4r+1)\frac{\pi}{2}$

$= \boxed{i\left(4m+1\right)\frac{\pi}{2} + \log(4r+1)\frac{\pi}{2}}$

Real part — Imaginary part

H.W.:- ① $\text{Log}\left(\text{Log}\left(\cos\theta + i\sin\theta\right)\right)$.

Q. ② $\log_i(i) = \frac{\log i}{\log i} = \frac{2m\pi i + \log i}{2r\pi i + \log i}$ $\left\{\because \log_b^a = \frac{\log a}{\log b}\right.$

* General exponential function:-

$(a+ib)^{(x+iy)}$ → form.

Let $a^z = \log e^{(az)} = \log$

expansion of term (no) → self study

{ log a = 2nπi + log a

DATE
PAGE

Let $a^z = e^{\log a^z} = e^{z\log a} = e^{z(2n\pi i + \log a)}$

V. Imp

Ques:- Prove that $i^{(i)}$ is wholly real and find its principal value.

Soln:- $i^{(i)} = e^{\log i^{(i)}} = e^{i\log i} = e^{i\log(\text{cis}(\pi/2))}$

$= e^{i\log e^{i\pi/2}}$ [$\because i = \text{cis}(\pi/2)$]

$= e^{i \cdot i(\pi/2)}$

$= e^{-\pi/2}$ ($\because i^2 = -1$)

$= e^{-\pi/2}[\cos(2n\pi) + i\sin(2n\pi)]$

Soln:- $(i)^{(i)} = e^{\log i^{(i)}} = e^{i\log(i)}$

$= e^{i\log e^{i(2n\pi + \pi/2)}}$ [for $i = \text{cis}(\pi/2)$]

$= e^{i(2n\pi i + i\pi/2)}$

$= e^{-(4n+1)\pi/2}$

(i) General value $= e^{-(4n+1)\pi/2}$ = Purely real no.

(ii) principle value ($n=0$) $= e^{-\pi/2}$ Ans

Imp.

Ques:- $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$

To prove (i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

iii) $\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0$

iv) $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$

Soln:- (i) & (ii) Let $a = \cos\alpha + i\sin\alpha$

$b = \cos\beta + i\sin\beta$

$c = \cos\gamma + i\sin\gamma$

$a+b+c = (\cos\alpha + \cos\beta + \cos\gamma) + i[\sin\alpha + \sin\beta + \sin\gamma]$

$= 0 + i0 = 0$

$a+b+c = 0$

$a+b = -c$ → (1)

Recall: $a^3+b^3+c^3 = abc$

$\tanh^2 x + \operatorname{sech}^2 x = 1$

cube both side

$a^3+b^3+3ab(a+b) = -c^3$

$a^3+b^3+c^3+3ab(-c) = +c^3$ (∵ ①)

$a^3+b^3+c^3 = 3abc$

Now

$(\operatorname{cis}\alpha)^3 + (\operatorname{cis}\beta)^3 + (\operatorname{cis}\gamma)^3 = 3(\operatorname{cis}\alpha)(\operatorname{cis}\beta)(\operatorname{cis}\gamma)$

∵ = de Moivre's theorem

$\operatorname{cis}3\alpha + \operatorname{cis}3\beta + \operatorname{cis}3\gamma = 3\operatorname{cis}(\alpha+\beta+\gamma)$

~~add both side~~ $i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$\operatorname{cis}3\alpha + \operatorname{cis}3\beta + \operatorname{cis}3\gamma + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3\operatorname{cis}(\alpha+\beta+\gamma)$

$= (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i\sin(3\alpha+3\beta+3\gamma)$

$= 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)$

Now

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

hence proved

(ii) $a = \cos\alpha + i\sin\alpha \Rightarrow$

$a = \operatorname{cis}\alpha \Rightarrow \frac{1}{a} = (\operatorname{cis}\alpha)^{-1} = a^{-1} = (\operatorname{cis}\alpha)^{-1}$ — ①

$\frac{1}{b} = b^{-1} = (\operatorname{cis}\beta)^{-1}$ — ②

$c^{-1} = (\operatorname{cis}\gamma)^{-1}$ — ③

adding ① + ② + ③

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = (\operatorname{cis}\alpha)^{-1} + (\operatorname{cis}\beta)^{-1} + \operatorname{cis}(\gamma)^{-1}$

$= (\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma)$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 - i0 = 0$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 0$

$bc + ac + ba = 0$

$\frac{ab+bc+cd}{abc}$

$= \frac{1}{c}+\frac{1}{a}+\frac{1}{b} = 0$

angles adds when they are multiplied (cis)

DATE
PAGE

$\Rightarrow \quad \text{cis}\beta \cdot \text{cis}\gamma + \text{cis}\alpha\,\text{cis}\gamma + \text{cis}\beta\,\text{cis}\alpha = 0$

$(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma) + (\cos\alpha+i\sin\alpha)(\cos\gamma+i\sin\gamma) + (\cos\beta+i\sin\beta)(\cos\alpha+i\sin\alpha)$

$\Rightarrow \cos(\beta+\gamma)+i\sin(\beta+\gamma) + \cos(\alpha+\gamma)+i\sin(\alpha+\gamma) + \cos(\beta+\alpha)+i\sin(\beta+\alpha) = 0$

Hence imaginary part & real part are have to - compeer.

(v) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

vi) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$ $\rightarrow a+b+c=0$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

vii) $\sum \cos 4\alpha = 2\sum \cos 2(\beta+\gamma)$

viii) $\sum \sin 4\alpha = 2\sum \sin 2(\beta+\gamma)$

$\{\sin^2\alpha+\sin^2\beta+\sin^2\gamma = \cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{3}{2}$

(v) & (vi) :- $a+b+c=0$

so $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) = 0$

$= (\text{cis}\alpha)^2+(\text{cis}\beta)^2+(\text{cis}\gamma)^2+2[\text{cis}(\alpha+\beta)+\text{cis}(\beta+\gamma)+\text{cis}(\gamma+\alpha)]$

$\hookrightarrow 0 \quad (\because \text{if } i)$

so $\text{cis}2\alpha+\text{cis}2\beta+\text{cis}2\gamma = 0$

$= (\cos 2\alpha+i\sin 2\alpha)+(\cos(2\beta)+i\sin(2\beta))+(\cos 2\gamma+i\sin 2\gamma) = 0$

so separating real & imaginary part.

$\cos 2\alpha+\cos 2\beta+\cos 2\gamma = 0$

$\sin 2\alpha+\sin 2\beta+\sin 2\gamma = 0$

(vii & viii) $a+b+c=0$

$a+b=-c$

$a^2+b^2+2ab = c^2$

$a^2+b^2-c^2 = -2ab$ again squaring

$a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2 = 4a^2b^2$

$a^4+b^4+c^4 = 2a^2b^2+2b^2c^2+2c^2a^2$

$(\cos\alpha+i\sin\alpha)^4+(\cos\beta+i\sin\beta)^4+(\cos\gamma+i\sin\gamma)^4 = 2(\cos\alpha+i\sin\alpha)^2(\cos\beta+i\sin\beta)^2+2(\cos\beta+i\sin\beta)^2(\cos\gamma+i\sin\gamma)^2+2(\cos\gamma+i\sin\gamma)^2(\cos\alpha+i\sin\alpha)^2$

Ques

DATE 17-11-22
PAGE

$= (\cos 4\alpha + i\sin 4\alpha) + \dots = 2[(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta) \dots]$

$\Rightarrow$ Compaire real & img. parts.

$\Sigma \cos 4\alpha = 2\Sigma \cos 2(\alpha+\beta)$ &

$\Sigma \sin 4\alpha = 2\Sigma \sin(\beta+\gamma)$

Ques-1 Prove that $i^{\alpha+i\beta} = e^{-(4n+1)\beta\pi}$ & $\alpha^2+\beta^2 = e^{-(4n+1)\beta\pi}$

sol^n:- $i^{(\alpha+i\beta)} = e^{(\alpha+i\beta)\log i} = e^{(\alpha+i\beta)[\log e^{i(\pi/2)}]} = e^{(\alpha+i\beta)[2n\pi i + \log i]}$

$= e^{(\alpha+i\beta)(2n\pi i + i\pi/2)} = e^{(\alpha+i\beta)(2n\pi i + \log e^{i(\pi/2)})}$

$= e^{i(4n+1)\pi/2\alpha + (-1)(4n+1)\pi/2\beta} = e^{(\alpha+i\beta)(4n+1)i\pi/2}$

Let $(4n+1)\frac{\pi}{2}\alpha = q$, $-(4n+1)\frac{\pi}{2}\beta = p$. — ①

$i^{(\alpha+i\beta)} = e^{iq+p} = e^p \cdot e^{iq}$

$= e^p[\cos q + i\sin q] = \underbrace{e^p\cos q}_{\text{Real part}} + \underbrace{i e^p \sin q}_{\text{Imaginary part}}$

Real part of $i^{(\alpha+i\beta)} = e^p\cos q$

imaginary " " " $= ie^p \sin q$

& $i^{\alpha+i\beta} = \alpha+i\beta \Rightarrow \text{real}(i^{(\alpha+i\beta)}) = \alpha$ & $\text{img}(i^{\alpha+i\beta}) = \beta$

$\Rightarrow \alpha = e^p\cos q$, $\beta = e^p \sin q$ — ②

square & adding eq^n ②

$\alpha^2+\beta^2 = e^{2p}\cos^2 q + \beta^2 e^{2p}\sin^2 q = e^{2p}[\sin^2 q + \cos^2 q]$

$\alpha^2+\beta^2 = e^{2p} = e^{2[-(4n+1)\frac{\pi}{2}\beta]} = e^{-(4n+1)\beta\pi}$ = RHS

(∵ ①)

hence proved

Ques 2:- If $i^{i^{i\dots\infty}} = A+iB$. Prove that $A^2+B^2 = e^{-\pi B}$ [principle value]

$\tan\frac{\pi}{2}\frac{A}{B} = \frac{B}{A}$

Take $i^{i^{A+iB}} = A+iB$ → same as Ques ① sub $n=0$ (∵ Principle value.

$i^{(A+iB)} = e^{i\pi/2 A + (-1)\pi/2 B}$

DATE
PAGE

So $i^{(A+iB)} = e^{i\frac{\pi}{2}(A+iB)} = e^{(i\frac{\pi}{2}A)+(-1)\frac{\pi}{2}B} = e^{-\frac{\pi}{2}B} \cdot e^{(\frac{\pi}{2}A)i}$

$i^{(A+iB)} = e^{-\frac{\pi}{2}B} \cdot \left(\cos\frac{\pi}{2}A + i\sin\frac{\pi}{2}A\right) = \underbrace{e^{-\frac{\pi}{2}B} \cdot \cos\frac{\pi}{2}A}_{\text{Real part}} + \underbrace{e^{-\frac{\pi}{2}B} i\sin\frac{\pi}{2}A}_{\text{Imaginary part}}$

So comparing real & Imaginary parts

$i^{(A+iB)} = A+iB = e^{-\frac{\pi}{2}B} \cdot \cos\frac{\pi}{2}A + e^{-\frac{\pi}{2}B} i\sin\frac{\pi}{2}A$

$A = e^{-\frac{\pi}{2}B}\cos\frac{\pi}{2}A$ ... ① & $B = e^{-\frac{\pi}{2}B}\sin\frac{\pi}{2}A$ ... ②

(i) squaring & adding

$A^2+B^2 = e^{-\pi B}\cos^2\left(\frac{\pi}{2}A\right) + e^{-\pi B}\sin^2\left(\frac{\pi}{2}A\right) = e^{-\pi B}(\sin^2\theta+\cos^2\theta)$

$= e^{-\pi B}$

(ii) Divide ② by ①

$\frac{B}{A} = \frac{e^{-\frac{\pi}{2}B}\sin\frac{\pi}{2}A}{e^{-\frac{\pi}{2}B}\cos\frac{\pi}{2}A} = \tan\left(\frac{A\pi}{2}\right)$

$\frac{B}{A} = \tan\left(\pi\cdot\frac{A}{2}\right)$ hence proved.

Ques:-3 Prove that $(\sqrt{i})^{\sqrt{i}} = e^{\frac{-\pi}{4\sqrt{2}}}\left[\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right]$ → principle value.

sol^n:- Consider $\sqrt{i} = \left(\text{cis}\frac{\pi}{2}\right)^{1/2} = \text{cis}\frac{\pi}{4}$ {By demoivre's theorm}

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \left(\frac{1+i}{\sqrt{2}}\right)$ ✓ ... ①

So $(\sqrt{i})^{\sqrt{i}} = e^{\log(\sqrt{i})^{\sqrt{i}}} = e^{\sqrt{i}\log\sqrt{i}} = e^{\sqrt{i}\log(i)^{1/2}} = e^{\sqrt{i}\frac{1}{2}\log i}$

$= e^{\sqrt{i}\cdot\frac{1}{2}\log e^{i\frac{\pi}{2}}} = e^{\frac{\sqrt{i}}{2}\cdot i\frac{\pi}{2}} = e^{i\frac{\pi}{4}\sqrt{i}}$

$= e^{\frac{i\pi}{4}\left(\frac{1+i}{\sqrt{2}}\right)}$ (∵ (i)) $= e^{\frac{-\pi}{4\sqrt{2}}}\cdot\text{cis}\left(\frac{\pi}{4\sqrt{2}}\right)$

$= e^{\frac{-\pi}{4\sqrt{2}}}\cdot e^{\frac{i\pi}{4\sqrt{2}}}$

$= e^{\frac{\pi}{4\sqrt{2}}}\cdot\left[\cos\left(\frac{\pi}{4\sqrt{2}}\right) + i\sin\left(\frac{\pi}{4\sqrt{2}}\right)\right]$ Hence

DATE
PAGE

ques:- If $(a+ib)^p = m^{(x+iy)}$ then prove that $\frac{y}{x} = \frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}$

when only principal value is considered. $\forall p, m \in \mathbb{R}$

sol^n Taking log on both sides —

$$\log(a+ib)^p = \log m^{(x+iy)}$$

$$p\log(a+ib) = (x+iy)\log m$$

from formula $\Rightarrow p\left[\frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)\right] = x\log m + iy\log m$

$$x\log m = \frac{p}{2}\log(a^2+b^2) \quad \& \quad y\log m = p\tan^{-1}\left(\frac{b}{a}\right)$$

(①) (②)

(∵ comparing real & img. part)

divided ② by ①

$$\frac{y\log m}{x\log m} = \frac{p\tan^{-1}(b/a)}{\frac{p}{2}\log(a^2+b^2)} = \frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)} = R.H.S$$

Hence proved.

