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An analytic model for left invertible weighted translation semigroups

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Abstract. M. Embry and A. Lambert initiated the study of a semigroup of operators $\{S_t\}$ indexed by a non-negative real number t and termed it as weighted translation semigroup. The operators S_t are defined on $L^2(\mathbb{R}_+)$ by using a weight function. The operator S_t can be thought of as a continuous analogue of a weighted shift operator. In this paper, we show that every left invertible operator S_t can be modeled as a multiplication by z on a reproducing kernel Hilbert space $\mathcal{H}$ of vector-valued analytic functions on a certain disc centered at the origin and the reproducing kernel associated with $\mathcal{H}$ is a diagonal operator. As it turns out that every hyperexpansive weighted translation semigroup is left invertible, the model applies to these semigroups. We also describe the spectral picture for the left invertible weighted translation semigroup. In the process, we point out the similarities and differences between a weighted shift operator and an operator S_t.

1. Introduction

With a view to develop a continuous analogue of weighted shift operators, M. Embry and A. Lambert initiated the study of operators that can be defined by using a *weight function* (rather than a weight sequence)(refer to [8]). In fact, for a positive measurable function φ, they constructed a semigroup $\{S_t\}$ of bounded linear operators on $L^2(\mathbb{R}_+)$, parametrized by a non-negative real number t and termed it as *weighted translation semigroup*. In [13], we continued the work carried out in [8, 9] and obtained characterizations of hyperexpansive semigroups in terms of their

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symbols. In this paper, we further explore the semigroup $\{S_t\}$ and record some important properties. In our approach the emphasis is on the study of an operator S_t for a fixed parameter t and not on the semigroup structure of $\{S_t\}$. In the process we highlight the similarities and differences between a weighted shift operator and an operator S_t. In Section 2, we set the notation and record some definitions required in the sequel. As described in [13] the term *hyperexpansive weighted translation semigroups* include completely hyperexpansive, 2-hyperexpansive, 2-isometric and alternatingly hyperexpansive weighted translation semigroups. In Section 3, following the process given by S. Shimorin in [18], we construct an analytic model for a left invertible semigroup $\{S_t\}$. We say that the *semigroup* $\{S_t\}$ *is left invertible* if every operator S_t in that semigroup is left invertible. Recall that a bounded linear operator T on a Hilbert space H is said to be *analytic* if $\cap_{n\geq 0} T^n H = \{0\}$. The multiplication operator $\mathcal{M}_z$ on z-invariant reproducing kernel Hilbert space of analytic functions defined on a disc in $\mathbb{C}$ is an example of an analytic operator. A result of S. Shimorin [18] asserts that any left invertible analytic operator is unitarily equivalent to the multiplication operator $\mathcal{M}_z$ on a reproducing kernel Hilbert space $\mathcal{H}$ of vector-valued analytic functions defined on a certain disc. In this paper, we prove that the semigroup $\{S_t\}$ is analytic in the sense that the operator S_t is analytic for each $t > 0$ and obtain the corresponding reproducing kernel Hilbert space, so as to realize each such left invertible S_t as a multiplication by z. We observe that the reproducing kernel associated with $\mathcal{H}$ is diagonal. Also, $\mathcal{H}$ admits an orthonormal basis consisting of polynomials in z. We explore the reproducing kernels associated with $\mathcal{H}$ in some special types of semigruops $\{S_t\}$. In this section, we also prove that the semigroup $\{S_t\}$ has the wandering subspace property. As a consequence, we prove that a left invertible operator S_t, $t > 0$, is an operator-valued weighted shift. We recall that every hyperexpansive weighted translation semigroup is left invertible ([13, Remark 4.2]). Thus the analytic model developed in this section applies to hyperexpansive weighted translation semigroups allowing us to view these operators as the multiplication by z on a suitable reproducing kernel Hilbert space. In Section 4, we describe the spectral picture of a left invertible weighted translation semigroup $\{S_t\}$. We know that for a weighted shift operator the point spectrum is empty and the spectrum is always a disc. We observe that these properties are also shared by a left invertible operator $S_t, t > 0$. We also obtain a formula for the spectral radius of S_t in terms of the symbol φ. However, we point out that unlike a weighted shift operator, for an operator S_t, $t > 0$, the kernel of ${S_t}^*$ is infinite-dimensional.

2. Preliminaries

Let $\mathbb{R}_+$ be the set of non-negative real numbers and let $L^2(\mathbb{R}_+)$ denote the Hilbert space of complex-valued square integrable Lebesgue measurable functions on $\mathbb{R}_+$. Let $\mathcal{B}(L^2)$ denote the algebra of bounded linear operators on $L^2(\mathbb{R}_+)$.

