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Completely hyperexpansive tuples of finite order

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We introduce and discuss a class of operator tuples, which we call completely hyperexpansive tuples of finite order. This class is in some sense antithetical to the class of completely hypercontractive tuples of finite order studied in the prequel of this paper. Motivated by Shimorin's notion of Cauchy dual operator, we also discuss a transform which sends certain completely hyperexpansive multishifts of finite order k to completely hypercontractive multishifts of respective order.

1. Preliminaries

The present work is a sequel to the paper [10]. In that paper, we discussed a class of functions, referred to as completely monotone functions of finite order, which in particular includes polynomials and completely monotone functions. The objective of the said paper was to study the operator-theoretic analogue of that class of functions. We observed that the new class of operator tuples, referred to as completely hypercontractive tuples of finite order, includes joint m-isometries and joint subnormal contractions. In some sense, that class of operator tuples provides a unified treatment of both these well known classes of operator tuples. However, among several examples of the completely hypercontractive tuples of finite order, there are examples of tuples which are neither m-isometries nor subnormals.

We recall the notation used in [10] for the ready reference. The symbol $\mathbb{N}$ stands for the set of non-negative integers; $\mathbb{N}$ forms a semigroup under addition. Let $\mathbb{N}^m$ denote the Cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (m times). Let $p \equiv (p_1, \cdots, p_m)$ and $n \equiv (n_1, \cdots, n_m)$ be in $\mathbb{N}^m$. We write $|p| := \sum_{i=1}^m p_i$ and $p \leq n$ if $p_i \leq n_i$ for $i = 1, \cdots, m$. For $n \in \mathbb{N}^m$, we let $n! := \prod_{i=1}^m n_i!$.

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For a real-valued map φ on $\mathbb{N}$, we define (backward and forward) difference operators ∇ and Δ as follows: $(\nabla\varphi)(n) = \varphi(n) - \varphi(n+1)$ and $(\Delta\varphi)(n) = \varphi(n+1) - \varphi(n)$. The operators ∇^n and Δ^n are inductively defined for all $n \in \mathbb{N}$ through the relations

$$\nabla^0\varphi = \Delta^0\varphi = \varphi,\ \nabla^n\varphi = \nabla(\nabla^{n-1}\varphi)(n \geq 1),\ \Delta^n\varphi = \Delta(\Delta^{n-1}\varphi)(n \geq 1).$$

A real-valued map φ on $\mathbb{N}$ is said to be *completely monotone* if $(\nabla^k\varphi)(n) \geq 0$ for all $n \geq 0$, $k \geq 0$. A real-valued map ψ on $\mathbb{N}$ is said to be *completely alternating* if $(\nabla^k\psi)(n) \leq 0$ for all $n \geq 0$, $k \geq 1$. Completely monotone maps on $\mathbb{N}$ form an extreme subset of the set of positive definite functions on $\mathbb{N}$, while completely alternating maps form an extreme subset of the set of negative definite functions on $\mathbb{N}$ (refer to [6]).

For a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$.

Completely hyperexpansive operators were introduced independently in [2] and [5]. An operator T in $B(\mathcal{H})$ is *completely hyperexpansive* if

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*k}T^k \leq 0 \ \text{ for all } \ n \geq 1. \tag{1.1}$$

It was observed in [5] that the condition (1.1) is equivalent to requiring, for every h in $\mathcal{H}$, the map $\psi_h(n) = \|T^n h\|^2$ to be completely alternating on $\mathbb{N}$. The symbiotic relationship between completely monotone and completely alternating maps carries over to subnormal contractions and completely hyperexpansive operators respectively, and this theme was focused upon in [5]. The reader is referred to [25] and [18] for the basic theory of hyperexpansive operators.

By a *commuting m-tuple* T on $\mathcal{H}$, we mean a tuple $(T_1, \cdots, T_m)$ of commuting bounded linear operators $T_1, \cdots, T_m$ on $\mathcal{H}$. For a commuting m-tuple T on $\mathcal{H}$, we interpret T^* to be $(T_1^*, \cdots, T_m^*)$, and T^p to be $T_1^{p_1} \cdots T_m^{p_m}$ for $p = (p_1, \cdots, p_m) \in \mathbb{N}^m$.

For the definitions and the basic theory of various spectra including the Taylor spectrum, the reader is referred to [14]. For a commuting m-tuple T, we reserve the symbols $\sigma(T), \sigma_l(T)$ and $\sigma_{ap}(T)$ for the Taylor spectrum, left-spectrum and approximate point spectrum of T respectively. The symbols $r(T)$ and $r_l(T)$ stand for the geometric spectral radii of $\sigma(T)$ and $\sigma_l(T)$ respectively. We recall that $r(T)$ and $r_l(T)$ are given by

$$r(T) := \sup\{\|z\|_2 : z \in \sigma(T)\},\ r_l(T) := \sup\{\|z\|_2 : z \in \sigma_l(T)\}, \tag{1.2}$$

where $\|z\|_2 := \left(|z_1|^2 + \cdots + |z_m|^2\right)^{\frac{1}{2}}$ is the Euclidean norm of $z = (z_1, \cdots, z_m)$. It is worth noting that $\sigma_l(T) \subseteq \sigma(T)$ and $r_l(T) \leq r(T)$.

Recall that a commuting m-tuple $T = (T_1, \cdots, T_m)$ on a Hilbert space $\mathcal{H}$ is said to be *joint subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a commuting m-tuple $N = (N_1, \cdots, N_m)$ of normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that

$$N_i h = T_i h \text{ for every } h \in \mathcal{H} \text{ and } 1 \leq i \leq m.$$

An m-tuple $S = (S_1, \cdots, S_m)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is *joint hyponormal* if the $m \times m$ matrix $([T_j^*, T_i])_{1 \leq i,j \leq m}$ is positive definite, where $[A, B]$ stands for the commutator $AB - BA$ of A and B. A joint subnormal tuple is always joint hyponormal [3].