Date :- 22-11-22

* Circular functions :-

$$e^{iz} = \cos z + i\sin z \quad -①$$

$$e^{-iz} = \cos z - i\sin z \quad -②$$

on adding

$$e^{iz} + e^{-iz} = 2\cos z$$

① $$\boxed{\cos z = \frac{e^{iz}+e^{-iz}}{2}}$$

on subtraction

$$e^{iz} - e^{-iz} = 2i\sin z$$

② $$\boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$$

Now

$$\tan z = \frac{\sin z}{\cos z} = \frac{\frac{e^{iz}-e^{-iz}}{2i}}{\frac{e^{iz}+e^{-iz}}{2}} = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$$

③ $$\boxed{\tan z = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}}$$

DATE
PAGE

④ $\cot z = \dfrac{i(e^{iz}+e^{-iz})}{(e^{iz}-e^{-iz})}$

⑤ $\sec z = \dfrac{1}{\cos z} = \dfrac{2}{e^{iz}+e^{-iz}}$

⑥ $\operatorname{cosec} z = \dfrac{1}{\sin z} = \dfrac{2i}{e^{iz}-e^{-iz}}$

* $\cos^2 z + \sin^2 z = 1$

* Hyperbolic functions :→

$\cosh z = \dfrac{e^z+e^{-z}}{2}$

$\sinh z = \dfrac{e^z-e^{-z}}{2}$

$\tanh z = \dfrac{e^z-e^{-z}}{e^z+e^{-z}}$

(∵ i is not nessary in Hyperbola)

$\coth z = \dfrac{e^z+e^{-z}}{e^z-e^{-z}}$

$\operatorname{sech} z = \dfrac{2}{e^z+e^{-z}}$

$\operatorname{cosech} z = \dfrac{2}{e^z-e^{-z}}$

Imp.

* Relationship b/w circular & hyperbolic functions

① $\sin z = \dfrac{e^{iz}-e^{-iz}}{2}$

DATE
PAGE

$$\sin(iz) = \frac{e^{(i)(iz)} - e^{-i(i)z}}{2i} = \frac{e^{-z} - e^{z}}{2i} = \frac{-1}{i}\left(\frac{e^{z} - e^{-z}}{2}\right)$$

$$= i\left[\frac{e^{z} - e^{-z}}{2}\right] \qquad \left(\because \frac{1}{i} = -i\right)$$

$$\boxed{\sin(iz) = i \sinh z}$$

(2) $$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cos(iz) = \frac{e^{i.iz} + e^{-i.(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cos(hz)$$

(3) $$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i \sinh z}{\cosh z} = i \tan(hz)$$

(4) $$\cot(iz) = \frac{1}{i \tan(hz)} = -i \cot hz \qquad \left(\because \frac{1}{i} = -i\right)$$

(5) $$\sec(iz) = \frac{1}{\cos(iz)} = \frac{1}{\cos(hz)} = \sec(hz)$$

(6) $$\operatorname{cosec}(iz) = \frac{1}{\sin(iz)} = \frac{1}{i \sin(hz)} = i \operatorname{cosec}(hz)$$

Sumary :-

(1) $$\boxed{\sin(iz) = i \sinh z}$$

$$\sinh z = \frac{1}{i} \sin(iz)$$

$$\boxed{\sinh z = -i \sin(iz)}$$ ✓

DATE
PAGE

② $\boxed{\cos(iz) = \cosh z}$ → it absorbs i

③ $\boxed{\tan(iz) = i\tan hz}$

$\tan(hz) = \frac{1}{i}\tan(iz)$

$\boxed{\tan hz = -i\tan(iz)}$

④ $\boxed{\cot(iz) = -i\cot(hz)}$

$\cot(hz) = \frac{-1}{i}\cot(iz)$

$\boxed{\cot(hz) = i\cot(iz)}$

⑤ $\boxed{\sec(iz) = \sec(hz)}$

⑥ $\operatorname{cosec}(iz) = -i\operatorname{cosec}(hz)$

$\boxed{\operatorname{cosec}(hz) = i\operatorname{cosec}(iz)}$

Ques 1: Separate real & imaginary part of $\sin(x+iy)$.

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy)$

$= \sin x\cos(hy) + \cos x \cdot i\sin(hy)$

$= \sin x\cosh y + i\cdot\cos x\sin(hy)$

↓ Real part ↓ img. part

∵ ① $\sin(iz) = i\sin$

② $\cos(iz) = \cos$

Ques 2: $\cos(x+iy)$

$\cos(x+iy) = \cos x\cos iy - \sin x\sin iy$

$= \cos x\cos(hy) - \sin x\, i\sin(hy)$

$= \cos x\cos(hy) - i\sin x\sin hy$

real part $= \cos x\cdot\cos(hy)$

img. part $= -\sin x\sin hy$

DATE
PAGE

Q. $\sinh(x+iy) = -i\sin[i(x+iy)]$ $\left(\because \sin(iz) = i\sinh z \Rightarrow \sinh z = \frac{1}{i}\sin(iz)\right)$

$= -i\sin(ix-y)]$

$= -i(\sin ix \cdot \cos y - \cos(ix)\sin(y))$

$= -i(i\sin(hx)\cdot\cos y - \cos(hx)\sin y)$

$= \underbrace{\sin(hx)\cos y}_{\text{Real part}} + i\underbrace{\cos(hx)\sin y}_{\text{Img. part.}}$

Q. $\cos h(x+iy) = \cos i(x+iy) = \cos(ix-y)$ $\{\because \cos(iz) = \cosh z\}$

$= \cos y\cos ix + \sin y \sin ix$

$= \cos y \cos(hx) + \sin y\, i\sin(hx)$

$= \underbrace{\cos y\cos(hx)}_{\text{Real part}} + i\underbrace{\sin y \sin(hx)}_{\text{ima. part } (+\sin y \sinh x)}$

Periods of hyperbolic functions :

$\sin(hz) = 2\pi i$

$\cos(hz) = 2\pi i$

$\tan(hz) = \pi i$

Formulas :→

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

OR

$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$

Ques. Separate real & imaginary parts of $\tan(x+iy)$

$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$

multiply & divide by 2 & $\cos(x-iy)$

$$= \frac{2\sin(x+iy)}{2\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)}$$

Using $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$= \frac{\sin(x+iy+x-iy) + \sin(x+iy-x+iy)}{\cos(x+iy+x-iy) + \cos(x+iy-x+iy)}$$

$$= \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos(2iy)} \quad \left\{ \begin{array}{l} \sin iz = i\sin hz \\ \cos iz = \cos hz \end{array} \right.$$

$$= \frac{\sin 2x + i\sin(h2y)}{\cos 2x + \cos(h2y)}$$

$$= \boxed{\frac{\sin 2x}{\cos 2x + \cos(h2y)} + \frac{i\sin(2hy)}{\cos 2x + \cos(h2y)}} \quad \text{Ans}$$

Real part ↓ Imaginary part ↓

Ques: $\cot(x+iy)$

$$\cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)}$$

Now multiply & divide by 2

$$= \frac{2\cos(x+iy)}{2\sin(x+iy)} \times \frac{\sin(x-iy)}{\sin(x-iy)}$$

$$= \frac{\sin(x+iy+x-iy) - \sin(x+iy-x+iy)}{\cos(x+iy-x+iy) - \cos(x+iy+x-iy)}$$

$$= \frac{\sin 2x - \sin 2iy}{\cos 2iy - \cos 2x} = \frac{\sin 2x - i\sin h2y}{\cos(h2y) - \cos 2x}$$

DATE
PAGE

$$\frac{\sin 2x}{\cos(h2y)-\cos 2x} - \frac{i\sin(h2y)}{\cos(h2y)-\cos 2x}$$

↓ Real part ↓ img. part.

Q. separate real & img part of $\sec(x+iy)$

$$\sec(x+iy) = \frac{1}{\cos(x+iy)} = \frac{2}{2}\,\frac{1}{\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{2(\cos x\cos iy + \sin x \sin iy)}{\cos(x+iy+x-iy)+\cos(x+iy-x+iy)} \quad \{2\cos A\cos B = \cos(A+B)+\cos(A-B)\}$$

$$= \frac{2\cos x\cos(hy) + 2\sin x\, i\sin(hy)}{\cos 2x + \cos(2iy)}$$

$$= \frac{2\cos x\cos(hy) + i\,2\sin x\sin(hy)}{\cos 2x + \cos(h2y)}$$

$$= \left[\frac{2\cos x\cos(hy)}{\cos 2x+\cos(h2y)} + i\,\frac{2\sin x\sin(hy)}{\cos 2x+\cos(h2y)}\right]$$

↓ Real part ; Img. part

Q. $\operatorname{cosec}(x+iy) = \frac{1}{\sin(x+iy)} = \frac{2\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$

$$= \frac{2[\sin x\cos iy - \cos x\sin iy]}{\cos(x+iy-x+iy)-\cos(x+iy+x-iy)} = \frac{2\sin x\cosh y - 2i\cos x\sinh y}{\cos 2iy - \cos 2x}$$

$$= \frac{2\sin x\cosh y - 2i\cos x\sin(hy)}{\cos(h2y)-\cos 2x}$$

$$= \frac{2\sin x\cosh y}{\cos(h2y)-\cos 2x} - \frac{i2\cos x\sin(hy)}{\cos(h2y)-\cos 2x}$$

Real part ← ; → img. part

DATE
PAGE

**

Ques: $\tanh(x+iy) = -i\tan i(x+iy)$

$= -i\left[\frac{\sin(ix-y)}{\cos(ix-y)}\right]$

$\tan iz = i\tanh z$

$\tanh z = \frac{1}{i}\tan iz$

$\boxed{\tanh z = -i\tan iz}$

$= -i\left[\frac{2\sin(ix-y)}{2\cos(ix-y)} \times \frac{\cos(ix+y)}{\cos(ix+y)}\right]$

$= -i\left[\frac{\sin(ix-y+ix+y) + \sin(ix-y-ix-y)}{\cos(ix-y+ix+y) + \cos(ix-y-ix-y)}\right]$

$= -i\left[\frac{\sin(2ix) + \sin(-2y)}{\cos(2ix) + \cos(-2y)}\right]$

$= -i\left[\frac{i\sin(2hx) - \sin(2y)}{\cos h2x + \cos 2y}\right]$

$= \frac{\sin(h2x)}{\cos(h2x) + \cos 2y} + \frac{i\sin 2y}{\cos h2z + \cos 2y}$ → img part

Real part ← $\frac{\sin(h2x)}{\cos(h2x) + \cos 2y}$

Ques: $\operatorname{sech}(x+iy) = \sec[i(x+iy)]$

$= \sec(ix-y)$ $(\because \operatorname{sech} z = \sec(iz))$

$= \frac{1}{\cos(ix-y)}$

$= \frac{2\cos(ix+y)}{2\cos(ix-y)\cos(ix+y)} = \frac{2[\cos ix\cos y - \sin ix\sin y]}{\cos(ix-y+ix+y) + \cos(ix-y-ix+y)}$

$= \frac{2\cos hx\cos y - 2i\sin(hx)\sin y}{\cos 2ix + \cos 2y}$

↳ $\cos 2hx$

$= \frac{2\cos hx\cos y}{\cos 2hx + \cos 2y} - \frac{i2\sin(hx)\sin y}{\cos 2hx + \cos 2y}$

↓ Real part

↓ img part

DATE
PAGE

Task for Theory

Ques:- If $u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, then Prove that

(i) $\tanh \frac{u}{2} = \tan \frac{\theta}{2}$ (ii) $\cosh u = \sec\theta$

(iii) $\sinh u = \tan\theta$ (iv) $\tan(hu) = \sin\theta$

(v) $\theta = -i \log\left(\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right)\right)$

$\tanh \frac{u}{2} = \frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}}$

(i) $u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$e^u = e^{\log[\tan(\frac{\pi}{4} + \frac{\theta}{2})]}$

$e^{u/2} e^{u/2} = e^u$

$\frac{e^{u/2}}{e^{-u/2}} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$\therefore \frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)}$

$\left(\frac{N}{D}\right)\left(\frac{N-D}{N+D}\right)$

Apply componendo & dividendo rule.

$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{1 + \tan(\theta/2) - 1 + \tan(\theta/2)}{1 + \tan(\theta/2) + 1 - \tan(\theta/2)}$

$\tanh\left(\frac{u}{2}\right) = \frac{2\tan(\theta/2)}{2}$

(ii) $\cosh u = \sec\theta$

$\cosh(u) = \cos(iu) = \cos\left(2 \cdot \frac{iu}{2}\right)$

$\left(\because \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}\right)$

$= \frac{1 - \tan^2(iu/2)}{1 + \tan^2(iu/2)}$

$\{\because \tan iz = i \tan(hz)$

$= \frac{1 - i^2 \tan^2(hu/2)}{1 + i^2 \tan^2(hu/2)}$

$= \frac{1 + \tan^2(hu/2)}{1 - \tan^2(hu/2)}$

$= \frac{1 + \tan^2(\theta/2)}{1 - \tan^2(\theta/2)}$ $(\because \cos\theta)$ $= \frac{1}{\cos 2(\theta/2)}$ $= \sec\theta$

= R.H.S

(iii) $\sin uh = \tan\theta$.

$\sin uh = -i\sin(iu) = -i\sin 2\left(\frac{iu}{2}\right)$ $\left(\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right)$

$-i\left(\frac{2\tan(iu/2)}{1+\tan^2(iu/2)}\right) = -i\left(\frac{2i\tan(hu/2)}{1+i^2\tan(hu/2)}\right)$

$= -i^2\left(\frac{2\tan(hu/2)}{1-\tan^2(hu/2)}\right) = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \tan\theta$ → (using 1st part)

$= R.H.S.$

$\left(\because \tan\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right)$

(iv) $\tan(hu) = \sin\theta$

$\tanh u = -i\tan(iu) = -i\tan 2\left(\frac{iu}{2}\right)$

$= -i\left[\frac{2\tan(iu/2)}{1-\tan^2(iu/2)}\right]$ $\left\{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right\}$

$= -i\left[\frac{2i\tan(hu/2)}{1-i^2\tan^2(hu/2)}\right]$ $\{\because \tan(iu) = i\tan(hu)$

$= \frac{2\tan(hu/2)}{1+\tan^2(hu/2)} = \frac{2\tan(\theta/2)}{1+\tan(\theta/2)} = \sin\theta = R.H.S.$

Hence proved.