Definition 2.1. For a measurable, positive function φ defined on $\mathbb{R}_+$ and $t \in \mathbb{R}_+$, define the function $\varphi_t \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\varphi_t(x) = \begin{cases} \sqrt{\frac{\varphi(x)}{\varphi(x-t)}}, & \text{if } \ x \geq t, \\ 0, & \text{if } \ x < t. \end{cases}$$

Suppose that φ_t is essentially bounded for every $t \in \mathbb{R}_+$. For each fixed $t \in \mathbb{R}_+$, we define S_t on $L^2(\mathbb{R}_+)$ by

$$S_t f(x) = \begin{cases} \varphi_t(x) f(x-t), & \text{if } \ x \geq t, \\ 0, & \text{if } \ x < t. \end{cases}$$

Remark 2.2. Substituting φ_t in the above definition, we get

$$S_t f(x) = \begin{cases} \sqrt{\frac{\varphi(x)}{\varphi(x-t)}} f(x-t), & \text{if } \ x \geq t, \\ 0, & \text{if } \ x < t. \end{cases}$$

It is easy to see that for every $t \in \mathbb{R}_+$, S_t is a bounded linear operator on $L^2(\mathbb{R}_+)$ with $\|S_t\| = \|\varphi_t\|_\infty$, where $\|\varphi_t\|_\infty$ stands for the essential supremum of φ_t given by

$$\|\varphi_t\|_\infty = \inf\{M \in \mathbb{R} : \varphi_t(x) \leq M \text{ almost everywhere}\}.$$

The family $\{S_t : t \in \mathbb{R}_+\}$ in $\mathcal{B}(L^2)$ is a semigroup with $S_0 = I$, the identity operator and for all $t, s \ \in \mathbb{R}_+$, $S_t \circ S_s = S_{t+s}$.

We say that φ_t is a *weight function corresponding to the operator* S_t. Further, the semigroup $\{S_t : t \in \mathbb{R}_+\}$ is referred to as the *weighted translation semigroup with symbol* φ. Throughout this article, we assume that the symbol φ is a continuous function on $\mathbb{R}_+$.

In [13, Corollary 3.3], characterizations of some special types of the semigroup $\{S_t\}$ such as subnormal contractions, completely hyperexpansive, 2-hyperexpansive, alternatingly hyperexpansive and m-isometries are obtained in terms of their symbols. The special classes of functions characterizing these classes of operators have been studied extensively in the literature (refer to [4, 16, 20]). We find it convenient to record the definitions of the classes of operators under consideration for ready reference. For a detailed account on these classes of operators, the reader is referred to [1, 3, 6, 10, 11, 19]. Let T be a bounded linear operator on a Hilbert space H and n be a positive integer. Let $B_n(T)$ denote the operator

$$B_n(T) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k. \tag{2.1}$$

An operator T is said to be

(1) *subnormal* if there exist a Hilbert space K containing H and a normal operator $N \in \mathcal{B}(K)$ such that $NH \subseteq H$ and $N|_H = T$;
(2) *completely hyperexpansive* if $B_n(T) \le 0$, for all integers $n \ge 1$;
(3) an *m-hyperexpansion* if $B_n(T) \le 0$, for all integers n, $1 \le n \le m$;
(4) *alternatingly hyperexpansive* if

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T^{*k} T^k \ge 0, \text{ for all integers } n \ge 1;$$

(5) an *m-isometry* if $B_m(T) = 0$;
(6) *hyponormal* if $T^*T - TT^* \ge 0$;
(7) a *contraction (expansion)* if $I - T^*T \ge 0$ $(I - T^*T \le 0)$.

3. Analytic model for the weighted translation semigroup $\{S_t\}$

Recall that a bounded linear operator T on a Hilbert space H is analytic if $\bigcap_{n\ge 0} T^n H = \{0\}$. We say that the semigroup $\{S_t\}$ is analytic, if for every $t > 0$, the operator S_t is analytic. In this section, we prove that the semigroup $\{S_t\}$ is analytic. We further show that every left invertible operator $S_t, t > 0$, can be modeled as a multiplication operator $\mathcal{M}_z$ on a reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ of vector-valued analytic functions on a disc centered at the origin. It turns out that the reproducing kernel associated with $\mathcal{H}$ is a diagonal operator. Also $\mathcal{H}$ admits an orthonormal basis consisting of polynomials in z. We compute the reproducing kernels in some concrete cases.

3.1. Analytic property of $\{S_t\}$

The following result is crucial in the construction of an analytic model for an operator S_t.

Theorem 3.1. *For every $t > 0$, the operator S_t is analytic.*

Proof. We first observe that $S_t^k(L^2(\mathbb{R}_+)) \subseteq \chi_{[kt,\infty)}L^2(\mathbb{R}_+)$ for $k = 0, 1, 2, \ldots, t > 0$, where χ_A stands for the characteristic function of the set A. For $f \in L^2(\mathbb{R}_+)$,

$$(S_t^k f)(x) = \begin{cases} \sqrt{\frac{\varphi(x)}{\varphi(x-kt)}}\, f(x-kt), & \text{if } \ x \geq kt, \\ 0, & \text{if } \ x < kt. \end{cases}$$

Therefore $S_t^k f \in \chi_{[kt,\infty)}L^2(\mathbb{R}_+)$. We now prove that for every $t > 0$ the operator S_t is analytic. In view of the observation above, it is sufficient to prove that $\bigcap_{k=0}^{\infty} \chi_{[kt,\infty)}L^2(\mathbb{R}_+) = \{0\}$. Let $f \in \bigcap_{k=0}^{\infty} \chi_{[kt,\infty)}L^2(\mathbb{R}_+)$. Then $f(x) = 0$ for all $x < kt$, for each $k \geq 0$. This forces that $f(x) = 0$ for all $x \in [0, \infty)$. Thus we have $\bigcap_{k=0}^{\infty} \chi_{[kt,\infty)}L^2(\mathbb{R}_+) = \{0\}$ implying that for every $t > 0$ the operator S_t is analytic. ∎

We now show that the semigroup $\{S_t\}$ possesses the wandering subspace property in the sense that for every $t > 0$ the operator S_t has the wandering subspace property. Recall that an operator T on a Hilbert space H possesses the *wandering subspace property* if $H = [E]_T$, where $[E]_T$ denotes the smallest T-invariant subspace containing $E = \ker T^*$. The following lemma gives a useful description of the kernel of the adjoint S_t^*, for $t > 0$.