Given a commuting m-tuple $T = (T_1, \cdots, T_m)$ on $\mathcal{H}$, we set

$$Q_T(X) := \sum_{i=1}^{m} T_i^* X T_i \ (X \in B(\mathcal{H})). \tag{1.3}$$

The operator Q_T^n is inductively defined for all $n \geq 0$ through the relations $Q_T^0(X) = X$ and $Q_T^n(X) = Q_T(Q_T^{n-1}(X))$ $(n \geq 1)$ for $X \in B(\mathcal{H})$. A routine verification shows that $Q_T^n(I) = \sum_{|p|=n} \frac{n!}{p!} T^{*p} T^p$ for any positive integer n.

Definition 1.1. Let Q_T be as given in (1.3). Set $B_0(Q_T) = I$. For an integer $q \geq 1$, let

$$B_q(Q_T) := \sum_{s=0}^{q} (-1)^s \binom{q}{s} Q_T^s(I). \tag{1.4}$$

If $B_q(Q_T) = 0$, then T is called as a *joint q-isometry.*

Fix an integer $q \geq 1$. We say that T is a *joint q-contraction (resp. joint q-expansion)* if

$$B_q(Q_T) \geq 0 \text{ (resp. } \leq 0).$$

We say that T is a *joint complete hypercontraction* (resp. *joint complete hyperexpansion*) if T is a joint q-contraction (resp. joint q-expansion) for all positive integers q. In all the definitions above, if $q = 1$ then we drop the prefix 1- and if $m = 1$ then we drop the term joint.

An m-variable weighted shift $T = (T_1, \cdots, T_m)$ with respect to an orthonormal basis $\{e_n\}_{n\in\mathbb{N}^m}$ of a Hilbert space $\mathcal{H}$ is defined by

$$T_i e_n := w_n^{(i)} e_{n+\epsilon_i} \ (1 \leq i \leq m, n \in \mathbb{N}^m),$$

where ϵ_i is the m-tuple with 1 in the ith place and zeros elsewhere. We indicate the m-variable weighted shift operator T with weight sequence

$$\left\{ w_n^{(i)} : 1 \leq i \leq m, n \in \mathbb{N}^m \right\}$$

by $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$. We always assume that the weight sequence of T is a bounded set consisting of positive numbers and that T is commuting. Notice that T_i commutes with T_j if and only if $w_n^{(i)} w_{n+\epsilon_i}^{(j)} = w_n^{(j)} w_{n+\epsilon_j}^{(i)}$ for all $n \in \mathbb{N}^m$.

This paper is partly motivated by Shimorin's notion of Cauchy dual operator. Recall that if T is a bounded left-invertible operator then the *Cauchy dual* T' is given by $T' := T(T^*T)^{-1}$. Note that if T is a (left-invertible) weighted shift operator with weights $\{w_n\}_{n\in\mathbb{N}}$ then T' is a weighted shift operator with weights $\{1/w_n\}_{n\in\mathbb{N}}$. More generally, the *spherical Cauchy dual* $T^{\mathfrak{s}}$ of $T : \{w_n^{(j)}\}_{n\in\mathbb{N}^m}$ is given by

$$T_i^{\mathfrak{s}} e_n := \frac{w_n^{(i)}}{\delta_n^2} e_{n+\epsilon_i} \ (1 \leq i \leq m, n \in \mathbb{N}^m),$$

where $\delta_n^2 := \sum_{i=1}^m \left(w_n^{(i)}\right)^2$ $(n \in \mathbb{N}^m)$ (for the general definition of spherical Cauchy dual for left-invertible tuples, refer to [8]).

The following result provides a connection between subnormal and completely hyperexpansive multishifts.

Theorem 1.2. *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ be a joint completely hyperexpansive m-variable weighted shift. If the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting then $T^{\mathfrak{s}}$ is joint subnormal.*

Remark 1.3. This result is obtained by A. Athavale [5, Proposition 6] in one variable, where the commutativity condition is vacuous. The general case can be deduced from this case as observed in [7, Proposition 3.4].

The following two definitions are reproduced from [10] with one change; we do not confine ourselves to maps taking positive real values.

Definition 1.4. Let k be a non-negative integer. A map $\psi : \mathbb{N} \to \mathbb{R}$ is said to be *completely monotone of order* k if $\nabla^k \psi$ is completely monotone. A completely monotone sequence of order 0 will be referred to as *completely monotone sequence*. The class of completely monotone sequences of order k will be denoted by $\mathcal{CM}_k$.

Definition 1.5. Let k be a non-negative integer. A C^∞ function $f : (0, \infty) \to \mathbb{R}$ is said to be *completely monotone function of order* k if

$$(-1)^l f^{(l)} \geq 0 \text{ for all } l \geq k,$$

where $f^{(l)}$ denotes the lth derivative of f. A completely monotone function of order 0 will be referred to as *completely monotone function*. The class of completely monotone functions of order k will be denoted by $\mathcal{L}_k$.

Let x be any real number. In keeping with the classical combinatorial theory, we define $(x)_0 = 1$, $(x)_1 = x$, and $(x)_k = x(x-1)\dots(x-k+1)$ for any integer $k \geq 2$.

We invoke the following two results from [10] for ready reference. Both the results hold true for real-valued functions as well.