(v) $\theta = i\log\left[\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right)\right]$

From part-1

$\tan\frac{hu}{2} = \tan\left(\frac{\theta}{2}\right)$ $\{\tan(iu) = i\tan(hu)$

$\frac{1}{i}\tan\left(\frac{iu}{2}\right) = \frac{\sin(\theta/2)}{\cos(\theta/2)}$

$\frac{1}{i}\tan\left(\frac{iu}{2}\right) = \frac{e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} + e^{-i\theta/2}}$

$$\tan(i\psi/2) = \frac{e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} + e^{i\theta/2}} \quad \cdots (1)$$

by reverse the eqn

$$\frac{1}{\tan(i\psi/2)} = \frac{e^{-i\theta/2} + e^{i\theta/2}}{e^{i\theta/2} - e^{i\theta/2}}$$

using componendo & dividendo rule

$$\frac{1+\tan(i\psi/2)}{1-\tan(i\psi/2)} = \frac{e^{i\theta/2} + e^{i\theta/2} + e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} + e^{i\theta/2} - e^{i\theta/2} + e^{-i\theta/2}}$$

$$\Rightarrow \tan\left(\frac{\pi}{4} + \frac{i\psi}{2}\right) = \frac{2e^{i\theta/2}}{2e^{-i\theta/2}}$$

$$\Rightarrow \tan\left(\frac{\pi}{4} + \frac{i\psi}{2}\right) = e^{i\theta}$$

$$\log \tan\left(\frac{\pi}{4} + \frac{i\psi}{2}\right) = i\theta \quad \text{hence proved.}$$

Inverse Trigonometric functions :-

If $w = \sin^{-1} z$

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$2zi = e^{iw} - \frac{1}{e^{iw}} \qquad \{\text{let } e^{iw} = p\}$$

$$2zi = p - \frac{1}{p}$$

$$2zip = p^2 - 1$$

$$p^2 - 2zip - 1 = 0$$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{so}$$

$$p = \frac{-(-2zi) \pm \sqrt{4z^2i^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2zi \pm 2\sqrt{-z^2 + 1}}{2}$$

$p = zi \pm \sqrt{1-z^2}$

we Take +ve sign only

$p = zi + \sqrt{1-z^2}$

$e^{iw} = zi + \sqrt{1-z^2}$ $\quad (\because p = e^{iw})$

take log on both side

$\log e^{iw} = \log^{zi+\sqrt{1-z^2}}$

$w = \frac{1}{i} \log (zi + \sqrt{1-z^2})$

① $\boxed{\sin^{-1} z = w = -i \log (zi + \sqrt{1-z^2})}$ Ans

Formulas :-

① $\sin^{-1} z = -i \log (zi + \sqrt{1-z^2})$

② $\cos^{-1} z = -i \log (z + \sqrt{1-z^2})$

③ $\tan^{-1} z = \frac{-i}{2} \log \left(\frac{1+iz}{1-iz}\right) = \frac{i}{2} \log \left(\frac{i+z}{i-z}\right)$, $z \neq \pm i$

④ $\operatorname{cosec}^{-1} z = \sin^{-1}\left(\frac{1}{z}\right) = -i \log \left(\frac{i+\sqrt{z^2-1}}{z}\right)$; $z \neq 0$

⑤ $\sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right) = -i \log \left(\frac{1+\sqrt{1-z^2}}{z}\right)$; $z \neq 0$

⑥ $\cot^{-1} z = \tan^{-1}\left(\frac{1}{z}\right) = \frac{-i}{2} \log \left(\frac{z+i}{z-i}\right)$; $z \neq \pm i$

Inverse of hyperbolic functions :-

Let $w = \sinh^{-1} z$

$\sinh w = z$

$\frac{e^w - e^{-w}}{2} = z$

$e^w - e^{-w} = 2z$ $\quad$ $\{p = e^w\}$

$p - \frac{1}{p} = 2z$

$p^2 - 1 = 2pz$

$p^2 - 2pz - 1 = 0$

$$p = \frac{2z \pm \sqrt{4z^2 - 4(1)(1)}}{2(1)} = \frac{2z \pm 2\sqrt{z^2-1}}{2} = z \pm \sqrt{z^2+1}$$

Take +ve value

$$\boxed{p = z + \sqrt{z^2+1}}$$

$$e^w = z + \sqrt{z^2+1}$$

Take log

$$\boxed{\sin^{-1}hz = w = \log(z + \sqrt{z^2+1})}$$

Formula's of hyperbolic functions (Inverse) :→

① $\sin^{-1}hz = \log(z + \sqrt{z^2+1})$

② $\cos^{-1}hz = \log(z + \sqrt{z^2-1})$

③ $\tanh^{-1}z = \frac{-1}{2}\log\frac{1-z}{1+z}$; $z \neq 1$

④ $\operatorname{cosec}^{-1}hz = \sinh^{-1}\left(\frac{1}{z}\right) = \log\frac{1+\sqrt{1+z^2}}{z}$, $z \neq 0$

⑤ $\sec^{-1}hz = \cosh^{-1}\left(\frac{1}{z}\right) = \log\frac{1+\sqrt{1-z^2}}{z}$, $z \neq 0$

⑥ $\cot^{-1}hz = \tanh^{-1}\left(\frac{1}{z}\right) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$, $z \neq \pm 1$

Ques: Separate real & imaginary part :→ of $\tan^{-1}(x+iy)$.

Let $\tan^{-1}(x+iy) = u+iv$ — ①

$\tan^{-1}(x-iy) = u-iv$ — ② ($\because$ change $i \to -i$)

add eqn ① & ②

$\tan^{-1}(x+iy) + \tan^{-1}(x-iy) = 2u$ $\quad \{\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$\Rightarrow \tan^{-1}\left(\frac{x+iy+x-iy}{1-(x+iy)(x-iy)}\right) = 2u$

$\Rightarrow \tan^{-1}\left[\frac{2x}{1-[x^2-i^2y^2]}\right] = 2u$

$\Rightarrow \tan^{-1}\left[\frac{2x}{1-x^2-y^2}\right] = 2u$.

$$\boxed{u = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)} = \text{Real part}.$$

Now subtract ② from ①

$$\tan^{-1}(x+iy) - \tan^{-1}(x-iy) = 2iv$$

$\left\{\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right.$

$$\Rightarrow \tan^{-1}\left(\frac{x+iy-x+iy}{1+(x+iy)(x-iy)}\right) = 2iv$$

$$\Rightarrow \tan^{-1}\left(\frac{2iy}{1+x^2-i^2y^2}\right) = 2iv$$

$$\Rightarrow \tan^{-1}\left(\frac{2iy}{1+x^2+y^2}\right) = 2iv$$

$$\Rightarrow \tan i2v = \frac{2iy}{1+x^2+y^2} \quad \{\tan iz = i\tan(hz)\}$$

$$\Rightarrow i\tan h2v = \frac{2iy}{1+x^2+y^2}$$

$$\Rightarrow \tan h^{-1}\left(\frac{2y}{1+x^2+y^2}\right) = 2v$$

$$\Rightarrow \boxed{v = \frac{1}{2}\tan h^{-1}\left(\frac{2y}{1+x^2+y^2}\right)} = \text{Img. part}$$

Ques: Find all the values of $\sin^{-1}2$, $\tan^{-1}$ taking 2 as a complex no.

Ans: $\sin^{-1}z = -i\log\left(zi + \sqrt{1-z^2}\right)$

so

$$\sin^{-1}2 = -i\log\left(2i + \sqrt{1-4}\right)$$

$$= -i\log\left(2i + \sqrt{-3}\right) = -i\log\left(2i + \sqrt{3}\,i\right)$$

$$= -i\log i(2+\sqrt{3})$$

$$= -i\left[\log i + \log(2+\sqrt{3})\right]$$

$$= -i\left[\log e^{i\pi/2} + \log(2+\sqrt{3})\right]$$

$$= -i\left[\frac{i\pi}{2} + \log(2+\sqrt{3})\right]$$

$\sin^{-1}2 = -i^2\frac{\pi}{2} - i\log(2+\sqrt{3})$

$$\boxed{\sin^{-1}2 = \frac{\pi}{2} - i\log(2+\sqrt{3})}$$

↓ Real part — ↓ Imaginary part.

Questions:

i) $1+i\tan\alpha$

$1+i\tan\alpha = r(\cos\theta + i\sin\theta)$

$r = \sqrt{1+\tan^2\alpha} = \sqrt{\sec^2\alpha} = \sec\alpha.$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\tan\alpha}{1}\right) = \alpha.$

$1+i\tan\alpha = \sec\alpha(\cos\alpha + i\sin\alpha)$

ii) $1-\sin\alpha+i\cos\alpha$

$x = 1-\sin\alpha, \quad y = \cos\alpha$

$r = \sqrt{(1-\sin\alpha)^2+\cos^2\alpha} = \sqrt{1+\sin^2\alpha-2\sin\alpha+\cos^2\alpha} = \sqrt{2-2\sin\alpha} = \sqrt{2(1-\sin\alpha)}$

$r = \sqrt{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \sqrt{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2} = \sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)$

$\theta = \tan^{-1}\left(\frac{\cos\alpha}{1-\sin\alpha}\right) = \tan^{-1}\left[\frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)}{\left(\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)}\right]$

$= \tan^{-1}\left[\frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}\right]$ $\left[\because \cos 2\theta = \cos^2\theta - \sin^2\theta\right]$

divide by $\cos\frac{\alpha}{2}$

$\theta = \tan^{-1}\left[\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right]$

$\therefore 1 - 2\sin\theta + i\cos\theta = i\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$ $\left[\because a(a_1+ib_1)+i\sin(a_1+ib_1)\right]$

Question: P.T. if $(a_1+ib_1)(a_2+ib_2)\cdots(a_n+ib_n) = A+iB$ then

(a) $(a_1^2+b_1^2)(a_2^2+b_2^2)\cdots(a_n^2+b_n^2) = A^2+B^2$

(b) $\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \cdots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$

Ans: (a) $|z| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2}$ $\quad a_1+ib_1 = \text{cis}\,\theta_1$

take modulus both sides.

$|(a_1+ib_1)(a_2+ib_2)\cdots(a_n+ib_n)| = |A+iB|$

$\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}\cdots\sqrt{a_n^2+b_n^2} = \sqrt{A^2+B^2}$

squaring both sides.

$(a_1^2+b_1^2)(a_2^2+b_2^2)\cdots(a_n^2+b_n^2) = A^2+B^2.$

(b) $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

take argument on both sides :-

$\arg[(a_1+ib_1)(a_2+ib_2)\cdots a_n+ib_n] = \arg(A+iB)$

$\arg(a_1+ib_1) + \arg(a_2+ib_2) + \cdots \arg(a_n+ib_n) = \arg(A+iB)$

$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \cdots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$

Hence proved.

Q.3: If $\cos\alpha + 2\cos\beta + 3\cos\gamma = 0$ & $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

(a) $\cos 3\alpha + 8\cos^3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

cube on both sides.

Let $a = \cos\alpha + i\sin\alpha$

$b = 2(\cos\beta + i\sin\beta)$

$c = 3(\cos\gamma + i\sin\gamma)$

$a+b+c = 0$

$a+b = -c$ cube both side

$a^3+b^3+3ab\underbrace{(a+b)}_{-c} = -c^3$

$a^3+b^3+c^3+3ab(-c)$

$a^3+b^3+c^3 = 3abc$

$(\cos\alpha + i\sin\alpha)^3 + (2\cos\beta + 2i\sin\beta)^3 + (3\cos\gamma + 3i\sin\gamma)^3 =$

using de-moivers theorm. $3[(\cos\alpha + i\sin\alpha)(2\cos\beta + 2i\sin\beta)(3\cos\gamma + 3i\sin\gamma)]$

$\Rightarrow \cos 3\alpha + i\sin 3\alpha + 8[\cos 3\beta + i\sin 3\beta] + 27(\cos 3\gamma + i\sin 3\gamma) = 18[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

on comparing real & imaginary parts.

(a) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

(b) $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$

(c) $\cos(2\alpha-\beta-\gamma) + 8\cos(2\beta-\gamma-\alpha) + 27\cos(2\gamma-\alpha-\beta) = 18$

(d) $\sin(2\alpha-\beta-\gamma) + 8\sin(2\beta-\gamma-\alpha) + 27\sin(2\gamma-\alpha-\beta) = 18$

From eq^n (1)

$$\frac{\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma}{\cos(\alpha+\beta+\gamma)} = 18$$

$$\frac{\cos 3\alpha}{\cos(\alpha+\beta+\gamma)} + \frac{8\cos 3\beta}{\cos(\alpha+\beta+\gamma)} + \frac{27\cos 3\gamma}{\cos(\alpha+\beta+\gamma)} = 18 + i0$$

$\cos(3\alpha)[\cos(\alpha+\beta+\gamma) + i\sin 0(0)]^{-1} + 8\cos 3\beta[\cos(-\alpha-\beta-\gamma)] + 27\cos 3\gamma[\cos(-\alpha-\beta-\gamma)]$

$\cos(3\alpha)\cos(-\alpha-\beta-\gamma) + 8\cos 3\beta(\cos(-\alpha-\beta-\gamma) + 27\cos^3(\cos(-\alpha-\beta-\gamma) = 18$

$\Rightarrow \cos(2\alpha-\beta-\gamma) + 8\cos(2\beta-\gamma-\alpha) + 27\cos(2\gamma-\alpha-\beta) = 18$ Hence proved

Similarly from eq^n (2)

$\sin(2\alpha-\beta-\gamma) + 8\sin(2\beta-\gamma-\alpha) + 27\sin(2\gamma-\alpha-\beta) = 18$

1:- Find all values of $(1+i)^{1/3}$ & obtain its product.

$(1+i)^{1/3}$

Q.5, 7, 9, 11, 13

DATE
PAGE

Ans: $z = (1+i)^{1/3} = r[\cos\theta + i\sin\theta]$

$r = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}\left(\frac{\pi}{4}\right)$

$z = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^{1/3}$

$z = (\sqrt{2})^{\frac{1}{3}}\left(\text{cis}\frac{\pi}{4}\right)^{1/3} = \left(\sqrt{2}\right)^{\frac{1}{3}}\left[\text{cis}\left(2n\pi+\frac{\pi}{4}\right)\right]^{1/3} = \sqrt{2}\left[\text{cis}\left(2n+\frac{1}{4}\right)\pi\right]$

using de-Moivers theorm:-

$z = (\sqrt{2})^{\frac{1}{3}}\left[\cos\left(2n+\frac{1}{4}\right)\frac{\pi}{3} + i\sin\left(2n+\frac{1}{4}\right)\frac{\pi}{3}\right]$

at $n=0$

putting $n = 0, 1, 2$

$z = 2^{1/6}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right] = 2^{1/6}\, e^{i\pi/12}$

at $n=1$ $z = 2^{1/6}\left[\text{cis}\left[\frac{2+1}{4}\right]\frac{\pi}{3}\right] = 2^{1/6}\left[\text{cis}\left[\frac{9}{4}\times\frac{\pi}{3}\right]\right] = 2^{1/6}\, e^{i\frac{3\pi}{4}}$

at $n=2$ $z = 2^{1/6}\left[\text{cis}\left[\frac{4+1}{4}\right]\frac{\pi}{3}\right] = 2^{1/6}\,\text{cis}\left[\frac{17\pi}{12}\right] = 2^{1/6}\, e^{i\frac{17\pi}{12}}$

product $= 2^{3/6}\cdot e^{\frac{27\pi}{12}}$ Ans

Q.5: solve the equation

$(4+x)^5 - (4-x)^5 = 0$

Q.6: separating $\log\left(\frac{3-i}{3+i}\right)$ into real & imaginary part.