Lemma 3.2. *For* $t > 0$, $\ker S_t^* = E = \chi_{[0,t)}L^2(\mathbb{R}_+)$. *In particular,* $\ker S_t^*$ *is infinite-dimensional.*

Proof. Let $g \in \chi_{[0,t)}L^2(\mathbb{R}_+)$. Then $g(x) = \chi_{[0,t)}(x)f(x)$ almost everywhere on $\mathbb{R}_+$ for some function $f \in L^2(\mathbb{R}_+)$. Now

$$(S_t^* g)(x) = (S_t^* \chi_{[0,t)} f)(x) = \sqrt{\frac{\varphi(x+t)}{\varphi(x)}}\chi_{[0,t)}(x+t)f(x+t) = 0 \text{ for all } x \geq 0.$$

Therefore $g \in E$. Conversely, assume that $f \in E$. Then

$$(S_t^* f)(x) = \sqrt{\frac{\varphi(x+t)}{\varphi(x)}}f(x+t) = 0 \text{ for all } x \geq 0.$$

This implies $f(x + t) = 0$ almost everywhere on $\mathbb{R}_+$. Hence f vanishes almost everywhere on the interval $[t, \infty)$. Therefore $f \in \chi_{[0,t)}L^2(\mathbb{R}_+)$. ∎

Remark 3.3. In light of the fact that for a weighted shift operator T $\ker T^*$ is one-dimensional [6, Proposition 6.3], Lemma 3.2 indicates that the theories of weighted shift operators and weighted translation semigroups are intrinsically different.

Proposition 3.4. *For every $t > 0$, the operator S_t possesses the wandering subspace property.*

Proof. It is sufficient to prove that $(\bigvee_{n=0}^{\infty} S_t^n E)^{\perp} = \{0\}$. Let $f \in (\bigvee_{n=0}^{\infty} S_t^n E)^{\perp}$. Thus f is orthogonal to $(S_t^n E)$ for all $n \geq 0$. For a non-negative integer n, we now prove that $f(x) = 0$ almost everywhere on the interval $[nt, (n+1)t)$. Let $g(x) = f(x + nt)$ almost everywhere on $[nt, (n+1)t)$. By Lemma 3.2, $\chi_{[0,t)} g \in E$ and $S_t^n \chi_{[0,t)} g \in S_t^n E$. Now

$$(S_t^n \chi_{[0,t)} g)(x) = \begin{cases} \sqrt{\frac{\varphi(x)}{\varphi(x-nt)}} \chi_{[0,t)}(x - nt) g(x - nt), & \text{if } \ x \geq nt, \\ 0, & \text{if } \ x < nt. \end{cases}$$

So

$$(S_t^n \chi_{[0,t)} g)(x) = \begin{cases} \sqrt{\frac{\varphi(x)}{\varphi(x-nt)}} g(x - nt), & \text{if } \ x \in [nt, (n+1)t), \\ 0, & \text{otherwise.} \end{cases}$$

Now $\langle f, S_t^n \chi_{[0,t)} g \rangle = 0$ implies that

$$\int_{nt}^{(n+1)t} \sqrt{\frac{\varphi(x)}{\varphi(x - nt)}} f(x) \overline{g(x - nt)} dx = 0.$$

As $\overline{g(x - nt)} = \overline{f(x)}$, we have

$$\int_{nt}^{(n+1)t} \sqrt{\frac{\varphi(x)}{\varphi(x - nt)}} |f(x)|^2 dx = 0.$$

This implies that $|f(x)|^2 = 0$ almost everywhere on the interval $[nt, (n+1)t)$. This result is true for every non-negative integer n. Hence $f(x) = 0$ almost everywhere on the interval $[0, \infty)$. Hence for every $t > 0$ the operator S_t possesses the wandering subspace property. ∎

3.2. Construction of an analytic model

Let $\{S_t\}$ be a left invertible weighted translation semigroup. We now proceed to construct an analytic model for every operator in such a semigroup. A notion of the *Cauchy dual* of a left invertible operator was introduced by S. Shimorin in [18]. Recall that for a left invertible operator T, the *Cauchy dual* T' of T is defined as $T' = T(T^*T)^{-1}$. Note that a condition that for every $t \in \mathbb{R}_+, \inf_x \frac{\varphi(x+t)}{\varphi(x)} > 0$, ensures the left invertibility of the semigroup $\{S_t\}$. Now it is easy to see that the Cauchy dual S'_t of S_t is given by

$$S'_t f(x) = \begin{cases} \frac{1}{\varphi_t(x)} f(x-t), & \text{if } x \geq t, \\ 0, & \text{if } x < t. \end{cases}$$

Observe that for $t \in \mathbb{R}_+$ the family of operators $\{S'_t\}$ also forms a semigroup. We say that the weighted translation semigroup $\{S'_t\}$ is a *Cauchy dual of the weighted translation semigroup* $\{S_t\}$. Recall that a result of S. Shimorin [18] asserts that any left invertible analytic operator T is unitarily equivalent to the multiplication operator $\mathcal{M}_z$ on a reproducing kernel Hilbert space $\mathcal{H}$ of vector-valued analytic functions defined on a disc D, with center origin and radius $r(L)^{-1}$, where $L = T'^*$ and $r(L)$ is the spectral radius of L. Let $L_t = S_t'^*$ for every $t > 0$. We now give a formula for the spectral radius of S_t, $r(S_t)$, in terms of the symbol φ. Recall that $\|S_t\| = \|\varphi_t\|_\infty$. Observe that

$$\|S_t^n\| = \|S_{nt}\| = \left\| \sqrt{\frac{\varphi(x)}{\varphi(x-nt)}} \right\|_\infty .$$