Theorem 1.6. *[10, Theorem 3.6] Let $\psi : \mathbb{N} \to \mathbb{R}$ be given and let k be a positive integer. Then the following statements are equivalent:*

(1) *ψ is a completely monotone sequence of order k.*
(2) *There exist a polynomial p_k of degree $k-1$ and a positive Radon measure μ on $[0,1]$ such that for every $n \in \mathbb{N}$,*

$$\begin{aligned}\psi(n) &= p_k(n) + (-1)^k \binom{n}{k} \mu(\{1\}) + \int_{[0,1)} \frac{1}{(1-x)^k} \Big(x^n - \sum_{j=0}^{k-1} \frac{(x-1)^j}{j!} (n)_j \Big) d\mu(x) \\ &= p_k(n) + \int_{[0,1]} \sum_{j=0}^{n-k} \Big((-1)^k \frac{(x-1)^j}{(j+k)!} (n)_{j+k} \Big) d\mu(x),\end{aligned}$$

where the integral in the last expression is absent if $n < k$.

If (2) holds then the integral representation in (2) is unique in the sense that the coefficients of p_k and measure μ are completely determined by ψ.

Theorem 1.7. *[10, Theorem 3.13] Let k be a positive integer and $f : (0, \infty) \to \mathbb{R}$ be a C^∞ function such that $f^{(l)}(0+) < \infty$ for $0 \leq l \leq k-1$. Then f is a completely monotone function of order k if and only if it admits the representation*

$$f(x) = p_k(x) + \int_{(0,\infty)} \frac{e^{-tx} - q_{k-1}(tx)}{t^k} d\mu(t), \tag{1.5}$$

where μ is a measure on $[0,\infty)$, p_k is a polynomial of degree at most k with leading coefficient equal to $\dfrac{(-1)^k\mu(\{0\})}{k!}$, and q_{k-1} is the polynomial of degree $k-1$ given by $q_{k-1}(xt) = \sum_{n=0}^{k-1}(-1)^n\dfrac{(xt)^n}{n!}$, the kth partial sum of the series expansion of e^{-xt}.

In this paper, we study a class of functions associated to the completely monotone functions of finite order, in the manner that completely alternating functions get tied up with completely monotone functions. The new class of functions will be referred to as the completely alternating functions of finite order. Motivated by Theorem 1.2, we introduce a notion of Cauchy dual map C_k of order k, which sends completely alternating functions $\psi : \mathbb{N} \to \mathbb{R}$ of order k to completely monotone functions of order k under some modest assumptions. In case $k = 1$, our definition yields $C_1(\psi) = \frac{1}{\psi}$. Thus the Cauchy dual map of order 1 sends completely alternating functions to completely monotone functions. In the last and third section, we use the function theory developed in Section 2 to study a class of operator tuples to be referred to as completely hyperexpansive tuples of finite order. One of the main results of this paper obtains an analogue of Theorem 1.2 for completely hyperexpansive tuples of finite order (Proposition 3.20). In [9], we raised the question: whether there exists a joint hyponormal q-isometry tuple which is not a joint isometry? In the present paper, we settle this question in negative (Corollary 3.12).

2. Function theory

In this section, we define completely alternating sequences of finite order and Bernstein functions of finite order. We also discuss a transform which sends a completely alternating sequence of order k to a completely monotone sequence of order k. The operator theoretic analogue of these sequences is studied in the next section.

2.1. Completely alternating sequences of finite order

Definition 2.1. Let k be a non-negative integer. A map $\psi : \mathbb{N} \to \mathbb{R}$ is said to be *completely alternating of order k* if $\nabla^k\psi$ is completely alternating. The class of completely alternating sequences of order k will be denoted by $\mathcal{CA}_k$.

Remark 2.2. A completely alternating sequence of order 0 is completely alternating.

We will see in the next subsection that a completely alternating sequence of order 1 turns out to be completely alternating (see Corollary 2.17).

The following fact relates the class of completely alternating sequences of order k with the class of completely monotone sequences of order k.

Proposition 2.3. *A map $\psi \in \mathcal{CA}_k$ if and only if $\Delta\psi \in \mathcal{CM}_k$.*

Proof. It is well-known that $\phi \in \mathcal{CA}$ if and only if $\Delta\phi \in \mathcal{CM}$ (see [6, Chapter 4, Lemma 6.3]). Note that $\psi \in \mathcal{CA}_k$ if and only if $\Delta\nabla^k\psi \in \mathcal{CM}$. Since Δ and ∇ are commuting, this happens if and only if $\nabla^k\Delta\psi \in \mathcal{CM}$, that is, $\Delta\psi \in \mathcal{CM}_k$. $\square$

We now give an example of a sequence in $\mathcal{CA}_2$ which is not a polynomial perturbation of a completely alternating sequence.

Example 2.4. Consider the sequence

$$\psi(n) = \frac{n(n+1)}{2} - (n+2)H(n+1) \ (n \in \mathbb{N}),$$

where $H(n)$ is nth partial sum of the harmonic series $\sum_{k=1}^{\infty} 1/k$. Note that $\Delta\psi(n) = n - H(n+2)$. It has been recorded in [10, Example 3.5] that $\Delta\psi(n-1)+2 = n+1-H(n+1)$ is completely monotone of order 2, and hence so is $\Delta\psi(n)$. By previous proposition, $\psi \in \mathcal{CA}_2$.

We claim that ψ is not a polynomial perturbation of a completely alternating sequence. Assume the contrary, so that $\psi(n) = p(n) + \phi(n)$, where p is a polynomial and ϕ is completely alternating. Now applying Δ on both sides, we obtain

$$n - H(n+2) = q(n) + \Delta\phi(n) \text{ for every integer } n \geq 0,$$

where q is a polynomial and $\Delta\phi$ is completely monotone. But then $n+1-H(n+1)$ is also a polynomial perturbation of a completely monotone sequence, which is impossible in view of the discussion in [10, Example 3.5].

We characterize all completely alternating sequences of finite order.