Ans: $\log\left(\frac{3-i}{3+i}\right)$

(i) $3-i = r[\cos\theta + i\sin\theta]$ $\Rightarrow r = \sqrt{9+1} = \sqrt{10}$

$\theta = \tan^{-1}\left(\frac{-1}{3}\right) = \tan^{-1}$

$\log\left(\frac{3-i}{3+i}\right) = \log(3-i) - \log(3+i)$

real $\log(3-i) = \frac{1}{2}\log\left((3)^2+(-1)^2\right) = \frac{1}{2}\log(8)$

real log

X

tanh Nid

hi

If $\tanh(u+iv) = x+iy$ ① show that

$\tanh 2u = \frac{2x}{1-x^2-y^2}$ $\Rightarrow \tanh(u+iv) = \tan^{-1}(x+iy)$ ①

convert $i \to -i$

$\tanh(u-iv) = x-iy$ ② $\Rightarrow \tanh(u-iv) = \tan^{-1}(x-iy)$ ②

add ① & ②

$\tanh(u+iv) + \tanh(u-iv) = 2x$

$2u = \tanh^{-1}(x+iy) + \tanh^{-1}(x-iy)$

$2u = \tanh^{-1}\left[\frac{x+iy+x-iy}{1-(x+iy)(x-iy)}\right], \tan^{-1}\left[\frac{2x}{1-(x^2-i^2y^2)}\right]$

$2u = \tanh^{-1}\left[\frac{2x}{1-x^2-y^2}\right] \Rightarrow \tan(h2u) = \frac{2x}{1-x^2-y^2}$

$\tan 2v = \frac{2y}{1+x^2+y^2}$

subtract eqn ① - ②

$\tan^{-1}(x+iy) - \tan^{-1}(x-iy) = h2iv$

$\tan^{-1}\left(\frac{x+iy-x+iy}{1+(x+iy)(x-iy)}\right) = h2iv$

$= \tan^{-1}\left[\frac{2iy}{1+x^2-i^2y^2}\right] = h2iv$

$\Rightarrow \tan^{-1}\left[\frac{2iy}{1+x^2+y^2}\right] = h2iv$

$\Rightarrow \boxed{\tan(2iv) = \frac{2iy}{1+x^2+y^2} \Rightarrow i\tan(h2v) = \frac{2iy}{1+x^2+y^2}}$ X

$\Rightarrow \tan(h(2iv)) = \left[\frac{2iy}{1+x^2+y^2}\right]$

$\Rightarrow -i\tan(i(2iv)) = \frac{2iy}{1+x^2+y^2}$

$\Rightarrow -\tan(-2v) = \frac{2y}{1+x^2+y^2}$

$3-i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

on back side

DATE
PAGE

Q.6 :- separate real & imaginary part of $\log\left(\frac{3-i}{3+i}\right)$

$\log\left[\frac{(3-i)(3-i)}{(3+i)(3-i)}\right] = \log\left(\frac{(3-i)^2}{(3)^2-(i)^2}\right) = \log\left(\frac{9+i^2-2(3)i}{10}\right)$

$\log\left(\frac{8-6i}{10}\right) = \log\left(\frac{4-3i}{5}\right)$

Now $\frac{4}{5} - \frac{3i}{5}$

$r = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = 1$

$\tan\theta = \frac{3/5}{4/5} = \frac{3}{4}$; $(\theta = 37°)$

$r\cos\theta = \frac{4}{5}$, $r\sin\theta = \frac{-3}{5}$

using direct formula

$\log\left(\frac{4}{5} - \frac{3i}{5}\right) = \frac{1}{2}\log\left[\left(\frac{4}{5}\right)^2 + \left(\frac{-3}{5}\right)\right] + i\left[\tan^{-1}\left(\frac{3}{4}\right) + 2n\pi\right]$

$= \frac{1}{2}\log(1) + i\left[\tan^{-1}\left(\frac{3}{4}\right) + 2n\pi\right]$

principal part

Ques 7 :- P.T. $\tan^{-1}x = \frac{1}{2i}\log\left(\frac{1+ix}{1-ix}\right)$ ... (1)

$1+ix = r[\cos\theta + i\sin\theta]$

$r\cos\theta = 1$

$r\sin\theta = x$

$r = \sqrt{1+x^2}$

$\tan\theta = x \Rightarrow \theta = \tan^{-1}(x)$

$\cos\theta = \frac{1}{\sqrt{1+x^2}}$, $\sin\theta = \frac{x}{\sqrt{1+x^2}}$

$1+ix = re^{i\theta}$

$1-ix = re^{-i\theta}$

put in (1)

$\frac{1}{2i}\log\left(\frac{re^{i\theta}}{re^{-i\theta}}\right) = \frac{1}{2i}\log e^{2i\theta} = \frac{1}{2i}\cdot 2i\theta = \theta$

$\frac{1}{2i}\log\left(\frac{1+ix}{1-ix}\right) = \tan^{-1}x$ hence proved.

Q.15: If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, Prove that $x_1 \cdot x_2 \cdot x_3 \cdots \infty = -1$

$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \text{cis}\left(\frac{\pi}{2}\right) = P$

$x_2 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \text{cis}\left(\frac{\pi}{4}\right) \cdot \text{cis}\left(\frac{\pi}{2}\right)^{1/2} = P^{1/2}$

$x_3 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} = \text{cis}\left(\frac{\pi}{8}\right) = P^{\frac{1}{3}}$

$\vdots$

$x_\infty = \cos\frac{\pi}{2^\infty} + i\sin\frac{\pi}{2^\infty} = \text{cis}\left(\frac{\pi}{\infty}\right) = \text{cis}\,0 = P^\infty$

$P \cdot P^{1/2} \cdot P^{1/3} \cdots P^\infty = P^{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{\infty}}$

$\text{cis}\left(\frac{\pi}{2}\right) \cdot \text{cis}\left(\frac{\pi}{4}\right) \text{cis}\left(\frac{\pi}{8}\right) \cdots \text{cis}\left(\frac{\pi}{\infty}\right)$

$= \text{cis}\left(\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{8}\cdots\frac{\pi}{\infty}\right) = \text{cis}\left[\frac{\pi}{2}\left(1+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{\infty}\right)\right]$

Sum $= \text{cis}\left[\frac{\pi}{2}\left(\frac{1-\frac{1}{2^\infty}}{1-\frac{1}{2}}\right)\right] = \text{cis}\left[\frac{\pi}{2}\left(\frac{1}{\frac{1}{2}}\right)\right] = \text{cis}\left[\frac{\pi}{2}\times 2\right]$

Sum $= \text{cis}\,\pi = \boxed{-1}$

Q.16: Solve $(4+x)^5 - (4-x)^5 = 0$

Sol^n $(4+x)^5 = (4-x)^5 \Rightarrow \left(\frac{4+x}{4-x}\right)^5 = 1$

$\frac{4+x}{4-x} = (1)^{1/5} = \text{cis}(0)^{1/5} = \text{cis}(2k\pi+0)^{1/5} = \text{cis}\left(\frac{2k\pi}{5}\right)$

Let $\frac{2k\pi}{5} = \theta$ by using componendo & dividendo rule.

$$\frac{y+x+y-x}{y+x-y+x} = \frac{\text{cis}\theta+1}{\text{cis}\theta-1}$$

$$= \frac{2y}{2x} \cdot \frac{\text{cis}\theta+1}{\text{cis}\theta-1} = \frac{\cos\theta+i\sin\theta+1}{\cos\theta+i\sin\theta-1}$$

$$\Rightarrow y(\text{cis}\theta-1) = x[\text{cis}\theta+1] = \frac{2\cos^2(\theta/2)+i\sin\theta}{-[2\sin^2(\theta/2)+i\sin\theta]}$$

$$\Rightarrow \frac{2\cos^2(\theta/2)+i\,2\sin(\theta/2)\cos(\theta/2)}{-[2\sin^2(\theta/2)+i\,2\sin(\theta/2)\cos(\theta/2)]} = \frac{2\cos(\theta/2)[\cos(\theta/2)+i\sin(\theta/2)]}{-2\sin(\theta/2)[\sin(\theta/2)+i\cos(\theta/2)]}$$

$$\Rightarrow -\cot(\theta/2)\left[\frac{\text{cis}(\theta/2)}{-i[\cos\theta/2+i\sin\theta/2]}\right] = \cot(\theta/2)\left[\frac{1}{i}\,\frac{\text{cis}(\theta/2)}{\text{cis}(\theta/2)}\right]$$

$$\Rightarrow -i\cot(\theta/2)\left[\frac{e^{i\theta/2}}{e^{+i\theta/2}}\right] = -i\cot(\theta/2)\left[\quad\right]$$

$$= -i\cot\theta/2[\cos\theta/2+i\sin\theta/2] = -i\cot(\theta/2)$$

Q.13:- If $\sin(u+iv) = x+iy$ then prove

(i) $\frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1$ (ii) $\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1$

Sol^n $\sin(u+iv) = \sin(h)(u+iv) = i\sin u\cos iv + \cos u \cdot \sin(iv)$

$= \sin u \cdot \cos(hv) + \cos u\, i\sin(hv)$

$= \sin u \cosh v + i\sin(hv)\cos u = x+iy$

(Real) (img.)

Now compairing.

$x = \sin u \cosh v$, $y = \sin(hv)\cos u$

$\sin u = \frac{x}{\cosh v}$ — (1), $\cos u = \frac{y}{\sin hv}$ — (2)

(i) Squaring & adding (1) & (2)

$\sin^2 u + \cos^2 u = \frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v}$. hence proved

(1)

$\cosh^2 u - \sinh^2 u = 1$

DATE
PAGE

(ii) Now $\cosh v = \frac{x}{\sin u}$ — (3) & $\sinh v = \frac{y}{\cos u}$ — (4)

Now squaring & subtracting eq^n (3) & (4)

$\cosh^2 v - \sinh^2 v = \frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u}$

$\downarrow$

1

hence proved.

$\{\because \cos^2 hu - \sin^2 hu = 1$

Imp ques:- $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$ i take (i)

i.e- $\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)] = \frac{\tan(\theta+i\theta) + \tan(\theta+i\phi)}{1 - \tan(\theta+i\phi)\tan(\theta-i\phi)}$

$= \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$ $\Rightarrow \tan 2\theta = \frac{2\cos\alpha}{1 - \cos^2\alpha + \sin^2\alpha}$

reverse it $\Rightarrow$ $\cot 2\theta = \frac{1-1}{2\cos\alpha} = 0$

$\cot 2\theta = \cot\frac{\pi}{2}$ Period of cot is π **

$\cot 2\theta = \cot(n\pi + \frac{\pi}{2})$

$2\theta = \cot(n\pi + \frac{\pi}{2}) \Rightarrow \boxed{\theta = \frac{n\pi}{2} + \frac{\pi}{4}}$ hence proved

(ii) Now $\tan(2i\phi) = \tan[(\theta + i\phi) - (\theta - i\phi)]$

$= \frac{\tan(\theta+i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta+i\phi)\tan(\theta-i\phi)} = \frac{\cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$

$= \frac{2i\sin\alpha}{2} = i\sin\alpha$

$\Rightarrow i\tan h\, 2\phi = i\sin\alpha$

$\tan h\, 2\phi = \sin\alpha \Rightarrow \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$

$\frac{\text{Componendo \& dividendo rule}}{\frac{e^{2\phi}}{e^{-2\phi}}}$

$e^{4\phi} = \frac{1+\sin\alpha}{1-\sin\alpha} = \frac{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2})^2}{(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2})^2}$

$\Rightarrow e^{2\phi} = \frac{1 + \tan(\alpha/2)}{1 - \tan(\alpha/2)}$

$\Rightarrow e^{2\phi} = \tan(\frac{\pi}{4} + \frac{\alpha}{2})$ Ans

DATE 29-11-22
PAGE

* C+iS Method of summation :-

Very important

Ques 1:- Sum to n terms of series $1 + x\cos\alpha + x^2\cos 2\alpha + x^3\cos 3\alpha + \dots$ (C-series)
Also find the sum to infinity, $|x| < 1$.

$C = 1 + x\cos\alpha + x^2\cos 2\alpha + x^3\cos 3\alpha + \dots$ — (1)

$S = x^0\sin(0\alpha) + x\sin\alpha + x^2\sin 2\alpha + x^3\sin 3\alpha + \dots$ — (2)

Multiply S with i and then (1) + (2)

$C + iS = (1+0) + x[\cos\alpha + i\sin\alpha] + x^2[\cos 2\alpha + i\sin 2\alpha] + x^3[\cos 3\alpha + i\sin 3\alpha]$

$= 1 + xe^{i\alpha} + x^2e^{i2\alpha} + x^3e^{i3\alpha} + \dots$

$= 1 + xe^{i\alpha} + (xe^{i\alpha})^2 + (xe^{i\alpha})^3 + \dots$

Let $\boxed{xe^{i\alpha} = t}$

$C + iS = 1 + t + t^2 + t^3 + \dots$ → G.P.

$r = t, \quad a = 1, \quad \text{sum} = \frac{a(1-r^n)}{1-r}$

$= \frac{1(1-t^n)}{1-t}$ upto n-terms.

$= \frac{1(1-x^ne^{in\alpha})}{1-xe^{i\alpha}} \times \frac{1-xe^{-i\alpha}}{1-xe^{-i\alpha}}$ → conjugate

$= \frac{1 - xe^{-i\alpha} - x^ne^{in\alpha} + x^{n+1}e^{i(n-1)\alpha}}{1 - xe^{-i\alpha} - xe^{i\alpha} + x^2e^{i\alpha}\cdot e^{-i\alpha}}$

$(\because x(e^{-i\alpha} + e^{i\alpha}))$
$= x[\cos\alpha - i\sin\alpha + \cos\alpha + i\sin\alpha]$
$[= x2\cos\alpha]$

$[e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha]$

$1 - xe^{i\alpha}$
$= 1 - x\cos\alpha - ix\sin\alpha$
$\bar{z} = 1 - x\cos\alpha + ix\sin\alpha$
$1 - x[\cos\alpha - i\sin\alpha]$
$= [1 - xe^{-i\alpha}]$ → If we take the conjugate of e funct then power become -ve.

$C + iS = \frac{1 - x[\cos\alpha - i\sin\alpha] - x^n[\cos n\alpha + i\sin n\alpha] + x^{(n+1)}[\cos(n-1)\alpha + i\sin(n-1)\alpha]}{1 - 2x\cos\alpha + x^2}$

$= \frac{(1 - x\cos\alpha - x^n\cos n\alpha + x^{(n+1)}\cos(n-1)\alpha) + i[x\sin\alpha - x^n\sin n\alpha + x^{n+1}\sin(n-1)\alpha]}{1 - 2x\cos\alpha + x^2}$

$\therefore C+iS = \frac{1-x\cos\alpha - x^n\cos n\alpha + x^{n+1}\cos(n-1)\alpha}{1-2x\cos\alpha+x^2} + i\left[\frac{x\sin\alpha - x^n\sin n\alpha + x^{n+1}\sin(n-1)\alpha}{1-2x\cos\alpha+x^2}\right]$

Equating real & Img part:

$$\Rightarrow C = \frac{1-x\cos\alpha - x^n\cos n\alpha + x^{n+1}\cos(n-1)\alpha}{1-2x\cos\alpha+x^2}$$

Now $x^{n+1} \to 0$ & $x^n \to 0$ as $n\to\infty$ as $|x|<1 \Rightarrow -1<x<1$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{10}$

$$\text{at } n\to\infty \quad C = \frac{1-x\cos\alpha}{1-2x\cos\alpha+x^2}$$ Ans

Ques: Sum to infinity

(i) $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \dots$

$S = \sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \dots$

$C = \cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + \dots$

Now

$C+iS = (\cos\alpha+i\sin\alpha) + x(\cos(\alpha+\beta)+i\sin(\alpha+\beta)) + \frac{x^2}{2!}(\cos(\alpha+2\beta)+i\sin(\alpha+2\beta)) + \dots$

$= e^{i\alpha} + \frac{xe^{i(\alpha+\beta)}}{1!} + \frac{x^2e^{i(\alpha+2\beta)}}{2!} + \dots$

$= e^{i\alpha} + \frac{xe^{i\alpha}e^{i\beta}}{1!} + \frac{x^2e^{i\alpha}(e^{i\beta})^2}{2!} + \dots$

$= e^{i\alpha}\left[1 + \frac{xe^{i\beta}}{1!} + \frac{(xe^{i\beta})^2}{2!} + \dots\right]$

$= e^{i\alpha}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots\right] \quad (\because xe^{i\beta} = z)$

$\{\because e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots$

$= e^{i\alpha}\cdot e^z$

$= e^{i\alpha}\cdot e^{x\cdot e^{i\beta}}$

$= e^{i\alpha}\cdot e^{x(\cos\beta+i\sin\beta)}$

DATE
PAGE

$= (\cos\alpha + i\sin\alpha) \cdot e^{x\cos\beta} \cdot e^{ix\sin\beta}$

$= e^{x\cos\beta} [\cos\alpha + i\sin\alpha] [\cos(x\sin\beta) + i\sin(x\sin\beta)]$

$= e^{x\cos\beta} [\cos\alpha \cdot \cos(x\sin\beta) + i\cos\alpha \sin(x\sin\beta) + i\sin\alpha \cdot \cos(x\sin\beta) + i^2 \sin\alpha \sin(x\sin\beta)]$

$= e^{x\cos\beta} [\cos\alpha \cdot \cos(x\sin\beta) - \sin\alpha \cdot \sin(x\sin\beta) + i[\sin(x\sin\beta)\cos\alpha + \sin\alpha\cos(x\sin\beta)]]$

↓ Real part

↓ Img part.