Hence, by the spectral radius formula,

$$r(S_t) = \lim_{n\to\infty} \|S_t^n\|^{1/n} = \lim_{n\to\infty} \left\| \sqrt{\frac{\varphi(x)}{\varphi(x-nt)}} \right\|_\infty^{1/n} .$$

Now

$$r(L_t) = r(S_t'^*) = r(S'_t) = \lim_{n\to\infty} \left\| \sqrt{\frac{\varphi(x-nt)}{\varphi(x)}} \right\|_\infty^{1/n} .$$

Recall that $E = \ker S_t^*$. By the injectivity of S_t, $S_t^n E \neq \{0\}$ for all positive integers n. Note that for a left invertible operator S_t the subspace $S_t^n E$ is a closed subspace of $L^2(\mathbb{R}_+)$ for every positive integer n. We now prove a lemma.

Lemma 3.5. *The subspaces* $\{S_t^n E\}_{n=0}^\infty$ *are mutually orthogonal.*

Proof. We first prove that $S_t^n E \subseteq \chi_{[nt,(n+1)t)} L^2(\mathbb{R}_+)$ for all $n \geq 0$. Let $f \in E$. By Lemma 3.2, $f = \chi_{[0,t)} g$ almost everywhere on $\mathbb{R}_+$ for some function $g \in L^2(\mathbb{R}_+)$. Then for any $n \geq 0$

$$(S_t^n f)(x) = (S_t^n \chi_{[0,t)} g)(x) = \begin{cases} \sqrt{\frac{\varphi(x)}{\varphi(x-nt)}} \chi_{[0,t)}(x-nt) g(x-nt), & \text{if } x \geq nt, \\ 0, & \text{if } x < nt. \end{cases}$$

Therefore

$$(S_t^n f)(x) = \begin{cases} \sqrt{\frac{\varphi(x)}{\varphi(x-nt)}}\, g(x-nt), & \text{if } \ x \in [nt,(n+1)t), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $S_t^n f \in \chi_{[nt,(n+1)t)} L^2(\mathbb{R}_+)$. Hence $S_t^n E \subseteq \chi_{[nt,(n+1)t)} L^2(\mathbb{R}_+)$ for all $n \geq 0$. Note that for $m \neq n$ the intervals $[mt,(m+1)t)$ and $[nt,(n+1)t)$ are disjoint. Therefore the functions $\chi_{[mt,(m+1)t)}$ and $\chi_{[nt,(n+1)t)}$ are orthogonal. Hence for $m \neq n$, $(S_t^m E) \perp (S_t^n E)$. ■

Remark 3.6. For a left invertible semigroup $\{S_t\}$, the application of Lemma 3.5 to a semigroup $\{S_t'\}$ yields that the subspaces $\{S_t'^n E\}_{n=0}^{\infty}$ are also mutually orthogonal.

The following theorem describes the analytic model for the left invertible weighted translation semigroup $\{S_t\}$. The reader may compare this theorem with [5, Theorem 2.7] for a similar model developed in the context of weighted shifts on directed trees.

Theorem 3.7. *Let $\{S_t\}$ be a weighted translation semigroup with symbol φ. Assume that for any given $t \in \mathbb{R}_+$, $\inf_x \frac{\varphi(x+t)}{\varphi(x)} > 0$. Let S_t' be a Cauchy dual of the operator S_t and $L_t = S_t'^*$. Let $E = ker\, S_t^*$. Then there exist a reproducing kernel Hilbert space $\mathcal{H}$ of E-valued analytic functions defined on a disc D_r, a disc with center origin and radius $r(L_t)^{-1}$ and a unitary operator $U\colon L^2(\mathbb{R}_+) \to \mathcal{H}$ such that $\mathcal{M}_z U = U S_t$, where the operator $\mathcal{M}_z$ denotes the multiplication by z on $\mathcal{H}$. Further, U maps E onto the subspace $\mathcal{E}$ of E-valued constant functions in $\mathcal{H}$ satisfying $(Ue)(z) = e$ for all $z \in D_r$ and for every $e \in E$. We have the following:*

(i) *The reproducing kernel $k_{\mathcal{H}}\colon D_r \times D_r \to B(E)$ associated with $\mathcal{H}$ satisfies for any $e \in E$ and $\lambda \in D_r$, $k_{\mathcal{H}}(.,\lambda)e \in \mathcal{H}$ and for any $f \in L^2(\mathbb{R}_+)$,*

$$\langle (Uf)(\lambda), e\rangle_E = \langle Uf, k_{\mathcal{H}}(.,\lambda)e\rangle_{\mathcal{H}}.$$

(ii) *The reproducing kernel $k_{\mathcal{H}}(z,\lambda)$ is the diagonal operator on E in the following sense:*

$$(k_{\mathcal{H}}(z,\lambda)e)(x) = \Big(\sum_{n=0}^{\infty} \frac{\varphi(x)}{\varphi(x+nt)} z^n \overline{\lambda}^n\Big) e(x), \quad e \in E,\ x \in \mathbb{R}_+.$$

(iii) *The E-valued polynomials in z are dense in $\mathcal{H}$.*

(iv) *The Hilbert space $\mathcal{H}$ admits an orthonormal basis consisting of E-valued polynomials in z.*

Proof. The application of Shimorin's construction [18] to the left invertible analytic operator S_t gives the proof of part (i).