Theorem 2.5. *Let $\psi : \mathbb{N} \to \mathbb{R}$ be given and let k be a positive integer. Then the following statements are equivalent:*

(1) *ψ is a completely alternating sequence of order k.*
(2) *There exist a polynomial p_k of degree k and a positive Radon measure μ on $[0,1]$ such that for every $n \in \mathbb{N}$,*

$$\begin{aligned}\psi(n) &= p_k(n) + (-1)^k \binom{n}{k+1}\mu(\{1\}) + \int_{[0,1)} \frac{1}{(1-x)^{k+1}}\Big(\sum_{j=0}^{k} \frac{(x-1)^j}{j!}(n)_j - x^n\Big)d\mu(x) \\ &= p_k(n) + \int_{[0,1]} \sum_{j=0}^{n-k-1} \Big((-1)^k \frac{(x-1)^j}{(j+k+1)!}(n)_{j+k+1}\Big)d\mu(x),\end{aligned}$$

where the integral in the last expression is absent if $n < k+1$.

If (2) holds then the integral representation in (2) is unique in the sense that the coefficients of p_k and measure μ are completely determined by ψ.

Proof. By Proposition 2.3, ψ is a completely alternating sequence of order k if and only if $-\psi$ is a completely monotone sequence of order $k+1$. The desired conclusions now follow from Theorem 1.6. $\square$

We skip the proof of the following corollary as it is similar to that of [10, Corollary 3.7].

Corollary 2.6. *If ψ is a completely alternating sequence of order k such that $\nabla^p\psi = 0$ for some integer $p \geq 1$ then ψ is a polynomial of degree less than or equal to $\min\{k+1, p-1\}$.*

Note that the sum of a completely alternating sequence and a polynomial of degree at most k belongs to $\mathcal{CA}_k$. In particular, we have the following:

Example 2.7. Consider the *Ramanujan's theta function* $\theta : \mathbb{N} \to \mathbb{R}$

$$\theta(n) = \frac{n!}{n^n}\Big(\frac{e^n}{2} - \sum_{j=0}^{n-1} \frac{n^j}{j!}\Big).$$

It may be concluded from [19, Theorem 1] that $\lambda(n) = n(\theta(n) - 1/3)$ is completely alternating. By the discussion prior to this example, $n\theta(n)$ belongs to $\mathcal{CA}_1$.

2.2. Bernstein functions of finite order

Definition 2.8. Let k be a non-negative integer. A C^∞ function $f : (0, \infty) \to \mathbb{R}$ is said to be *completely alternating of order k or Bernstein function of order k* if $(-1)^l f^{(l+1)} \geq 0$ for all $l \geq k$, where $f^{(l)}$ denotes the lth derivative of f. A Bernstein function of order 0 will be referred to as *completely alternating or Bernstein function*. The class of Bernstein functions of order k will be denoted by $\mathcal{B}_k$.

Remark 2.9. Note that f belongs to $\mathcal{B}_k$ if and only if f' belongs to $\mathcal{L}_k$, where $\mathcal{L}_k$ denotes the set of completely monotone functions of order k.

It is well-known that $e^{-t\psi} \in \mathcal{L}_1$ for every $t > 0$ if and only if $\psi \in \mathcal{B}_0$ [6, Chapter 4, Proposition 6.10]. Here is a counter-part of this fact for the class of Bernstein functions of finite order.

Proposition 2.10. *Let k be a positive integer. If $e^{-t\psi} \in \mathcal{L}_k$ for every $t > 0$ then $\psi \in \mathcal{B}_{k-1}$.*

Proof. Suppose $e^{-t\psi}$ is completely monotone of order k for every $t > 0$. Then

$$e^{-t\psi} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} t^j \psi^j,$$

and hence

$$(-1)^l (e^{-t\psi})^{(l)} = \sum_{j=1}^{\infty} \frac{(-1)^{j+l}}{j!} t^j (\psi^j)^{(l)}.$$

As the above equation holds for every $t > 0$, for any positive integer l, we get that

$$\lim_{t \to 0+} \frac{(-1)^l}{t} (e^{-t\psi})^{(l)} = (-1)^{l+1} \psi^{(l)}.$$

Now as $e^{-t\psi} \in \mathcal{L}_k$ for every $t > 0$, we have

$$\lim_{t \to 0+} \frac{(-1)^l}{t} (e^{-t\psi})^{(l)} \geq 0 \text{ for } l \geq k.$$

Thus $(-1)^l \psi^{(l+1)} \geq 0$ for $l \geq k - 1$. Thus ψ is a completely alternating function of order $k - 1$. $\square$

Remark 2.11. Note that the converse of the above proposition is not true. For example, $\psi(x) = x^3$ is completely alternating of order 2, but e^{-x^3} is not completely monotone of order 3 because $(e^{-x^3})^{(3)} > 0$ at $x = 1$.

Example 2.12. A monic polynomial of degree at most k is a Bernstein function of order l, where $l = k - 1$ if k is an odd integer and $l = k$ if k is an even integer. A completely alternating function certainly belongs to $\mathcal{B}_0$. Further, if f is a perturbation of a Bernstein function by a polynomial of degree at most k, then f belongs to either $\mathcal{B}_{k-1}$ or $\mathcal{B}_k$.

Let us see a particular instance in which the observation above is applicable. For an odd integer $n \geq 1$, consider the function

$$f_n(t) = \frac{t^n}{1+t}, \ t \in (0, \infty).$$

It has been recorded in [23, Remark 3.11] that f_1 belongs to $\mathcal{B}_0$. Assume that $n \geq 3$. Since

$$f_1(t) - f_n(t) = \frac{t}{1+t}\left(1 - t^{n-1}\right) = t(1 - t + t^2 - \cdots + t^{n-3} - t^{n-2}),$$

$f_1 - f_n$ is a polynomial of degree equal to $n-1$. By the preceding discussion, f_n belongs to $\mathcal{B}_{n-1}$. We claim that for $n \geq 3$, $f_n \notin \mathcal{B}_{n-2}$. Indeed,

$$\begin{aligned} f_n^{(n-1)}(t) &= f_1^{(n-1)}(t) + (n-1)! \\ &= \left(1 - \frac{1}{1+t}\right)^{(n-1)} + (n-1)! \\ &= \frac{(n-1)!(-1)^n}{(1+t)^n} + (n-1)!, \end{aligned}$$

which is positive for any $t \in (0, \infty)$. This shows that $\mathcal{B}_k \subsetneq \mathcal{B}_{k+1}$ if k is an odd integer.