Q.8 :- $(1+i\sqrt{3})^{(1+i\sqrt{3})}$

$e^{\log (1+i\sqrt{3})^{(1+i\sqrt{3})}}$

$e^{(1+i\sqrt{3})\log(1+i\sqrt{3})}$

$= e^{(1+i\sqrt{3})\log 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$

$= e^{(1+i\sqrt{3})\log 2[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}]}$

$= e^{(1+i\sqrt{3})\log(2\,\mathrm{cis}\frac{\pi}{3})}$

$= e^{(1+i\sqrt{3})[\log 2 + \log e^{i\pi/3}]}$

$= e^{(1+i\sqrt{3})[\log 2 + i\frac{\pi}{3}]}$

$= e^{[\log 2 + i\frac{\pi}{3} + i\sqrt{3}\log 2 - \sqrt{3}\frac{\pi}{3}]}$ $= e^{\log 2} \cdot e^{i\pi/3} \cdot \left(e^{i\sqrt{3}\log 2} + e^{\log i}\right)$

$= e^{\left(\log 2 - \frac{\pi}{\sqrt{3}}\right)} \cdot e^{\left(\frac{\pi}{3} i + i\sqrt{3}\log 2\right)} \cdot e^{-\frac{\pi}{\sqrt{3}}}$

$\Rightarrow e^{(\log 2 - \frac{\pi}{3})} \cdot e^{i(\frac{\pi}{3} + \sqrt{3}\log 2)}$

$\Rightarrow e^{(\log 2 - \pi/3)} \cdot \mathrm{cis}\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)$

$\Rightarrow \frac{e^{\log 2}}{e^{\pi/3}} \cdot \left[\cos\left(\frac{\pi}{3} + \sqrt{3}\log 2\right) + i\sin\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)\right]$

$\Rightarrow 2e^{-\pi/3} \cos\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)$ — Real part.

$\Rightarrow 2e^{-\pi/3} \sin\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)$ → Img. part.

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Q. Ans :- $\sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha + \ldots$ to infinitely,

angle is changing w.r.t cos so this is C-series.

$C = \sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha \ldots$

$S = \sin\alpha\sin\alpha + \sin^2\alpha\sin 2\alpha + \sin^3\alpha\sin 3\alpha \ldots$

$C + iS = \sin\alpha[\cos\alpha + i\sin\alpha] + \sin^2\alpha[\cos 2\alpha + i\sin 2\alpha] + \sin^3\alpha[\cos 3\alpha + i\sin 3\alpha] + \ldots$

$C+iS = \sin\alpha . e^{i\alpha} + \sin^2\alpha\, e^{i2\alpha} + \sin^3\alpha . e^{i3\alpha} + \dots$

$= \sin\alpha . e^{i\alpha} + (\sin\alpha . e^{i\alpha})^2 + (\sin\alpha . e^{i\alpha})^3 \dots$

Let $\sin\alpha . e^{i\alpha} = t$

$= t + t^2 + t^3 \dots$ → G.P.

$= \frac{t}{1-t} = \frac{\sin\alpha . e^{i\alpha}}{1 - \sin\alpha . e^{i\alpha}}$ $\left\{S_\infty = \frac{a}{1-r}\right\}$

take conjugate

$= \frac{\sin\alpha e^{i\alpha} \times 1 - \sin\alpha e^{-i\alpha}}{(1-\sin\alpha . e^{i\alpha})(1-\sin\alpha . e^{-i\alpha})}$

$\left\{1 - \sin\alpha . e^{i\alpha} \rightarrow 1 - \sin\alpha e^{-i\alpha}\right.$ (conjugate)

$= \frac{\sin\alpha e^{i\alpha} - \sin^2\alpha . e^{i\alpha} e^{-i\alpha}}{1 - \sin\alpha e^{-i\alpha} - \sin\alpha e^{i\alpha} + \sin^2\alpha . e^{i\alpha} . e^{-i\alpha}}$

$= \frac{\sin\alpha(\cos\alpha + i\sin\alpha) - \sin^2\alpha}{1 - \sin\alpha(e^{-i\alpha} + e^{i\alpha}) + \sin^2\alpha}$

$\left\{e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha\right\}$

$\left\{\cos\alpha = \frac{e^{-i\alpha} + e^{i\alpha}}{2}\right.$ $\left\{\sin 2\theta = 2\sin\theta\cos\theta\right.$

$= \frac{(\sin\alpha\cos\alpha - \sin^2\alpha) + i\sin^2\alpha}{1 - 2\sin\alpha\cos\alpha + \sin^2\alpha}$

$C+iS = \frac{\sin\alpha\cos\alpha - \sin^2\alpha}{1 - \sin 2\alpha + \sin^2\alpha} + i\frac{\sin^2\alpha}{1 - \sin 2\alpha + \sin\alpha}$

on compairing :-

$$C = \frac{\sin\alpha.\cos\alpha - \sin^2\alpha}{1 - \sin 2\alpha + \sin^2\alpha}$$ Ans

$(x+a)^5 = {}^5C_0 x^5 a^0 + {}^5C_1 x^4 a^1 + {}^5C_2 x^3 a^2 + {}^5C_3 x^2 a^3 + {}^5C_4 x^1 a^4 + {}^5C_5 x^0 a^5$

Pascal Table :

n	$(x+a)^n$	Binomial coeff.						
1		1	1					
2		1	2	1				
3		1	3	3	1			
4		${}^4C_0=1$	${}^4C_1=4$	6	4	1		
5		1	5	10	10	5	1	
6		1	6	15	20	15	6	1
7		1	7	21	35	35	21	7

$\left\{\left(\frac{x+1}{x}\right)^n\right.$ → In this power reduced by 1 (of x)

DATE
PAGE

Ques: P.T $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

$x = \cos\theta + i\sin\theta \Rightarrow \frac{1}{x} = x^{-1} = \cos\theta - i\sin\theta$ — (2)
↳ (1)

(1)+(2)

$x + \frac{1}{x} - 2\cos\theta$ (∵ we want cos terms)

take 6 power on both side

$\left(x+\frac{1}{x}\right)^6 = 2^6\cos^6\theta$

power decrease by 2

$\Rightarrow 1.x^6 + 6x^4 + 15x^2 + 20x^0 + 15x^{-2} + 6x^{-4} + 1x^{-6} = 64\cos^6\theta$

$\Rightarrow \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20 = 64\cos^6\theta$

$\Rightarrow 2\cos 6\theta + 12\cos 4\theta + 15(2\cos 2\theta) + 20 = 64\cos^6\theta$

$\Rightarrow 2[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10] = 64\cos^6\theta$

$\Rightarrow \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 = \cos^6\theta$.

$\cos m\theta + i\sin m\theta = x^m$

$\frac{1}{x^m} = \cos m\theta - i\sin m\theta$

$x^m + \frac{1}{x^m} = 2\cos m\theta$

Date : 01-12-22

Ques 2:- Expand $\sin^7\theta$ in a series of multiple of θ.

Ans: $x = \cos\theta + i\sin\theta \Rightarrow x^m = \cos m\theta + i\sin m\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta \Rightarrow x^{-m} = \cos m\theta - i\sin m\theta$

$x - \frac{1}{x} = 2i\sin\theta \Rightarrow x^m + \frac{1}{x^m} = 2\cos m\theta$

$x^m - \frac{1}{x^m} = 2i\sin\theta$.

$\left(x - \frac{1}{x}\right)^7 = (2i)^7\sin^7\theta$

we get alternate signs.

power reduced by 2 in $x^m + \frac{1}{x^m}$

$1.x^7 - 7x^5 + 21x^3 - 35x + 35x^{-1} - 21x^{-3} + 7x^{-5} - 1x^{-7} = -i\sin^7\theta(128)$

$\Rightarrow \left[x^7 - \frac{1}{x^7}\right] - 7\left[x^5 - \frac{1}{x^5}\right] + 21\left[x^3 - \frac{1}{x^3}\right] - 35\left(x - \frac{1}{x}\right) = -i\,2i\sin 7\theta$ (i)

$\Rightarrow 2i\sin 7\theta - (7)2i\sin 5\theta + (21)2i\sin 3\theta - (35)2i\sin\theta = -i\,2i\sin 7\theta$ (2i)

$\Rightarrow 2i\left[\sin 7\theta + 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right] = 2i\sin 7\theta$

$-i\left(\frac{1}{2}\right)^6$

$\Rightarrow \frac{-1}{64}\left[\sin 7\theta + 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right] = \sin 7\theta$

Ans

Ques- $\sin^6\theta \cdot \cos^2\theta$

$n=6$, coff = 1, -6, 15, -20, 15, -6, 1

$n=2$, coff = 1 2

1 -6 15 -20 15 -6 1

\+ — 2 -12 30 -40 30 -12 2

\+ — — 1 -6 15 -20 15 -6 1

Final coff → 1 -4 4 4 -10 4 4 -4 1

Total no. of terms = 9 {(6+1) & (2+1)}

If sin will have even power ans. will be in form of cos & if power is odd ans. in the for of sinθ

In case of sin coff are alternative & in cos it is +ve.

$\left(x + \frac{1}{x}\right)^2 \left(x - \frac{1}{x}\right)^6 = (\cos\theta)^2 (2i\sin\theta)^6$

cosθ 2 | 2isinθ 6

$= 1x^8 - 4x^6 + 4x^4 + 4x^2 - 10x^0 + 4x^{-2} + 4x^{-4} - 4x^{-6} + 1x^{-8}$

$= 2^8 \cdot i^6 \sin^6\theta \cos^2\theta$

$\Rightarrow 1\left(x^8 + \frac{1}{x^8}\right) - 4\left[x^6 + \frac{1}{x^6}\right] + 4\left[x^4 + \frac{1}{x^4}\right] + 4\left[x^2 + \frac{1}{x^2}\right] - 10 = 256(-1)\sin^6\theta\cos^2\theta$

$\Rightarrow 2\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 10 = 256\sin^6\theta\cos^2\theta$

$\Rightarrow \frac{2}{256}\left[\cos 8\theta - 2\cos 6\theta + 2\cos 4\theta + 2\cos 2\theta - 5\right] = \sin^6\theta\cos^2\theta$

$\frac{1}{128}$

$= \frac{1}{128}\left[\cos 8\theta - 2\cos 6\theta + 2\cos 4\theta + 2\cos 2\theta - 5\right] = \sin^6\theta \cdot \cos^2\theta$.

DATE
PAGE

Ques→ $\cos^4\theta \sin^3\theta$ { power of sin is odd so ans. in form of sine }

n=4	1	4	6	4	1			
n=3	+1	−3	+3	−1				
	1	4	6	4	1			
	−3	−12	−18	−12	−3			
		3	12	18	12	3		
			−4	−4	−4	−4	−1	
Final coeff.	1	1	−3	−3	3	3	−1	−1

$$\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3 = [2\cos\theta]^4[2i\sin\theta]^3$$

$$= x^7 + x^5 - 3x^3 - 3x^1 + 3x^{-1} + 3x^{-3} - x^{-5} - x^{-7}$$

$$= i^3 16 \times 8 \times \cos^4\theta \cdot \sin^4\theta$$

$$= 1\left(x^7 - \frac{1}{x^7}\right) + 1\left(x^5 - \frac{1}{x^5}\right) - 3\left[x^3 - \frac{1}{x^3}\right] - 3\left[x^1 - \frac{1}{x^1}\right]$$

$$= -i \times 128 \times \cos^4\theta \cdot \sin^3\theta$$

$$= 2i\sin 7\theta + 1.2i\sin 5\theta - 3[2i\sin 3\theta] - 3[2i\sin\theta]$$

$$= -i(128) \cdot \cos^4\theta \cdot \sin^3\theta$$

$$\Rightarrow \frac{-2\times i}{128\, i}[2\sin 7\theta + 2\sin 5\theta - 3\sin 3\theta - 3\sin\theta] = \cos^4\theta\sin^3\theta$$

$$\frac{128}{64}$$

$$\Rightarrow -\frac{1}{64}[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta] = \cos^4\theta\sin^3\theta$$

Date :- 6-12-22

Properties of Complex numbers.

① The sum, difference, product and quotient of the complex no. is a complex no.

② If a complex no. is zero that means real and imaginary part of that no. are also zero

3) :- If two complex no. are equal than their
(a) - there corresponding real & img. parts will also equal.
(b) - there conjugates will also be equal.

4) :- Properties :→

(i) :- $\overline{z_1+z_2} = \overline{z}_1 + \overline{z}_2$

(ii) :- $\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$

(iii) :- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$, $z_2 \neq 0$

(iv) :- $|z| = |\overline{z}|$

(v) :- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

(vi) :- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(vii) :- $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

(viii) :- $|z_1+z_2| \leq |z_1| + |z_2|$ {Triangular inequality}

(ix) :- $|z_1| - |z_2| \leq |z_1 - z_2|$

Ques: Sum upto infinity

$$\frac{\cos\theta}{0!} + \frac{\sin\theta\cos 2\theta}{1!} + \frac{\sin^2\theta\cos 3\theta}{1.2} + \text{-----}$$

Angle change wrt. cos so this is C-series

$$C = \frac{\cos\theta}{0!} + \frac{\sin\theta\cos 2\theta}{1!} + \frac{\sin^2\theta\cos 3\theta}{1.2} + \text{----}$$

$$S = \frac{\sin\theta}{0!} + \frac{\sin\theta\sin 2\theta}{1!} + \frac{\sin^2\theta\sin 3\theta}{1.2} + \text{----}$$

$$C+iS = \frac{1}{0!}[\cos\theta + i\sin\theta] + \frac{\sin\theta}{1!}[\cos 2\theta + i\sin 2\theta] + \frac{\sin^2\theta}{2!}[\cos 3\theta + i\sin 3\theta] + \text{----}$$

$$= \frac{e^{i\theta}}{0!} + \frac{\sin\theta\, e^{2i\theta}}{1!} + \frac{\sin^2\theta \cdot e^{3i\theta}}{2!} + \text{----}$$

$$= e^{i\theta}\left[1 + \sin\theta \cdot e^{i\theta} + \sin^2\theta \cdot e^{2i\theta} + \text{----}\right]$$

Let $\sin\theta \cdot e^{i\theta} = t$

$= e^{i\theta}\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \dots\right]$ $\left\{S_\infty = \frac{a}{1-r}\right.$

$= \left[e^{i\theta}\left[\frac{1}{1-t}\right] = e^{i\theta}\left[\frac{1}{1-\sin\theta \cdot e^{i\theta}}\right]\right.$

We know that

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$= e^{i\theta} \cdot e^{t} = e^{i\theta} \cdot e^{\sin\theta \cdot e^{i\theta}}$

$= (\cos\theta + i\sin\theta)\, e^{\sin\theta[\cos\theta + i\sin\theta]}$

$= [\cos\theta + i\sin\theta]\, e^{\sin\theta\cos\theta + i\sin^2\theta}$

$= [\cos\theta + i\sin\theta]\, e^{i\theta} e^{\sin\theta\cdot\cos\theta} \cdot e^{i\sin^2\theta}$

$= e^{\sin\theta\cos\theta}\, e^{i[\theta + \sin^2\theta]}$

$= e^{\sin\theta\cos\theta}\left[\cos[\theta + \sin^2\theta] + i\sin[\theta + \sin^2\theta]\right]$

$\Rightarrow \boxed{e^{\sin\theta\cos\theta}\cos[\theta + \sin^2\theta] + i e^{\sin\theta\cos\theta}\sin[\theta + \sin^2\theta]}$

↓ Real part

↓ img part.