A proof of (ii) involves a computation of the reproducing kernel for the RKHS $\mathcal{H}$ associated to the operator $S_t, t > 0$. The following formula for the reproducing kernel is given in [18, Corollary 2.14]. For $e \in E,\ z, \lambda \in D$,

$$k_{\mathcal{H}}(z,\lambda)e = \sum_{n,k\geq 0} (PL^n(L^*)^k i_E)ez^n\overline{\lambda}^k$$

where i_E is the embedding of E into $L^2(\mathbb{R}_+)$. Observe that

$$(L_t f)(x) = \sqrt{\frac{\varphi(x)}{\varphi(x+t)}} f(x+t) \ \text{ for all } x \geq 0$$

and

$$L_t^* f(x) = \begin{cases} \sqrt{\frac{\varphi(x-t)}{\varphi(x)}} f(x-t), & \text{if } \ x \geq t, \\ 0, & \text{if } \ x < t. \end{cases}$$

Note that $L_t L_t^* f(x) = \frac{\varphi(x)}{\varphi(x+t)} f(x)$ for all $x \geq 0$. It is easy to see that for any positive integer k

$$L^k(L^*)^k f(x) = \frac{\varphi(x)}{\varphi(x+kt)} f(x) \ \text{ for all } x \geq 0.$$

Using Remark 3.6, we prove that $(PL_t^j L_t^{*k})|_E = 0$ for all non-negative integers $j \neq k$. For any $f, g \in E$ and non-negative integers $j \neq k$

$$\langle PL_t^j L_t^{*k} f, g\rangle_E = \langle PS_t'^{*j} S_t'^k f, g\rangle_E = \langle S_t'^{*j} S_t'^k f, g\rangle_{L^2(\mathbb{R}_+)} = \langle S_t'^k f, S_t'^j g\rangle_{L^2(\mathbb{R}_+)} = 0.$$

Therefore $(PL_t^j L_t^{*k})|_E = 0$ for all non-negative integers $j \neq k$.

We now compute the reproducing kernel for the RKHS $\mathcal{H}$ associated to the operator $S_t, t > 0$. Note that $E = \ker S_t^* = \ker L_t$ is infinite-dimensional. For $e \in E,\ z, \lambda \in D_r$ and $x \in \mathbb{R}_+$,

$$\begin{aligned} k_{\mathcal{H}}(z,\lambda)e(x) &= \sum_{n,k\geq 0} (PL_t^n(L_t^*)^k i_E)e(x)z^n\overline{\lambda}^k = \sum_{n,k\geq 0} (PL_t^n(L_t^*)^k)e(x)z^n\overline{\lambda}^k \\ &= \sum_{n=0}^{\infty} (PL_t^n(L_t^*)^n)e(x)z^n\overline{\lambda}^n = \sum_{n=0}^{\infty} \frac{\varphi(x)}{\varphi(x+nt)} Pe(x)z^n\overline{\lambda}^n \\ &= \sum_{n=0}^{\infty} \frac{\varphi(x)}{\varphi(x+nt)} e(x)z^n\overline{\lambda}^n = e(x)\sum_{n=0}^{\infty} \frac{\varphi(x)}{\varphi(x+nt)} z^n\overline{\lambda}^n. \end{aligned}$$

Hence

$$k_{\mathcal{H}}(z,\lambda)e(x) = \Big(\sum_{n=0}^{\infty} \frac{\varphi(x)}{\varphi(x+nt)} z^n \overline{\lambda}^n\Big) e(x).$$

To prove (iii), recall that the operator S_t, $t > 0$, possesses the wandering subspace property [Proposition 3.4]. Therefore $L^2(\mathbb{R}_+) = \bigvee_{n\geq 0} S_t^n(E)$. Since $\mathcal{M}_z$ is unitarily equivalent to S_t, it follows that $\mathcal{H} = \bigvee_{n\geq 0} \mathcal{M}_z^n(\mathcal{E})$. This proves that the E-valued polynomials in z are dense in $\mathcal{H}$.

For proving (iv), we need to describe the orthonormal basis of $L^2(\mathbb{R}_+)$. We begin with the following function on $\mathbb{R}$. Let

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \leq x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The function ψ is referred to in the literature as the basic Haar function [14]. Then the family of Haar functions given by $\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)\}$, where j, k are integers, forms an orthonormal basis of $L^2(\mathbb{R})$ [14, Theorem 6.3.4]. Define $\widetilde{\psi_{jk}} : \mathbb{R}_+ \to \mathbb{R}$ by $\widetilde{\psi_{jk}}(x) = \psi_{jk}(x)$ for an integer j and a non-negative integer k. It can be verified that the set $\mathcal{B} = \{\widetilde{\psi_{jk}}\}$, where j is an integer and k is a non-negative integer, is an orthonormal basis of $L^2(\mathbb{R}_+)$. A unitary operator $U : L^2(\mathbb{R}_+) \to \mathcal{H}$ is defined as follows: for each $f \in L^2(\mathbb{R}_+)$, define an E-valued function U_f on the disc D_r as follows:

$$(U_f)(z) = \sum_{n=0}^{\infty} (PL_t^n f) z^n$$

where P is an orthogonal projection on E and $U(f) = U_f$. Since U is a unitary operator and $\mathcal{B}$ is an orthonormal basis of $L^2(\mathbb{R}_+)$, the set $\{U\widetilde{\psi_{jk}}\}$ where j is an integer and k is a non-negative integer is an orthonormal basis of $\mathcal{H}$. Note that

$$(U\widetilde{\psi_{jk}})(z) = \sum_{n=0}^{\infty} (PL_t^n \widetilde{\psi_{jk}}) z^n$$

for some integer j and some non-negative integer k. We now prove that the expression $(U\widetilde{\psi_{jk}})(z)$ contains only finitely many non-zero terms. Note that

$$L_t^n \widetilde{\psi_{jk}}(x) = \sqrt{\frac{\varphi(x)}{\varphi(x+nt)}}\, \widetilde{\psi_{jk}}(x+nt) \text{ for all } x \geq 0.$$

Therefore $L_t^n \widetilde{\psi_{jk}} = 0$ if $nt > \frac{k+1}{2^j}$. Hence $(U\widetilde{\psi_{jk}})(z)$ is a polynomial of degree at most $[\frac{k+1}{t2^j}]$. Thus the Hilbert space $\mathcal{H}$ admits an orthonormal basis consisting of E-valued polynomials in z. This proves (iv).

∎

The following diagram describes the unitary equivalence of a left invertible operator S_t on $L^2(\mathbb{R}_+)$ with an operator M_z on a reproducing kernel Hilbert space $\mathcal{H}$ as stated in Theorem 3.7:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{M_z} & \mathcal{H} \\ U\uparrow & & \downarrow U^{-1} \\ L^2(\mathbb{R}_+) & \xrightarrow{S_t} & L^2(\mathbb{R}_+) \end{array}$$

We now compute the reproducing kernels in some particular types of weighted translation semigroup $\{S_t\}$. Recall that the reproducing kernel is a $B(E)$-valued function defined on $D_r \times D_r$, where D_r is a disc with center origin and radius $r(L_t)^{-1}$. The formula for $r(L_t)$ is given by

$$r(L_t) = \lim_{n\to\infty} \left\| \sqrt{\frac{\varphi(x-nt)}{\varphi(x)}} \right\|_\infty^{1/n}.$$

Example 3.8. Let $\{S_t\}$ be a weighted translation semigroup with symbol φ.
(1) Let $\varphi(x) = c$, where c is a constant. In this case $\frac{\varphi(x)}{\varphi(x+nt)} = 1$. Then $r(L_t) = 1$ and D_r is the open unit disc. It is easy to see that for $e \in E$ and $z, \lambda \in D_r$

$$k_{\mathcal{H}}(z,\lambda)e(x) = \Big(\sum_{n=0}^{\infty} z^n \overline{\lambda}^n\Big)e(x) = \frac{1}{1-z\overline{\lambda}}e(x).$$

(2) Let $\varphi(x) = x+1$. In this case

$$\frac{\varphi(x)}{\varphi(x+nt)} = \frac{x+1}{x+1+nt} = 1 - \frac{nt}{x+1+nt}.$$

Therefore $r(L_t) = 1$ and D_r is the open unit disc. For $e \in E$ and $z, \lambda \in D_r$, the computations reveal that

$$k_{\mathcal{H}}(z,\lambda)e(x) = \Big(\frac{1}{1-z\overline{\lambda}} - \sum_{n=0}^{\infty} \frac{nt}{x+1+nt} z^n \overline{\lambda}^n\Big)e(x).$$

(3) Let $\varphi(x) = \frac{1}{x+1}$. In this case

$$\frac{\varphi(x)}{\varphi(x+nt)} = \frac{x+1+nt}{x+1} = 1 + \frac{nt}{x+1}.$$

Therefore $r(L_t) = 1$ and D_r is the open unit disc. For $e \in E$ and $z, \lambda \in D_r$, it can be seen that

$$k_{\mathcal{H}}(z,\lambda)e(x) = \Big(\frac{1}{1-z\overline{\lambda}} + \frac{t}{x+1}\frac{z\overline{\lambda}}{(1-z\overline{\lambda})^2}\Big)e(x).$$

(4) Let

$$\varphi(x) = \begin{cases} x+1, & \text{if } 0 \le x \le 1, \\ 2, & \text{if } x > 1. \end{cases}$$

If $x \ge 1$, then $x + nt \ge 1$ for all $n \in \mathbb{N}, t \in \mathbb{R}_+$. In this case $\frac{\varphi(x)}{\varphi(x+nt)} = 1$. For $x < 1$, there exists $N \in \mathbb{N}$ such that $x + Nt \le 1$, but $x + (N+1)t > 1$. Then

$$\frac{\varphi(x)}{\varphi(x+nt)} = \begin{cases} \frac{x+1}{x+nt+1}, & \text{if } n \le N, \\ \frac{x+1}{2}, & \text{if } n > N. \end{cases}$$