Remark 2.13. It is clear from the definition that $\mathcal{B}_k \subseteq \mathcal{B}_{k+1}$. However, we shall see that the classes $\mathcal{B}_k$ and $\mathcal{B}_{k+1}$ coincide if k is even (Proposition 2.16). The examples above show that the containment is strict if k is odd.

The following result characterizes all Bernstein functions of finite order. Also, it can be used to generate examples in this class.

Theorem 2.14. *Let k be a positive integer and $f : (0, \infty) \to \mathbb{R}$ be a C^∞ function such that $f^{(l)}(0+) < \infty$ for $0 \leq l \leq k$. Then f is a completely alternating function of order k if and only if it admits the representation*

$$f(x) = p_{k+1}(x) + \int\limits_{(0,\infty)} \frac{q_k(tx) - e^{-tx}}{t^{k+1}} d\mu(t), \tag{2.6}$$

where μ is a measure on $[0, \infty)$, p_{k+1} is a polynomial of degree at most $k+1$ with leading coefficient equal to $\dfrac{(-1)^{k+1}\mu(\{0\})}{(k+1)!}$, and q_k is the polynomial of degree k given by $q_k(xt) = \displaystyle\sum_{n=0}^{k}(-1)^n \frac{(xt)^n}{n!}$, the $(k+1)$th partial sum of the series expansion of e^{-xt}.

Proof. The proof follows from the simple observation that f belongs to $\mathcal{B}_k$ if and only if f' is a completely monotone function of order k. Appealing to the integral representation of a completely monotone function of finite order as stated in Theorem 1.7, we get that

$$f'(x) = p_k(x) + \int\limits_{(0,\infty)} \frac{e^{-tx} - q_{k-1}(tx)}{t^k} d\mu(t),$$

where μ is a measure on $[0,\infty)$, p_k is a polynomial of degree at most k with leading coefficient equal to $\dfrac{(-1)^k\mu(\{0\})}{k!}$, and q_{k-1} is the polynomial of degree $k-1$ given by $q_{k-1}(xt) = \sum_{n=0}^{k-1}(-1)^n\dfrac{(xt)^n}{n!}$, the kth partial sum of the series expansion of e^{-xt}. Now the desired result follows by integrating f' and appealing to Fubini–Tonelli theorem. □

We invoke the following lemma from [10] for ready reference.

Lemma 2.15. *[10, Lemma 3.16] Let $k \geq 2$ be an integer. Then there does not exist a C^∞ function $f : (0,\infty) \to (0,\infty)$ such that $f^{(k)}(x) \leq 0$ for all x and $f^{(k-1)}(x_0) < 0$ for some x_0.*

Proposition 2.16. *For an integer $k \geq 1$, if $f : (0,\infty) \to (0,\infty)$ belongs to $\mathcal{B}_k$ and if k is odd, then f belongs to $\mathcal{B}_{k-1}$.*

Proof. If f is completely alternating of order k then $(-1)^k f^{(k+1)} \geq 0$. As k is odd, we have $f^{(k+1)} \leq 0$. Now the Lemma 2.15 implies that $f^{(k)} \geq 0$, that is, $(-1)^{k-1}f^{(k)} \geq 0$. This in turn implies that f is completely alternating of order $k-1$. □

Here is the discrete analogue of Proposition 2.16, which may be obtained by imitating the proof of Proposition 2.16.

Corollary 2.17. *For an integer $k \geq 1$, if $\psi : \mathbb{N} \to (0,\infty)$ is a completely alternating sequence of order k and if k is odd, then ψ is completely alternating of order $k-1$.*

Remark 2.18. The proof of Proposition 2.16 actually shows that for an even integer $k \geq 1$, $\nabla^k\psi \leq 0$ implies $\nabla^{k-1}\psi \leq 0$.

At this stage, we find it necessary to explain our notation in order to achieve clarity in the sequel. Observe that though in statements of [10, Corollary 3.18] and Corollary 2.17, k is an odd integer, the reader should note the disparity in the following sense. For instance, in case $k = 3$, the first result ensures the implication

$$\nabla^3\psi \geq 0 \Longrightarrow \nabla^2\psi \geq 0,$$

whereas the second result forces

$$\nabla^4\psi \leq 0 \Longrightarrow \nabla^3\psi \leq 0.$$

However, it will be seen that the disparity vanishes at the operator level.

2.3. A Cauchy dual of finite order

Recall that the transform $\phi \longmapsto 1/\phi$ sends a completely alternating function to a completely monotone function. In this subsection, we discuss a generalized transform which sends a completely alternating function of order k to a completely monotone function of order k. The need of such a transform lies in the fact that the transform $\phi \longmapsto 1/\phi$ fails to have this property unless $k = 1$ as shown below.

Example 2.19. Consider the sequence $\psi(n) = n^2 + 1$ for $n \in \mathbb{N}$. Then ψ is completely alternating of order 3. However, $\phi(n) = \frac{1}{\psi(n)}$ is not completely monotone of any finite order. In fact, since ϕ is non-increasing, by [10, Corollary 3.20], ϕ is completely monotone of finite order if and only if ϕ is completely monotone. However, $\nabla^4\phi(0) < 0$, and hence ϕ is not completely monotone.