Test Ques

Ques 6: $\log\left(\frac{3-i}{3+i}\right)$

Date :- 7-12-22

$\log\left(\frac{3-i}{3+i}\right) = 2n\pi i + \log\left(\frac{3-i}{3+i}\right)$

$3+i = r\cos\theta = re^{i\theta}$

$r = 2, \quad \theta = \tan^{-1}\frac{1}{3}$

$= 2n\pi i + \log e^{-2i\theta}$

$= 2n\pi i + (-2i\tan^{-1}\frac{1}{3})$

$= 2n\pi i - 2i\tan^{-1}\frac{1}{3}$

$= 2i\left(n\pi - \tan^{-1}\frac{1}{3}\right)$ → purely img

DATE
PAGE

First formulas

Function	Period
$\sin\theta, \cos\theta$	2π
$\tan\theta$	π
$e^z, \log_e^z$	$2\pi i$
$\sinh\theta, \cosh\theta$	$2\pi i$
$\tanh\theta$	πi

log is many valued function.

* $\log N = x + 2n\pi i$ (where $e^x = N$)
* $\log(ij) = \log j + i\pi/2$
* $\log(-N) = \pi i + \log N$, (where N is +ve)

Ques:- Prove that $(\cosh x + \sinh x)^n = (\cosh nx + \sinh nx)^n$

$(\cosh x + \sinh x)^n = (\cos ix - i\sin ix)^n$

by de-moivers theorm

$= (\cos n(ix) - i\sin(n)ix)$

$= (\cosh nx + i\sinh nx)$ Ans

Ques:- $\left(\frac{1+\tanh v}{1-\tanh v}\right)^3 = \left(\frac{1+\frac{\sinh x}{\cosh x}}{1-\frac{\sinh x}{\cosh x}}\right)^3 = \left[\frac{\cosh x + \sinh x}{\cosh x - \sinh x}\right]^3$

$= \left[\frac{\cos iv + -i\sin iv}{\cos iv + i\sin iv}\right]^3 = \frac{\cos iv - i\sin iv}{\cos iv + i\sin iv}\left(\frac{e^{-iv}}{e^{iv}}\right)^3$

$= (e^{-2iv})^3 = e^{-6iv} = \cos 6iv - i\sin 6iv \Rightarrow \cosh 6v + \sinh 6v$ Ans

$(\because \sinh v = -i\sin iv)$

Q.3:- What is the real part of $\log(1 + i\tan\alpha)$

$1 + i\tan\alpha = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$r = \sqrt{1+\tan^2\alpha} = \sec^2\alpha$

$\theta = \tan^{-1}\left(\frac{\tan\alpha}{1}\right) = \alpha$

real part $= r\cos\theta = \alpha(\sec^2\alpha)$

$\log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta$

↓ Real part.

Q:- What is the real part of $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^{n\theta}$

$= \frac{\cos\theta + i\sin\theta}{(-i)^n(\cos\theta - i\sin\theta)} = \frac{e^{i\theta}}{(i)^{n\theta}\, e^{-i\theta}} = \frac{e^{2i\theta}}{-1}$

$= -(\cos 2\theta + i\sin 2\theta)$

Ques:- Prove that $(1+i\tan\alpha)^{-i} = e^{(2n\pi+\alpha) - i\log(\sec\alpha)}$

$1 + i\tan\alpha = r(\cos\theta + i\sin\theta) = e^{i\theta}$

$r = \sqrt{1+\tan^2\alpha} = \sqrt{\sec^2\alpha} = \sec\alpha$

$\theta = \tan^{-1}\left(\frac{\tan\alpha}{1}\right) = \alpha$

take $\log(1+i\tan\alpha)^{-i}$

$= e^{-i\log(1+i\tan\alpha)} = e^{-i\log(\sec\alpha[\cos\alpha + i\sin\alpha])}$

$= e^{-i\log(\sec\alpha) - i\log(\cos\alpha + i\sin\alpha)}$

$(1+i\tan\alpha)^{-i} = e^{2n\pi} \cdot e^{\log(1+i\tan\alpha)^{-i}} = e^{-i\log(1+i\tan\alpha)}$

$= e^{-i(2n\pi i + \log(1+i\tan\alpha))}$

$= e^{2n\pi - i\log\left[1 + i\frac{\sin\alpha}{\cos\alpha}\right]} = e^{2n\pi - i\log\left[\frac{\cos\alpha + i\sin\alpha}{\cos\alpha}\right]}$

$= e^{2n\pi - i\log(\cos\alpha + i\sin\alpha) - i\log(\sec\alpha)}$

$(1+i\tan\alpha)^{-i} = e^{-i\log(1+i\tan\alpha)}$

$= e^{-i(2n\pi i + \log(1+i\tan\alpha))}$

$= e^{-i\left[2n\pi i + \log\left(\frac{\cos\alpha + i\sin\alpha}{\cos\alpha}\right)\right]}$

$= e^{-i[2n\pi i + \log(\cos\alpha + i\sin\alpha) - \log(\cos\alpha)]}$

$= e^{-i[2n\pi i + \log(\cos\alpha + i\sin\alpha) - \log(\cos\alpha)]}$

$= e^{2n\pi - i\log e^{i\alpha} + \log(\cos\alpha)}$

$= \left[e^{2n\pi + \alpha} - \log(\sec\alpha)\right]$ Ans

3rd Unit :- Differential eqn

DATE 13-12-22
PAGE

Order & degree :

* $\left(\frac{d^2y}{dx^2}\right)^7 + 5\frac{dy}{dx} + 3y = 0$

Order = 2, degree = 7

1. **Order :** The order of a differential eq^n is the order of the highest differential coeff. occuring in it.

2. **Degree :-** Degree of the differential eq^n is the degree of the highest differential coeff. when the eq^n has been ~~made~~ made free from fractions & radicals as per as differential coeff. are concerned.

eg: $\frac{\left[1+\frac{dy}{dx}\right]^{1/2}}{\frac{d^2y}{dx^2}} = 2 \Rightarrow \left[1+\frac{dy}{dx}\right]^{5/2} = 2\cdot\frac{d^2y}{dx^2}$

$\Rightarrow \left[1+\frac{dy}{dx}\right]^5 = 2^2\cdot\left(\frac{d^2y}{dx^2}\right)^2$

Order = 2, degree = 2.

* **Ordinary differential eq^n :→ (O.D.E) :**
One independent variable, one dependent variable or more.

Differential eq^n

- Ordinary D.E.
  one dependent
  or
  One Independent
  more
  $\frac{dy}{dx}$
- Partial D.E.
  one or more dependent <
  " " Independent <
  $\frac{\partial y}{\partial x}$, $\frac{\partial y}{\partial t}$

The Ascent Circle

Physics

DATE
PAGE

order = no. of constants

Exact differential equations :→

$$M\,dx + N\,dy = 0$$

functions of x & y.

partial diff eq^n ← $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ → For exact diff. eq^n.

eg:- $y.dx + x.dy = 0$

$d(x.y) = y.dx + x.dy$

* $\int_{y\text{-constant}} M.dx + \int$ terms in N not containing $x\ dy = c$

Ques:- 1 solve $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$

Compair with $Mdx + Ndy = 0$

$M = 3x^2 + 4xy$, $N = 2x^2 + 2y$

$\frac{\partial M}{\partial y} = 0 + 4x$, $\frac{\partial N}{\partial x} = 4x + 0$

so $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x}$ so the given eq^n is exact eq^n.

$$\int_{y\text{-const}} (3x^2 + 4xy)dx + \int 2y.dy = c$$

$$\frac{3x^3}{3} + \frac{4yx^2}{2} + \frac{2y^2}{2} = c$$

$$\boxed{x^3 + 2yx^2 + y^2 = c}$$

Ques.2:- solve $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right).dy = 0$

$Mdx + N.dy = 0$

$M = 1 + e^{x/y}$, $N = e^{x/y}\left(1 - \frac{x}{y}\right)$

$\left\{ \frac{de^{2/x}}{dx} = e^{2/x} \cdot \frac{d}{dx}(2/x) = e^{2/x} \cdot 2(-1).x^{-2} = \frac{-2e^{2/x}}{x^2} \right\}$

$$\frac{\partial M}{\partial y} = 0 + -x\frac{e^{x/y}}{y^2} \quad ; \quad \frac{\partial N}{\partial x} = e^{x/y}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right).e^{x/y}.\frac{1}{y}$$

$$= \frac{-e^{x/y}}{y} + \frac{e^{x/y}}{y} - \frac{xe^{x/y}}{y^2}$$

$$\text{i.e. } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{-x.e^{x/y}}{y}$$

$$\int (1+e^{x/y})\,dx + \int 0\cdot dy = c \qquad \left[\because \int e^{ax}.dx = \frac{e^{ax}}{a}\right]$$

$$\Rightarrow x + \frac{e^{x/y}}{1/y} = c$$

$$\Rightarrow \boxed{x + y.e^{x/y} = c}$$

Date:- 14-12-22

* Integrating factor:- These are used to make the eqⁿ exact eqⁿ. by multiplying with not exact eqⁿ.

Q.1 Solve the D.eqⁿ $\underbrace{(x^2y - 2xy^2)}_{M}dx - \underbrace{(x^3 - 3x^2y)}_{N}dy = 0$

$$\frac{\partial M}{\partial y} = x^2 - 4xy \quad , \quad \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Not exact eqⁿ} \Rightarrow$$

I.F. $= \frac{1}{(x^2y - 2xy^2)x + \cdots}$

* Rules to find integrated factor:

Rule 1 & 2:- • If M & N both are homogeneous funct of x & y with same degree, then

$$\text{I.F} = \frac{1}{Mx + Ny} \quad ; \quad Mx + Ny \neq 0$$

{ f(tx, ty) = tⁿ f(x, y) }

→ Now we make given eqⁿ exact.

$$\text{I.F.} = \frac{1}{(x^2y - 2xy^2)x + \{-(x^3 - 3x^2y).y\}}$$

$$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

New eqn $= \frac{1}{x^2y^2}[(x^2y - 2xy^2)dx + (x^3 - 3x^2y)dy] = 0$

$= \left(\frac{1}{y} - \frac{2}{x}\right)dx + \left[\frac{-x}{y^2} + \frac{3}{y}\right]dy = 0$

$M'dx + N'.dy$

$M' = \left(\frac{1}{y} - \frac{2}{x}\right)$, $N' = \left[\frac{-x}{y^2} + \frac{3}{y}\right]$

Now we solve it by some method.

$\frac{\partial M'}{\partial y} = \frac{-1}{y^2}$; $\frac{\partial N'}{\partial x} = \frac{-1}{y^2} + 0$

$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow$ eqn is exact D.E.

$\int M'dx + \int$ (terms in N' not cont. x) $.dy = C$

$\int\left(\frac{1}{y} - \frac{2}{x}\right).dx + \int \frac{3}{y}.dy = C$

$\frac{x}{y} - 2\log x + 3\log y = C$

$= \frac{x}{y} - \log x^2 + \log y^3 = C$

$= \boxed{\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = C}$ → Sol^n of exact D. eqn.

Ques: $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

$M = xy^2 + 2x^2y^3$, $N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$, $\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

This is not homogeneous eqn because degrees are not eqⁿ equal.

DATE
PAGE

Rule 2 :→ If $Mdx + Ndy = 0$ is of the form $\boxed{yf_1(xy)dx + xf_2(xy)dy = 0}$

$$\boxed{I.F. = \frac{1}{Mx - Ny} \; ; \quad Mx - Ny \neq 0}$$

$$y[f_1(xy) + 2f_1(xy)^2]dx + x[f_1(xy) - f_1(xy)^2]dy = 0$$

$$I.F. = \frac{1}{Mx - Ny} = \frac{1}{(x^2y^2 + 2x^2y^3)x - (x^2y - x^3y^2)y}$$

$$= \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \frac{1}{3x^3y^3}$$

Now we multiply I.F. with given eqⁿ

$$\frac{1}{3x^3y^3}\left[(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy\right] = 0$$

$$= \left[\frac{1}{3x^2y} + \frac{2}{3x}\right]dx + \left[\frac{1}{3xy^2} - \frac{1}{3y}\right]dy = 0$$

Now solⁿ is

$$M' = \frac{1}{3x^2y} + \frac{2}{3x} \quad \& \quad N' = \frac{1}{3xy^2} - \frac{1}{3y}$$

$$\int_{y\,\text{const.}}\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = C$$

$$= \frac{-1}{3yx} + \frac{2}{3}\log x - \frac{1}{3}\log y = C$$

$$= \frac{-1}{3xy} + \log x^{2/3} - \log y^{1/3} = \frac{-1}{3xy} + \log\left(\frac{x}{y}\right)^{2/3} = C$$

DATE
PAGE

Ques.3: $y \cdot dx - (x + 2y^3) \cdot dy = 0$ — (1)

Rule 3: → If $M dx + N \cdot dy = 0$ is of the form

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$$ or a constant then I.F. is $e^{\int f(y) \cdot dy}$.

Ans so $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ → eqn is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-1-1}{y} = \frac{-2}{y}$$, funct. of y only

I.F. $e^{\int -\frac{2}{y} \cdot dx} = e^{-2 \log y} = e^{\log y^{-2}} = \left(\frac{1}{y}\right)^2$

multiply with eqn (1)

$\frac{1}{y^2} [y \cdot dx - (x + 2y^3) \cdot dy] = 0$

$\frac{1}{y} \cdot dx - \left[\frac{x}{y^2} + 2y\right] \cdot dy = 0$

Now soln

soln: $\int \frac{1}{y} \cdot dx - \int 2y \cdot dy = C$

$= \log \frac{x}{y} - y^2 = C$

$= \log \frac{x}{y} - y^2 = C \Rightarrow \boxed{\frac{x}{y} - y^2 = C}$ Ans

Q.4: $\cos x \cos y \cdot dx - 2 \sin x \sin y \cdot dy = 0$ — (1)

$\frac{\partial M}{\partial y} = -\cos x \sin y$, $\frac{\partial N}{\partial x} = -2 \cos x \sin y$.