Observe that $r(L_t) = 1$ and D_r is the open unit disc. It can be seen that for $e \in E$ and $z, \lambda \in D_r$, $x \ge 1$

$$k_{\mathcal{H}}(z,\lambda)e(x) = \Big(\sum_{n=0}^{\infty} z^n \overline{\lambda}^n\Big) e(x) = \frac{1}{1 - z\overline{\lambda}} e(x)$$

and for $x < 1$

$$\begin{aligned} k_{\mathcal{H}}(z,\lambda)e(x) &= \Big(\sum_{n=0}^{N} \frac{x+1}{x+nt+1} z^n \overline{\lambda}^n + \sum_{n=N+1}^{\infty} \frac{x+1}{2} z^n \overline{\lambda}^n\Big) e(x) \\ &= \Big(\sum_{n=0}^{N} \Big(\frac{x+1}{x+nt+1} - \frac{x+1}{2}\Big) z^n \overline{\lambda}^n + \frac{x+1}{2} \frac{1}{1 - z\overline{\lambda}}\Big) e(x). \end{aligned}$$

(5) Let $\varphi(x) = a^x$, $a > 1$. In this case $\frac{\varphi(x)}{\varphi(x+nt)} = \frac{a^x}{a^{x+nt}} = a^{-nt}$. Therefore $r(L_t) = a^{-t}$ and D_r is the disc with radius a^t. For $e \in E$ and $z, \lambda \in D_r$, we observe that

$$k_{\mathcal{H}}(z,\lambda)e(x) = \Big(\sum_{n=0}^{\infty} a^{-nt} z^n \overline{\lambda}^n\Big) e(x) = \frac{1}{1 - a^{-t} z\overline{\lambda}} e(x).$$

In the light of [13, Corollary 3.3], we observe that in example (1) the semigroup is an isometry, in example (2) the semigroup is a 2-isometry , in example (3) the semigroup is a subnormal contraction, in example (4) the semigroup is 2-hyperexpansive and in example (5), the semigroup is alternatingly hyperexpansive.

3.3. Operator-valued weighted shift

An operator-valued weighted shift is a generalization of a weighted shift operator in the sense that the weight sequence is a sequence of operators. In this subsection, we prove that for every $t > 0$ a left invertible operator S_t is an operator-valued weighted shift. We begin with the definition of an operator-valued weighted shift.

Let H be a nonzero Hilbert space. The Hilbert space denoted by l^2_H is the Hilbert space of all vector sequences $\{h_n\}_{n=0}^{\infty} \subseteq H$ such that $\sum_{n=0}^{\infty} ||h_n||^2 < \infty$, equipped with the standard inner product

$$\langle \{g_n\}_{n=0}^{\infty}, \{h_n\}_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} \langle g_n, h_n \rangle, \qquad \{g_n\}_{n=0}^{\infty}, \{h_n\}_{n=0}^{\infty} \in l^2_H.$$

If $\{W_n\}_{n=0}^{\infty} \subseteq \mathcal{B}(H)$ is a uniformly bounded sequence of operators, then the operator $W \in B(l^2_H)$ defined by

$$W(x_0, x_1, \ldots) = (0, W_0 x_0, W_1 x_1, \ldots), \qquad (x_0, x_1, \ldots) \in l^2_H,$$

is called an *operator-valued weighted shift* with weights $\{W_n\}$. For the basic theory of operator-valued weighted shifts, the reader is referred to [11, 12].

Theorem 3.9. *For every $t > 0$, a left invertible operator S_t is an operator-valued weighted shift.*

Proof. Recall that for $t > 0$ the operator S_t possesses the wandering subspace property [Proposition 3.4]. That is $L^2(\mathbb{R}_+) = \bigvee_{n=0}^{\infty} S_t^n E$, where $E = \ker S_t^*$. In addition, the closed subspaces $\{S_t^n E\}$ are mutually orthogonal (Lemma 3.5). Hence $L^2(\mathbb{R}_+) = \bigoplus_{n=0}^{\infty} S_t^n E$. Now by a similar argument as given in [2, Theorem 2.5], the operator S_t, $t > 0$, is an operator-valued weighted shift. ∎

4. Spectral picture

In this section we describe the spectral picture of an operator in a left invertible semigroup $\{S_t\}$ and compute the spectrum in some particular examples. As in case of a weighted shift operator, it turns out that the spectrum of S_t, $t > 0$, is a disc and the point spectrum of S_t, $t > 0$, is empty. In this section, we assume that t is a positive real number.

We use the following notation in the sequel: $\sigma(T)$: the spectrum of T, $\sigma_{ap}(T)$: the approximate point spectrum of T, $\sigma_p(T)$: the point spectrum of T, $\sigma_e(T)$: the essential spectrum of T.

We begin with an observation about the circular symmetry of the spectrum of S_t.

Proposition 4.1. *The operator S_t is unitarily equivalent to the operator $e^{-i\theta t} S_t$ for any real number θ.*

Proof. For a real number θ, define the map $M_\theta\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ as $(M_\theta f)(x) = e^{i\theta x} f(x)$. Then clearly M_θ is a unitary operator. We claim that $M_\theta^* S_t M_\theta = e^{-i\theta t} S_t$. Note that

$$\begin{aligned} ((M_\theta^* S_t M_\theta) f)(x) &= M_\theta^* (S_t M_\theta f)(x) = e^{-i\theta x} (S_t M_\theta f)(x) \\ &= \begin{cases} e^{-i\theta x} \sqrt{\frac{\varphi(x)}{\varphi(x-t)}} (M_\theta f)(x-t), & \text{if } \ x \geq t, \\ 0, & \text{if } \ x < t, \end{cases} \\ &= \begin{cases} e^{-i\theta x} \sqrt{\frac{\varphi(x)}{\varphi(x-t)}} e^{i\theta(x-t)} f(x-t), & \text{if } \ x \geq t, \\ 0, & \text{if } \ x < t, \end{cases} \\ &= ((e^{-i\theta t} S_t) f)(x). \end{aligned}$$

Hence the operator S_t is unitarily equivalent to the operator $e^{-i\theta t} S_t$. ∎

Remark 4.2. From Proposition 4.1, the spectrum of S_t as well as various spectral parts have circular symmetry about the origin. Also note that the operator S_t is not onto. Indeed, the characteristic function of the interval $[0, t)$ does not belong to the range of S_t. Therefore $0 \in \sigma(S_t)$.