Definition 2.20. Let k be a non-negative integer. Let $\psi : \mathbb{N} \to \mathbb{R}$ be a completely alternating function of order k. We define a *Cauchy dual* $\mathcal{C}_k(\psi)$ *of order* k *of* ψ to be a solution $\phi : \mathbb{N} \to (0, \infty)$ of the following difference equation:

$$\left(\nabla^k \phi(n)\right)\left(\nabla^k \psi(n)\right) = 1 \ (n \in \mathbb{N}).$$

In this case, we set $\mathcal{C}_k(\psi) = \phi$.

Remark 2.21. The Cauchy dual of ψ of order zero is given by $C_0(\psi) = \frac{1}{\psi}$. In particular, C_0 sends a completely alternating sequence to a completely monotone sequence. This fact has been observed in the proof of [5, Proposition 6] (cf. [23, Theorem 3.6]).

We discuss below the existence and uniqueness of Cauchy dual map of finite order.

Theorem 2.22. *Let $\psi : \mathbb{N} \to \mathbb{R}$ be a completely alternating sequence of order k. Then Cauchy dual $\mathcal{C}_k(\psi)$ of order k of ψ exists in $\mathcal{CM}_k$ if and only if $\nabla^k \psi(n) > 0$ for every $n \in \mathbb{N}$.*

Further, if $C_k(\psi) = \phi$ for some $\phi : \mathbb{N} \to (0, \infty)$ then $\phi + p_k$ is also a Cauchy dual of ψ of order k for every polynomial p_k of degree less than k.

Proof. If a Cauchy dual ϕ of order k of ψ exists in $\mathcal{CM}_k$ then clearly $\nabla^k \phi(n) > 0$ $(n \in \mathbb{N})$. Since $\nabla^k \psi(n) = \frac{1}{\nabla^k \phi(n)}$, the condition $\nabla^k \psi(n) > 0$ $(n \in \mathbb{N})$ is necessary.

Conversely, assume that $\nabla^k \psi(n) > 0$ for every $n \in \mathbb{N}$. By Remark 2.21, $\frac{1}{\nabla^k \psi}$ is completely monotone. Thus there exists a positive Radon measure μ on $[0, 1]$ such that

$$\frac{1}{\nabla^k \psi(n)} = \int\limits_{[0,1]} x^n d\mu(x) \ (n \in \mathbb{N}).$$

Define $\phi : \mathbb{N} \to \mathbb{R}$ by

$$\begin{aligned} \phi(n) &= p_k(n) + (-1)^k \binom{n}{k} \mu(\{1\}) \\ &\quad + \int\limits_{[0,1)} \frac{1}{(1-x)^k}\Big(x^n - \sum_{j=0}^{k-1} \frac{(x-1)^j}{j!}(n)_j\Big) d\mu(x), \end{aligned} \tag{2.7}$$

where p_k is a polynomial of degree less than k. In the remaining part of the proof, we need the following fact [15, Theorem 8.4]: *For any polynomial p of degree less than k, $\nabla^k p = 0$.* It is now easy to see that ϕ satisfies the difference equation $\left(\nabla^k \phi(n)\right)\left(\nabla^k \psi(n)\right) = 1$ $(n \in \mathbb{N})$. Further, by Theorem 1.6, ϕ is completely monotone of order k. The uniqueness part follows at once from the aforementioned fact. $\square$

Remark 2.23. The Cauchy dual of order k is unique up to a polynomial perturbation. This also suggests us that if $\nabla^k \psi = 0$ then the Cauchy dual of order k of ψ may be set as ψ itself. This also suggests a possibility of an operator transform (analogous to the Cauchy dual transform), which sends a q-isometry to itself.

Note that if a Cauchy dual of finite order exists, then one can also choose a Cauchy dual of same order which is positive valued.

We examine the Cauchy dual of finite order in two concrete situations.

Example 2.24. Consider the completely alternating function ψ of order k given by $\psi(n) = \delta(n) + p_k(n)$, where δ is completely alternating and $p_k(t) = a_0 - a_1 t + \cdots + (-1)^k a_k t^k$ is a polynomial of degree k. By [6, Proposition 6.12], there exists a positive regular Borel measure μ on $[0, 1]$ such that

$$\delta(n) = \delta(0) + n\mu(\{1\}) + \int\limits_{[0,1)} (1 - x^n)\frac{d\mu(x)}{1-x} \text{ for } n \geq 0$$

(see also [5, Remark 2]). Notice that $\delta(1) = \delta(0) + \mu(\{1\}) + \mu([0, 1))$, and hence $\mu([0, 1]) = \delta(1) - \delta(0)$. Clearly, $\nabla^k\psi = \nabla^k\delta + (-1)^k a_k \nabla^k n^k$. If $k = 1$ then

$$\nabla^k\psi = -\mu(\{1\}) - \int\limits_{[0,1)} x^n d\mu(x) + a_1.$$

By Theorem 2.22, Cauchy dual $\mathcal{C}_1(\psi)$ of ψ of order 1 exists in $\mathcal{CM}_1$ if and only if

$$\mu(\{1\}) + \int\limits_{[0,1)} x^n d\mu(x) < a_1 \text{ for every } n \geq 0.$$

For instance, if one takes a_1 bigger than $\delta(1) - \delta(0)$ then Cauchy dual of order 1 exists in $\mathcal{CM}_1$.