So given eq^n is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-2\sin y\cos x + \cos x\sin y}{\cos x\cos y}$$

$$= \frac{-\cos x\sin y}{\cos x\cos y} = -\tan y$$

$$I.F. = e^{\int f(y)} = e^{\int -\tan y\,dy} = e$$

$$= e^{-(-\log|\cos y|)} = \cos y.$$

$\because \int \tan x = \log|\sec x| = -\log|\cos x|$

$\therefore$ function of y

Multiply with eq^n (1)

$$\underbrace{\cos x\cos^2 y\,dx}_{M'} + \underbrace{(-2\sin x\sin y\cdot\cos y)}_{N'}\,dy = 0$$

Now we integrate for sol^n

$$\int \cos x\cdot\cos^2 y\,dx - \int 2\sin y\cos y\,dy = C$$

$$= \int \sin x\cdot\cos^2 y - \int \sin 2y\,dy = C$$

$$= \sin x\cdot\cos^2 y - \left(\frac{-\cos 2y}{2}\right) = C$$

$$= \sin x\cdot\cos^2 y + \frac{\cos 2y}{2} = C$$ Ans

Rule 4:→ $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ or constant term

$$I.F. = e^{\int f(x)\,dx}$$

IInd Method $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-\cos x\sin y + 2\sin y\cos x}{-2\sin x\sin y}$

$$= \frac{\sin y\cos x}{-2\sin x\sin y} = \frac{-1}{2}\cot x.$$

↳

(∵ $\int \cot x = \log(\sin x)$

DATE
PAGE

I.F. $= e^{\int f(x)\,dx} = e^{\int -\frac{1}{2}\cot x} = e^{-\frac{1}{2}\log(\sin x)} = e^{\log(\sin x)^{-1/2}}$

$= \boxed{\frac{1}{\sqrt{\sin x}}}$ multi with eq^n ①

I.F $(M\,dx + N\,dy) \Rightarrow M'\,dx + N'\,dy$

$$\frac{1}{\sqrt{\sin x}}\left[\cos x \cos y\,.\,dx + (-2\sin x \sin y)\,.\,dy\right] = 0$$

Now we integrate for sol^n

sol^n = $\int_{y\,\text{constant}} M\,dx + \int_{x} \text{terms in } N \text{ without} = C$

$= \int \frac{\cos x \cos y}{\sqrt{\sin x}} + \int 0\,dy = 0$ ——— ②

Let $\sin x = t$

$\cos x\,dx = dt$ put in ②

$= \int \frac{\cos y}{\sqrt{t}}\,dt = \cos y \cdot \frac{t^{1/2+1}}{1/2+1} = C$

$= \boxed{\cos y \cdot \frac{2t^{3/2}}{3} = C}$

Ques 5:- $x^2 y\,dx - (x^3 + y^3)\,dy = 0$ → homogeneous

$\frac{\partial M}{\partial y} = x^2$; $\frac{\partial N}{\partial x} = -3x^2 - y^3$

Not exact diff. eq^n

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$

NP

DATE
PAGE

$= \frac{x^2 \cancel{4} 3x^2 + y^3}{+x^3 - y^3} = \frac{4x^2 + y^3}{-x^3 - y^3}$

$= \frac{x^2 y \cancel{+ 3x^2} + y^3}{\cancel{-3x^3} - y^3}$ =

$\therefore$ I.F. $= \frac{1}{Mx + Ny} = \frac{1}{(x^2 y_0)x + (-x^3 - y^3)y}$

$= \frac{1}{x^3 \cdot y - y x^3 - y^4}$ Now multiply it with eq^n (1)

$\frac{1}{\cancel{x^3 y - y x^3} - y^4} [x^2 \cdot y \cdot dx - (x^3 + y^3) \cdot dy] = 0$

$M dx + N dy = -\frac{x^2}{y^3} \cdot dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right) \cdot dy = 0$

sol^n $= \int \frac{-x^2}{y^3} \cdot dx + \int \frac{1}{y} \cdot dy = c$

$= \boxed{\frac{-x^3}{3y^3} + \log y = c}$ Ans

Date: 15-12-22

Rule 5: If the eq^n is of type $x^a y^b [my \cdot dx + nx dy] + x^{a'} y^{b'} [m' y dx + n' x dy] = 0$ then I.F. is $x^h y^k$

where $\frac{a+h+1}{m} = \frac{b+k+1}{n}$

$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$

Ques: $(2x^2 y^2 + y) dx - (x^3 y - 3x) dy = 0$

$\frac{\partial M}{\partial y} = 4x^2 y + 1$, $\frac{\partial N}{\partial x} = -[3x^2 y - 3]$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so eq^n is not exact.

Using rule no. 5:

$x^0y^0[2x^2$

$2x^2y^2.dx + y.dx - x^2y.dy + 3x.dy = 0$

$x^0y[2y.dx - x.dy] + x^0y^0[y.dx + 3x\,dy] = 0$

Now using $x^ay^b[my.dx + nx.dy] + x^{a'}y^{b'}[m'y.dx + n'x.dy]$

Comparing

$a=2,\ b=1,\ m=2,\ n=-1$

$a'=0,\ b'=0,\ m'=1,\ n'=3$

using $\frac{a+h+1}{m} = \frac{b+k+1}{n}$; $\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$

$\therefore \frac{2+h+1}{2} = \frac{1+k+1}{-1}$

$\frac{3+h}{2} = \frac{2+k}{-1} \Rightarrow -3-h = 4+2k$

$\Rightarrow \boxed{2k+h = -7}$ — (1)

$\therefore \frac{0+h+1}{1} = \frac{0+k+1}{3}$

$\Rightarrow 3(h+1) = k+1$

$\boxed{3h-k = -2}$ — (2)

multiply eq^n (1) with 3 & sub.

$3h + 6k = -21$

$3h - k = -2$

$- \quad + \quad +$

$7k = -19$

$\boxed{k = \frac{-19}{7}}$ put in (2)

$3h + \frac{19}{7} = -2$

$21h + 19 = -14$

$21h = -33$

$\boxed{h = \frac{-33}{21+7}} = \boxed{\frac{-11}{7}}$

I.F. $= x^{-11/7} \cdot y^{-19/7}$

Now mul. it with given eq^n

$$\frac{1}{x^{11/7} \cdot y^{19/7}} \left[(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy\right] = 0$$

$$= \left(2x^{2-11/7} \cdot y^{2-19/7} + x^{-11/7} \cdot y^{1-19/7}\right)dx - \left[x^{3-11/7} \cdot y^{1-19/7} - 3x^{1-11/7} \cdot y^{-19/7}\right]dy$$

$$= 0$$

Now its exact eq^n

sol^n : $\int \left(2x^{3/7} \cdot y^{-5/7} + x^{-11/7} \cdot y^{-12/7}\right)dx + \int 0\,dy = c$

$$= 2y^{-5/7} \cdot \frac{x^{3/7+1}}{\frac{3}{7}+1} + y^{-12/7} \cdot \frac{x^{\frac{-11}{7}+1}}{\frac{-11}{7}+1} = c$$

$$= 2y^{-5/7} \cdot \frac{7x^{10/7}}{10} + y^{-12/7} \cdot \frac{7x^{-4/7}}{-4} = c$$

$$= \boxed{\frac{21}{10} \cdot y^{-5/7} \cdot x^{10/7} - \frac{7}{4} y^{-12/7} \cdot x^{-4/7} = C}$$

DATE
PAGE

Leibnitz Linear eqⁿ :-

Linear $= \left(\frac{d^2y}{dx}\right)^1 + \left(\frac{dy}{dx}\right) + 1 = 0$

$$\frac{dy}{dx} + Py = Q$$

(P: funct. of independant variable 'x')

$$I.F. = e^{\int P.dx}$$

$$\text{sol}^n: \quad y.(I.F.) = \int Q.(I.F)\,dx + C$$

Ques:- $\frac{dy}{dx} + \frac{y}{x} = e^x \quad (x>0)$

$P = \frac{1}{x}$, $I.F. = e^{\int \frac{1}{x}.dx} = e^{\log x} = x$

sol^n: $y.x = \int e^x.x\,dx + C$ (e^x: v, x: u)

IELTS

$= x.\int e^x.dx - \int\left[\frac{dx}{dx}.\int e^x.dx\right].dx$ $\quad \left[\int u.v = u.\int v.dx - \int\left[\frac{du}{dx}.(\int v.dx)\right]\right]$

$= x.e^x - \int 1.e^x.dx$

$= xe^x - e^x + C = \boxed{e^x(x-1)+C}$ Ans

Ques:- $\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{1}{x}$

$P = \frac{1}{x\log x}$, $Q = \frac{1}{x}$

$e^{\int P.dx} = e^{\int \frac{1}{x\log x}dx}$

Let $\log x = t$

$\frac{1}{x}dx = dt$ put in (1)

$e^{\int \frac{1}{t}.dt} = e^{\log t} = t = \log x$

$I.F. = \log x$

$y \cdot \log x = \int \frac{1}{x} \cdot \log x \cdot dx + C$

Let $\log x = t$

$\frac{1}{x} \cdot dx = dt$

$\Rightarrow y \cdot \log x = \int t \cdot dx + C$

$y \cdot \log x = \frac{t^2}{2} + C$

$y \cdot \log x = \frac{(\log x)^2}{2} + C$

Q.No:- (1) $\frac{dy}{dx} + 2y \cdot \tan x = \sin x$

(2) $(x+y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{(x+y)}$

$\frac{dx}{dy} = x+y \Rightarrow \boxed{\frac{dx}{dy} - x = y}$

$\left\{ \frac{dx}{dy} - Px = Q \right\}$

$\therefore I.F. = e^{\int P \cdot dy}$

$Sol = x \cdot I.F. = \int Q \cdot I.F. dy + C$

$P = -1, \quad Q = y$

$e^{\int P \cdot dy} = e^{\int -1 dy} = e^{-y}$

sol^n $= x \cdot e^{-y} = \int \underset{u}{y} \cdot \underset{v}{e^{-y}}$

$= x \cdot e^{-y} = y \cdot \int e^{-y} dy - \int \left[\frac{dy}{dy} \left(\int e^{-y} dy \right) \right] dy$

$x \cdot e^{-y} = -y \cdot e^{-y} - \int \frac{e^{-y}}{-1} \cdot dy$

$= -y \cdot e^{-y} - \frac{e^{-y}}{(-1)(-1)} + C$

$x \cdot e^{-y} = -e^{-y} [y+1] + C$

Ans 2:- $\frac{dy}{dx} + 2y \cdot \tan x = \sin x$

$P = 2\tan x, \quad Q = \sin x$

$I.F. = e^{\int P\,dx} = e^{\int 2\tan x\,dx} = e^{2\log\sec x} = e^{\log\sec^2 x}$

$I.F. = \sec^2 x$

sol^n.

$$y\cdot\sec^2 x = \int \sin x\sec^2 x\,dx + C = \int \frac{\sin x}{\cos x}\cdot\sec x\,dx$$

$$\Rightarrow y\sec^2 x = \int \sec x\tan x\,dx = \int d(\sec x) + C$$

$$y\sec^2 x = \sec x + C$$ Ans

$\left[\because \frac{d}{dx}(\sec x) = \sec x\tan x\right]$

Date:- 20-12-22

V. Imp. (10 Marks Ques)

* Linear differential eq^n :-

(1) Second and higher order linear Diff. eq^n :-

* Second " " " " " " with constant coeffs

Let $D = \frac{d}{dx}$ $\Rightarrow Dy = \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = D^2y$; $\frac{d^3y}{dx^3} = D^3y$

eg:- $\frac{d^2y}{dx^3} - 5\frac{dy}{dx} + 6y = 0$

$D^2y - 5Dy + 6y = 0$

$[D^2 - 5D + 6]\cdot y = 0$

$f(D)\cdot y = 0$ → symbolic form of D.Eq^n.

* Auxiliary equation:

$f(D) = 0$

$D^2 - 5D + 6 = 0$

$D = 2, 3$ → Real & distinct

Gen. sol^n is $y^c = C_1e^{2x} + C_2e^{3x}$

** If on R.H.S is zero then we use only complementry sol^n otherwise P.I ≠ 0

Complete sol^n = C.S + P.I

↓ Particular integral

{ no. of constants = order of eqⁿ }

DATE
PAGE

Rules :-

Rule 1 :- If the roots of auxiliary eqⁿ are real and distinct m_1, m_2 so

C.S. is $C_1 e^{m_1 x} + C_2 e^{m_2 x}$ C_1 & C_2 = arbitrary const.

Q:- $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 4y = 0$

$D^2y - 3Dy + 4y = 0$

$(D^2 - 3D + 4)y = 0$

$f(D).y = 0$

A.eqⁿ is $D^2 - 3D + 4 = 0$

$D = -1, 4$

C.S. is $y^c = C_1 e^{-1x} + C_2 e^{4x}$

Rule 2 :- If roots are real and repeated m_1, m_1

C.S. is $(C_1 + C_2 x) e^{m_1 x}$

If m_1, m_2, m_3

C.S is $(C_1 + C_2 x + C_3 x^2) e^{m_1 x}$ and so on

Q:- roots 2, 4, 5, 5

$C_1 e^{2x} + C_2 e^{4x} + [C_3 + C_4 x] e^{5x}$ Ans

Q:- $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

$D = \frac{d}{dt}$

$D^2x + 6Dx + 9x = 0$

$(D^2 + 6D + 9)x = 0$

A.eqⁿ = $D^2 + 6D + 9 = 0$

Roots = $-3, -3$

C.S. is $x^c = [C_1 + C_2 t]e^{-3t}$

Rule 3: If roots are complex no. $\boxed{\alpha \pm i\beta}$

$\boxed{y^c = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]}$ $\because C_1 e^{\alpha+i\beta} + C_2 e^{\alpha-i\beta}$

Ques: $\frac{d^3y}{dx^3} + y = 0$

$(D^3+1)y = 0$

A.eq^n : $D^3+1=0$ $\quad \{A^3+B^3 = (A+B)(A^2+B^2-AB)\}$

$(D+1)(D^2-D+1)=0$

$D=-1$, $\quad \frac{-(-1) \pm \sqrt{(1)^2 - 4.1.1}}{2(1)}$

$D=-1$, $\boxed{\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}$

C.S. is $C_1 e^{-1x} + e^{\frac{1}{2}x}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right]$

date :- 21-12-22

$\star$ $\frac{d^n y}{dx^n} + a_0\frac{d^{(n-1)}y}{dx^{(n-1)}} + a_1\frac{d^{(n-2)}y}{dx^{(n-2)}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = X$ ↓ $f(x)$ or constant

If $X=0$, Homogeneous L.D.E $\Rightarrow$ sol^n = C.S.

If $X \neq 0$ (Non-zero) Non " " $\Rightarrow$ sol^n = C.S + P.I.

Ques: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = e^{5x}$ (Non-homogeneous)

Comp sol^n : C.S + P.I.

symbolic form $D^2y - 3Dy - 4y = e^{5x}$

$(D^2-3D-4)y = e^{5x} \Rightarrow f(D).y = e^{5x}$

A.eq^n : $D^2-3D-4=0$

$D = -1, 4$

$\boxed{C.S.\ y_c = C_1 e^{-1x} + C_2 e^{4x}}$

$\star$ Inverse operator : $\frac{1}{f(D)}$, $X = e^{ax}$

$\star$ Particular Integral = $\boxed{\frac{1}{f(D)}.X}$

Particular Integral :

Rule 1 :-

If $X = e^{ax}$

$$P.I. = \frac{1}{f(D)} \cdot e^{ax}$$

To evaluate P.I. Put $D = a$, $f(a) \neq 0$.