We now describe the spectrum of a left invertible operator S_t.

Proposition 4.3. *Let S_t be a left invertible operator.*

(1) *The point spectrum of the operator $S_t, \sigma_p(S_t)$ is empty.*
(2) *The point spectrum of S_t^* contains the disc D_r with center origin and radius $r(L_t)^{-1}$.*
(3) *The spectrum of the operator S_t, $\sigma(S_t)$ is a closed disc with center origin and radius $r(S_t)$.*

Proof.
(1) The proof readily follows from the fact that for any complex number λ the operator $S_t - \lambda I$ is injective.
(2) The fact that the operator S_t is unitarily equivalent to the operator $\mathcal{M}_z$ on the reproducing kernel Hilbert space $\mathcal{H}$ is used in the following proof. By Theorem 3.7 (i), for $f \in L^2(\mathbb{R}_+)$, $e \in E$ and $w \in D_r$,

$$\langle U_f, M_z^* k_{\mathcal{H}}(.,w)e\rangle_{\mathcal{H}} = \langle M_z U_f, k_{\mathcal{H}}(.,w)e\rangle_{\mathcal{H}} = \langle w U_f(w), e\rangle_E = \langle U_f, \overline{w} k_{\mathcal{H}}(.,w)e\rangle_{\mathcal{H}}.$$

Thus $M_z^* k_{\mathcal{H}}(.,w)e = \overline{w} k_{\mathcal{H}}(.,w)e$ for all $w \in D_r$ and $e \in E$. Hence the point spectrum of S_t^* contains the disc D_r.

(3) By part (2), $D_r \subseteq \sigma_p(S_t^*)$. The circular symmetry of the spectrum implies that $\sigma(S_t) = \sigma(S_t^*)$. Therefore $D_r \subseteq \sigma(S_t)$. By [5, Lemma 5.3], the spectrum of an analytic operator is always connected. Hence, $\sigma(S_t)$ is a closed disc with radius $r(S_t)$.
This completes the proof of the theorem. ∎

At this stage we point out that Proposition 4.3 (1) and (3), proved independently here, can be also looked upon as a consequence of [12, Lemma 2.2(iii)]. We now turn our attention towards the approximate point spectrum of an operator S_t. Recall that for any operator T

$$m(T) := \inf\{||Tf|| : ||f|| = 1\} \text{ and } r_1(T) = \lim_{n\to\infty} [m(T^n)]^{1/n}.$$

At this stage we record the following well-known result useful in determining the approximate point spectrum of an operator [15]. For any operator T, for the approximate point spectrum we have

$$\sigma_{ap}(T) \subseteq \{z \ : \ r_1(T) \le |z| \le r(T)\}.$$

We now compute the spectra of the semigroups $\{S_t\}$ associated to some special types of symbols φ.

Example 4.4. Let $\{S_t\}$ be a weighted translation semigroup with symbol φ.

(1) Let $\varphi(x) = c$. It is easy to see that $r_1(S_t) = r(S_t) = 1$. Therefore $\sigma_{ap}(S_t)$ is a unit circle and $\sigma(S_t)$ is a closed unit disc.

(2) Let $\varphi(x) = x + 1$. In this case $r_1(S_t) = r(S_t) = 1$. Hence $\sigma_{ap}(S_t)$ is a unit circle and $\sigma(S_t)$ is a closed unit disc.

(3) Let $\varphi(x) = e^{2x}$. In this case $r_1(S_t) = r(S_t) = e^t$. Thus $\sigma_{ap}(S_t)$ is a circle with radius e^t and $\sigma(S_t)$ is a closed disc with radius e^t.

Note that in example (1) the semigroup is an isometry, in example (2) the semigroup is a 2-isometry and in example (3), the semigroup is alternatingly hyperexpansive (we refer to [13, Corollary 3.3]).

Remark 4.5. Atkinson's theorem [7, Theorem 5.17] asserts that an operator $T \in \mathcal{B}(H)$ is essentially invertible if and only if $\ker T$ is finite-dimensional, $\ker T^*$ is finite-dimensional and the range of T is closed. Here we know that $\ker S_t = \{0\}$ and $\ker {S_t}^*$ is infinite-dimensional. As a consequence, the operator S_t is not essentially invertible and thus $0 \in \sigma_e(S_t)$. We know that for a weighted shift operator T, $\ker T = \{0\}$ and $\ker T^*$ is one-dimensional. Therefore if the range of a weighted shift operator T is closed, then it is essentially invertible. If T is a completely hyperexpansive weighted shift, then the essential spectrum of T, $\sigma_e(T)$, is a unit circle [3, Proposition 5]. If T is a hyponormal weighted shift, then $\sigma_e(T)$ is a circle with center origin and radius $||T||$ [6, Theorem 6.7(a)]. Thus in these cases the essential spectrum of a weighted shift operator and that of a weighted translation semigroup are not the same.

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