If $k \geq 2$, then

$$\nabla^k\psi = -\int\limits_{[0,1)} x^n(1 - x)^{k-1}d\mu(x) + a_k k!,$$

where we used the fact that

$$\Delta^k n^k = k!\ (k \in \mathbb{N}). \tag{2.8}$$

We derive the formula (2.8) by induction on $k \in \mathbb{N}$. If we assume $\Delta^k n^k = k!$ for some $k \geq 1$, then

$$\Delta^{k+1}n^{k+1} = \Delta^k((n + 1)^{k+1} - n^{k+1}) = \Delta^k((k + 1)n^k + r(n)),$$

where r is a polynomial in n of degree less than k. By the fact stated in the discussion following (2.7) and induction hypothesis, we obtain

$$\Delta^{k+1}n^{k+1} = \Delta^k((k + 1)n^k) = (k + 1)\Delta^k n^k = (k + 1)!,$$

which completes the derivation of (2.8). This also implies that $\nabla^k n^k = (-1)^k k!$ for $k \in \mathbb{N}$. Thus Cauchy dual $\mathcal{C}_k(\psi)$ of order k exists in $\mathcal{CM}_k$ if and only

$$\int\limits_{[0,1)} x^n(1 - x)^{k-1}d\mu(x) < a_k k! \text{ for every } n \geq 0.$$

In particular, this happens if μ is supported at 1 and a_k is positive. In that case, both ψ and $\mathcal{C}_k(\psi)$ are polynomials.

Remark 2.25. Since any completely monotone function $\psi : \mathbb{N} \to (0, \infty)$ of order 1 is completely monotone and since any completely alternating sequence $\phi : \mathbb{N} \to (0, \infty)$ of order 1 is completely alternating (Corollary 2.17), the preceding example provides a transform, *different* from the Cauchy dual transform, which sends completely alternating sequence to a completely monotone sequence.

Example 2.26. Consider the sequence $\psi(n) = \frac{n(n+1)}{2} - (n+2)H(n+1)$, where $H(n)$ is nth partial sum of the harmonic series $\sum_{k=1}^{\infty} 1/k$. We already recorded in Example 2.4 that $\psi \in \mathcal{CA}_2$. Note that $\frac{1}{\nabla^2\psi} = \frac{n+3}{n+2}$ is a completely monotone sequence with representing measure being the sum of Dirac delta measure supported at 1 and the weighted Lebesgue measure xdx on $[0,1]$. It is easy to see that a Cauchy dual of order 2 of ψ is given by

$$\begin{aligned}
\phi(n) &= 1 + 2n + \binom{n}{2} + \int\limits_{[0,1)} \frac{1}{(1-x)^2}\,(x^n - 1 + n(1-x))\,xdx \\
&= 1 + 2n + \binom{n}{2} + \sum_{j=1}^{n-1} \frac{n-j}{j+1} \\
&= 1 + \binom{n}{2} + (n+1)H(n)
\end{aligned}$$

for every integer $n \geq 2$.

3. Operator theory

In Section 2, we developed the basic theory of completely alternating sequences of finite order as well as Bernstein functions of finite order. This theory motivates one naturally to introduce a new class of operators, referred in the sequel as completely hyperexpansive tuples of finite order. The properties of sequences in $\mathcal{CA}_k$ get mirrored into the operator theoretic properties of completely hyperexpansive tuples of order k resulting into a rewarding fusion of the two theories. Association with the function theory not only transforms function theoretic statements into operator theoretic ones, but also allows one to obtain new proofs of some known results in operator theory (see, for instance, Remark 3.14 and the discussion prior to Corollary 3.16 below).

3.1. Completely hyperexpansive tuples of finite order

Let T be a commuting m-tuple of operators $T_1, \cdots, T_m \in B(\mathcal{H})$, and let $Q_T(X) = \sum_{i=1}^m T_i^* X T_i$ ($X \in B(\mathcal{H})$) be the spherical generating 1-tuple associated with T. For an integer $q \geq 1$, let

$$B_q(Q_T) := \sum_{s=0}^{q} (-1)^s \binom{q}{s} Q_T^s(I). \tag{3.9}$$

Definition 3.1. Let k be a positive integer. We say that T is a *joint complete hyperexpansion of order* k if $B_q(Q_T) \leq 0$ for all $q \geq k$. A joint complete hyperexpansion of order 0 is same as *joint complete hyperexpansion*.

Remark 3.2. Note that T is a joint complete hyperexpansion of order k if and only if for every $h \in \mathcal{H}$, the sequence $\{\langle Q_T^n(I)h,\, h\rangle\}$ is completely alternating of order $k-1$. Applying Proposition 2.3 to the case in which $k = 2$, we get the known fact that joint complete hyperexpansion of order 2 is a joint complete hyperexpansion.

Example 3.3. We note that the direct sum of a joint k-isometry and a joint complete hyperexpansion is a complete hyperexpansion of order k.

Proposition 3.4. *Let T be a complete hyperexpansion of order k. Then the following statements are equivalent:*

(1) *T is a joint p-isometry.*
(2) *T is joint q-isometry with $q = \min\{k, p\}$.*

Proof. This follows from Corollary 2.6. $\square$

Example 3.5. Let $\{\beta_n\}$ be a completely alternating sequence of order $k-1$ such that $\beta_0 = 1$. Assume further that $\{\beta_{n+1}/\beta_n\}$ is bounded and let m be a positive integer. Consider the positive definite kernel given by

$$\kappa_\beta(z, w) = \sum_{n=0}^{\infty} \frac{\binom{n+m-1}{n}}{\beta_n} \langle z,\, w\rangle^n$$

defined for z, w in the open unit ball in $\mathbb{C}^m$. Consider the reproducing kernel Hilbert space $\mathscr{H}(\kappa_\beta)$ associated with κ_β and let $M_{z,\beta}$ denote the m-tuple of bounded linear multiplication operators $M_{z_1}, \cdots, M_{z_m}$ defined on $\mathscr{H}(\kappa_\beta)$. One may argue as in [10, Example 4.5] to see that $M_{z,\beta}$ is a complete hyperexpansion of order k.

Here is a rigidity theorem, which answers the question raised in the discussion following [9, Proposition 4.12].