If $f(a) = 0$ (case of failure) then $\boxed{x \cdot \frac{1}{\frac{d(f(D))}{dD}} \cdot e^{ax}}$

→ $P.I. = \frac{1}{D^2 - 3D - 4} \cdot e^{5x}$, Put $D = 5$

$= \frac{1}{5^2 - 3.5 - 4} \cdot e^{5x} = \frac{1}{6} \cdot e^{5x}$

Complete soln = C.S + P.I

$= \boxed{C_1 e^{-x} + C_2 e^{4x} + \frac{e^{5x}}{6}}$ dy

Q2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = e^{4x}$

C.S. $= C_1 e^{-1x} + C_2 e^{4x}$

$P.I. = \frac{1}{D^2 - 3D - 4} \cdot e^{4x}$, Put $D = a = 4$

$= \frac{1}{16 - 12 - 4} e^{4x} = \frac{1}{0} \cdot e^{4x}$ (case of failure)

so

$x \cdot \frac{1}{\frac{d}{dD} f(D)} \cdot e^{ax} = x \cdot \frac{1}{2D - 3} \cdot e^{4x}$, Put $D = 4$

$= \frac{x}{8 - 3} \cdot e^{4x} = \frac{x \cdot e^{4x}}{5}$

Complete soln = $\boxed{C_1 e^{-x} + C_2 e^{4x} + \frac{x \cdot e^{4x}}{5}}$ dy

Rule 2 :-

$\frac{1}{D}f(x) = \int f(x)\,dx$

$D(f(x)) = \frac{d}{dx}[f(x)]$

DATE
PAGE

Rule 2.6 - If $X = \sin ax$ or $\cos ax$ / $\sin(ax+b)$ / $\cos(ax+b)$

$P.I. = \frac{1}{f(D^2)} \cdot X$, Put $D^2 = -a^2$, $f(-a^2) \neq 0$

If $f(-a^2) = 0$ case of failure

$x \cdot \frac{1}{\frac{df(D)}{dD}} \sin ax / \cos ax$ ---

Ques. :→ $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \sin 2x$, $a = 2$

$D^3y + D^2y - D - y = \sin 2x$

$(D^3 + D^2 - D - 1)y = \sin 2x \Rightarrow f(D) = \sin 2x$

A.E. = $D^3 + D^2 - D - 1 = 0$

roots $-1, -1, 1$

$C.S. = C_1e^x + [C_2 + C_3x]e^{-1x}$

$P.I. = \frac{1}{f(D)} \cdot \sin 2x = \frac{1}{D^3 + D^2 - D - 1} \cdot \sin 2x$

$= \frac{1}{D^2 \cdot D + D^2 - D - 1} \cdot \sin 2x$, $a = 2$, $D^2 = -a^2 = -4$

$= \frac{1}{-4D - 4 - D - 1} \cdot \sin 2x$

$= \frac{1}{-5D - 5} \sin 2x = \frac{-1}{5(D+1)} \cdot \sin 2x$ → use Rule to convert

$= \frac{-1}{5} \times \frac{(D-1)}{(D+1)(D-1)}$, $D^2 = -4$

$= \frac{-1}{5} \frac{(D-1)}{(D^2-1)} = \frac{-(D-1) \cdot \sin 2x}{5(-4-1)}$

$= \frac{1}{25}[D(\sin 2x) - \sin 2x]$ ↳ $D(f(x))$ → diff.

P.I. $= \frac{1}{25}(2\cos 2x - 4\sin 2x)$

Complete sol^n $= C_1 e^x + (C_2 + C_3 x)e^{-x} + \frac{1}{25}(2\cos 2x - 4\sin 2x)$ Ans

Ques: $\frac{d^2y}{dx^2} + y = \cos x$

$D^2y + y = \cos x$

A.E. $\Rightarrow D^2 + 1 = 0$

$(D^2 - i^2) = 0$

$(D+i)(D-i)$

roots, $D = \pm i$ imaginary.

$= 0 \pm i$ $\quad \alpha = 0$ & $\beta = 1$

C.S. $e^{0x}[C_1 \cos 1.x + C_2 \sin 1.x]$.

P.I. $= \frac{1}{D^2+1} \cdot \cos x$, $\quad D^2 = -(1)^2 = -1$

$= \frac{1}{-1+1} \cdot \cos x = \frac{1}{0}\cos x \rightarrow$ case of failer.

P.I. $= x \cdot \frac{1}{2D} \cdot \cos x$

$= \frac{x}{2}\int \cos x \, dx$

P.I. $= \frac{x}{2} \cdot \sin x$

$\frac{1}{D} f(x) = \int f(x) \, dx$

Complete sol^n $= e^{0x}[C_1 \cos x + C_2 \sin 1x] + \frac{x}{2}\sin x$ Ans

* Bernoulli's Eq^n:

$\frac{dy}{dx} + Py = Qy^n$

P: Funct. of independent variable ('x')

DATE
PAGE

Q0 $\frac{1}{y^n}\cdot\frac{dy}{dx} + P\cdot\frac{y}{y^n} = Q$

$\frac{1}{y^n}\cdot\frac{dy}{dx} + P\cdot\frac{1}{y^{n-1}} =$ $\boxed{y^{-n}\cdot\frac{dy}{dx} + P\cdot y^{1-n} = Q}$ — (1)

Now

$y^{1-n} = t$

$(1-n)y^{-n}\cdot\frac{dy}{dx} = \frac{dt}{dx}$

$(1-n)y^{-n}\cdot\frac{dy}{dx}\cdot\frac{dt}{dx}$

$y^{-n}\cdot\frac{dy}{dx} = \frac{1}{(1-n)}\cdot\frac{dt}{dx}$ put in (1)

$\frac{1}{(n-1)}\cdot\frac{dt}{dx} + P\cdot t = Q$

Now

$\boxed{\frac{dt}{dx} + (1-n)Pt = (1-n)Q}$ → Liberitz's eqn

Ques.1 $\frac{dy}{dx} + \frac{1}{x}\cdot y = x^9\cdot y^6$, $P = \frac{1}{x}$, $Q = x^3$

$\frac{1}{y^6}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot\frac{y}{y^6} = x^3$

$\frac{1}{y^6}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y^5} = x^3 = y^{-6}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot y^{-5} = x^3$ — (1)

$y^{-5} = t$

$-5\cdot y^{-6}\cdot\frac{dy}{dx} = \frac{dt}{dx}$

$y^{-6}\cdot\frac{dy}{dx} = -\frac{1}{5}\cdot\frac{dt}{dx}$ put in (1)

$\Rightarrow -\frac{1}{5}\cdot\frac{dt}{dx} + \frac{1}{x}\cdot t = x^3$

$= \frac{dt}{dx} - \frac{5}{x}t = -5x^2$, $P = -5/x$, $Q = 5x^2$

I.F. $e^{-\int 5/x\,dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$

$t \cdot \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5}\,dx + C$

$\frac{y^{-5}}{x^5} = -5 \cdot \frac{x^{-3+1}}{-3+1} + C$

$\boxed{\frac{1}{y^5 x^5} = \frac{-5 \cdot x^{-2}}{2} + C}$ Ans

Ques 2: $\frac{dy}{dx} + y = y^2$

$\frac{1}{y^2} \cdot \frac{dy}{dx} + \frac{y}{y^2} = 1$

$y^{-2} \cdot \frac{dy}{dx} + y^{-1} = 1$

Let $y^{-1} = t$

$-1 \cdot y^{-2} \frac{dy}{dx} = \frac{dt}{dx}$ put in (1)

$-\frac{dt}{dx} + t = 1$

$\boxed{\frac{dt}{dx} - t = -1}$

$P = -1$, $Q = -1$

I.F. $e^{\int P\,dx} = e^{\int -1\,dx} = e^{-x}$

Sol: $t \cdot (e^{-x}) = \int -1 \cdot (e^{-x})\,dx + C$

$= -y^{-1} e^{-x} = \int -e^{-x}\,dx + C = \frac{-e^{-x}}{-1} + C$

$\boxed{-y^{-1} \cdot e^{-x} = +e^{-x} + C}$ Ans

DATE
PAGE

Ques: $\frac{dy}{dx} - \frac{\tan y}{(1+x)} = (1+x)\, e^x \sec y$ (form of $\frac{dy}{dx} + Py = Q$)

$\frac{1}{\sec y}\cdot\frac{dy}{dx} - \frac{\tan y}{\sec y}\cdot\frac{1}{(1+x)} = (1+x)e^x$ ($\frac{\tan y}{\sec y} = \frac{\sin y}{\cos y}\times\frac{\cos y}{1} = \sin y$)

$\cos y \frac{dy}{dx} - \sin y \cdot \frac{1}{(1+x)} = (1+x)e^x$ — (1)

$\sin y = t$

$\cos y \cdot \frac{dy}{dx} = \frac{dt}{dx}$ put in (1)

$\frac{dt}{dx} - \frac{1}{(1+x)}\,t = (1+x)e^x$

$I.F. = e^{\int P dx} = e^{-\int \frac{1}{1+x}\,dx} = e^{-\log(1+x)} = e^{\log(1+x)^{-1}}$

$= (1+x)^{-1} = \frac{1}{(1+x)}$

Sol^n: $t\cdot\frac{1}{(1+x)} = \int \frac{1}{(1+x)}\cdot(1+x)e^x + c$

$\Rightarrow \boxed{\frac{\sin y}{(1+x)} = e^x + c}$

Ques: $\frac{dy}{dx} = 2y\tan x + y^2\tan^2 x$ { take max. power on RHS }

$\frac{dy}{dx} - 2y\tan x = y^2\tan^2 x$

$\frac{1}{y^2}\cdot\frac{dy}{dx} - \frac{2y}{y^2}\cdot\tan x = \tan^2 x$

$y^{-2}\cdot\frac{dy}{dx} - 2y^{-1}\tan x = \tan^2 x$ — (1)

Let $y^{-1} = t$

$-1\cdot y^{-2}\cdot\frac{dy}{dx} = \frac{dt}{dx}$ put in (1)

$\frac{-dt}{dx} = 2t \cdot \tan x = \tan^2 x \Rightarrow \frac{dt}{dx} + 2t \tan x = -\tan^2 x$

$I.F. = e^{\int P.dx} = e^{\int 2\tan x} = e^{2.\log \sec x} = e^{\log \sec^2 x}$

$= \sec^2 x$

$= \frac{d}{\sec^2 x}$

Sol^n = $\frac{d}{\sec^2 x} \cdot \frac{\sec^2 x}{\sec^2 x} = \int \frac{\sec^2 x \cdot \tan^2 x}{\sec^2 x} dx + C = \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos^2 x}$

$\frac{\sec^2 x}{y \sec^2 x} = \int \tan^2 x \, dt$ Let $\tan x = t$

$\sec^2 x \cdot dx = \frac{dt}{dx}$

$= \frac{\sec^2 x}{y \sec^2 x} = \int t^2 dt$

$= \frac{\sec^2 x}{y \sec^2 x} = \frac{-t^3}{3} + C =$

$\frac{\sec^2 x}{y} = \frac{-\tan^3 x}{3} + C$

$\boxed{\frac{\sec^2 x}{y} = \frac{-\tan^3 x + C}{3}}$

Date :-

* Rule 3 :→ If $X = x^m$ we will expand $f(D)$ after converting to funct $(1+P)^n$ upto D^m term.

$(1+P)^n = 1 + nP + \frac{n(n-1)}{2!} P^2 + \frac{n(n-1)(n-2)}{3!} P^3 + \cdots$

Ques:- $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$, $a = x^2$

$(D^3 + 3D^2 + 2)y = x^2$

$P.I. = \frac{1}{D^3 + 3D^2 + 2D} \cdot x^2$

{ minimum power of D coeff = 2 we take it common }

$$= \frac{1}{2D}\left[\frac{1}{1+\left(\frac{D^3+3D^2}{2D}\right)}\right]x^2$$

$$= \frac{1}{2D}\left[\left(1+\left(\frac{D^2}{2}+\frac{3D}{2}\right)\right)^{-1}\right].x^2$$

$$\Rightarrow \frac{1}{2D}\left[1+(-1)\left(\frac{D^2}{2}+\frac{3D}{2}\right)+\frac{(-1)(-1-1)}{2!}\left(\frac{D^2}{2}+\frac{3D}{2}\right)^2+\cdots\right]x^2$$

$$= \frac{1}{2D}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{D^2}{4}+\frac{9D^2}{2}+\frac{3D^3}{2}+\cdots\right]x^2$$

$$= \frac{1}{2D}\left[x^2-\frac{D^2x^2}{2}-\frac{3Dx^2}{2}+\frac{D^2x^2}{4}+\frac{9D^2x^2}{2}+\frac{3D^3x^2}{2}+\cdots\right]$$

$$= \frac{1}{2D}\left[x^2-\frac{2}{2}-\frac{3}{2}.2x+0+\frac{9}{4}x2+0\right]$$

$$= \frac{1}{2D}\left[x^2-1-3x+\frac{9}{2}\right]$$

$$= \frac{1}{2D}\left[x^2-3x+\frac{7}{2}\right]$$

$$= \frac{1}{2}\int\left(x^2-3x+\frac{7}{2}\right)dx = \boxed{\frac{1}{2}\left[\frac{x^3}{3}-\frac{3x^2}{2}+\frac{7}{2}x\right]}$$ dy

$\because D=\frac{d}{dx}$

$D^2=\frac{d^2}{dx^2}$

$\Rightarrow D(x)=2x$

$D^2(x^2)=2$

$D^3(x^2)=0$

$\because \frac{1}{D}=\int dx$

Rule 4:- If $\boxed{X=e^{ax}.V}$

So $\frac{1}{f(D)}.e^{ax}.V = e^{ax}.\frac{1}{f(D+a)}.V$

means we do not do anything with e^{ax}.

Ques:- $\frac{d^2y}{dx^2}-\frac{2.dy}{dx}+4y=e^x\cos x$

$(D^2-2D+4)y=e^x\cos x$; $a=1$

DATE
PAGE

$$P.I = \frac{1}{D^2-2D+4} \cdot e^x \cos x$$

$$= e^x \cdot \frac{1}{(D+1)^2-2(D+1)+4} \cdot \cos x$$

$$= e^x \cdot \frac{1}{D^2+1+2D-2D-2+4} \cdot \cos x = e^x \cdot \frac{1}{D^2+3} \cdot \cos x$$

Now $\boxed{D^2 = -(a)^2 = -(1)^2 = -1}$

$$\Rightarrow e^x \cdot \frac{1}{-1+3} \cdot \cos x$$

Rule 5: If $X = xV$, V is funct. of indep. variable.

$$P.I. = \frac{1}{f(D)} \cdot xV = x \cdot \frac{1}{f(D)} \cdot V + \frac{d}{dD}\left[\frac{1}{f(D)}\right] V.$$

Ques.1:- $\frac{d^2y}{dx^2} - y = x \sin x$

$$(D^2-1)y = x \sin x$$

$$P.I. = \frac{1}{D^2-1} \cdot x \sin x$$

$$= x \cdot \frac{1}{(D^2-1)} \cdot \sin x + x \cdot \frac{d}{dD}\left[\frac{1}{(D^2-1)}\right] \cdot \sin x$$

Now $D^2 = -(a)^2 = -1$

$$= x \cdot \frac{1}{D^2-1} \cdot \sin x + \frac{-2D}{(D^2-1)^2} \cdot \sin x$$

$$= \frac{x}{-1-1} \cdot \sin x - \frac{2D}{(-1-1)^2} \cdot \sin x$$

$$= -\frac{x}{2} \sin x - \frac{2}{4} (D \sin x) = \boxed{\frac{-x}{2} \sin x - \frac{1}{2} \cos x}$$ Ans

(differentiate)

DATE
PAGE

for complimentary sol^n

Rule 4: If we have repeated complex roots, $\alpha \pm i\beta$ $\alpha \pm i\beta$

C.S. $= e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x)$

Ques: $\frac{d^4y}{dx^4} + 8\cdot\frac{d^2y}{dx^2} + 16y = 0$

Roots $= \pm 2i, \pm 2i \Rightarrow 0 \pm 2i\,;\ \pm\ 0 \pm 2i$

C.S. $y_c = e^{0x}[(C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x]$ dy



THE ASCENT CIRCLE

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