Proposition 3.6. *Let T be a complete hyperexpansion of order k. Then the following statements are equivalent:*

(1) *T is a joint subnormal.*
(2) *T is a joint hyponormal.*
(3) *T is a joint isometry.*

Our proof of Proposition 3.6 relies on a number of observations. The first of which should be compared with [10, Lemma 4.6], where a complete hypercontraction of finite order may have left spectrum a proper subset of the closed unit ball.

Lemma 3.7. *The left spectrum of a complete hyperexpansion of finite order is contained in the unit sphere, and hence the Taylor spectrum is contained in the closed unit ball.*

Proof. The desired conclusions follow easily from the discussions before and after [16, Lemma 3.2] and the proof of [16, Lemma 3.2]. For the sake of completeness, we include the details. We recall the fact that for any tuple of Hilbert space operators, left spectrum and approximate spectrum coincide. The inclusion $\sigma_{ap}(T) \subseteq \sigma_l(T)$ is recorded in [14, Table 2.6] while the reverse inclusion may be deduced from the following fact: If $(S_1, \cdots, S_m)$ is bounded from below then $\sum_{j=1}^{m} (S_j^{\mathfrak{s}})^* S_j = I$, where $S^{\mathfrak{s}}$ is the spherical Cauchy dual of S. Let $\lambda \in \sigma_l(T)$. It is observed in the discussion prior to [16, Lemma 3.2] that there exists a sequence $\{x_n\}$ of unit vectors such that $\|T^\alpha x_n\| \to |\lambda^\alpha|$ for any $\alpha \in \mathbb{N}^m$. Now consider $\lim_{n\to\infty} \langle B_q(T)x_n, x_n\rangle \leq 0$ for even integer q to conclude that $\|\lambda\|_2 = 1$, that is, $\sigma_{ap}(T) \subseteq \partial\mathbb{B}$. Since the convex envelopes of $\sigma_{ap}(T)$ and $\sigma(T)$ coincide [14], the Taylor spectrum of T is contained in the closed unit ball. $\square$

Remark 3.8. By *spectral dichotomy*, we mean that $\sigma(T)$ is either part of the unit sphere or the entire closed unit ball. It turns out that a complete hyperexpansive m-tuple of finite order admits spectral dichotomy if and only if $m = 1$. The necessary part has been observed in [9, Example 3.9] while the sufficiency part may be concluded from the last lemma.

The essential part in our proof of Proposition 3.15 is Lemma 3.10, which is reminiscent of the well-known fact that norm and spectral radius are same for a hyponormal operator (see, for instance, [13, Proof of Proposition 4.6]). A basic tool in this proof is the following spectral radius formula for the Taylor spectrum ([20,12]; refer to the discussion prior to [16, Proposition 3.1]): Let T be a commuting m-tuple of operators in $B(\mathcal{H})$. Then

$$r(T) = \lim_{n\to\infty} \|Q_T^n(I)\|^{\frac{1}{2n}}. \tag{3.10}$$

In the proof of Lemma 3.10, we need a simple characterization of commuting tuples for which $r_l(T)$ coincides with the square-root of the norm of $Q_T(I)$.

Lemma 3.9. *Let T be a commuting m-tuple of operators $T_1, \cdots, T_m \in B(\mathcal{H})$. Then $r_l(T) = \|Q_T(I)\|^{1/2}$ if and only if $Q_T(I) \le r_l(T)^2 I$, where the radii $r(T)$ and $r_l(T)$ are as defined in (1.2).*

Proof. If $r_l(T) = \|Q_T(I)\|^{1/2}$ then for any h in $\mathcal{H}$,

$$\langle Q_T(I)h,\, h\rangle \le \|Q_T(I)\|\|h\|^2 = \langle r_l(T)^2 h,\, h\rangle,$$

and hence $Q_T(I) \le r_l(T)^2 I$.

Conversely, assume that $Q_T(I) \le r_l(T)^2 I$. It follows that $\|Q_T(I)\| \le r_l(T)^2$. On the other hand, by [22, Lemma 2.3], we have $r_l(T)^2 \le \|Q_T(I)\|$. For the sake of completeness, we provide an alternative proof of this fact. A simple inductive argument on $k \in \mathbb{N}$ shows that

$$Q_T^k(I) \le \|Q_T(I)\|^k I \text{ for every integer } k \ge 1.$$

Thus $\|Q_T^k(I)\| \le \|Q_T(I)\|^k I$, and hence by (3.10), we get $r(T)^2 \le \|Q_T(I)\|$. Since $r_l(T) \le r(T)$, we get $r_l(T) = \|Q_T(I)\|^{1/2}$. $\square$

Lemma 3.10. *Let T be a joint hyponormal m-tuple of operators $T_1, \cdots, T_m$ in $B(\mathcal{H})$. Then $r(T) = \|Q_T(I)\|^{1/2} = r_l(T)$.*

Proof. Let Q_T be as defined in (1.3). Since

$$Q_T^2(I) - Q_T(I)^2 = \sum_{i,j=1}^{m} T_i^*[T_j^*, T_i]T_j,$$

by the joint hyponormality of T, we obtain $Q_T(I)^2 \le Q_T^2(I)$ (see [9, Lemma 4.10] for more details). We claim that

$$\langle Q_T^n(I)h,\, h\rangle \le \langle Q_T^{n-1}(I)h,\, h\rangle^{\frac{1}{2}} \langle Q_T^{n+1}(I)h,\, h\rangle^{\frac{1}{2}} \ (h \in \mathcal{H}, n \ge 1). \tag{3.11}$$

Fix $h \in \mathcal{H}$, and note that for integer $n \ge 1$,

$$\begin{aligned}
\langle Q_T^n(I)h,\, h\rangle &= \langle Q_T^{n-1}(Q_T(I))h,\, h\rangle \\
&= \sum_{|\alpha|=n-1} \frac{(n-1)!}{|\alpha|!} \langle Q_T(I)T^\alpha h,\, T^\alpha h\rangle \\
&\le \left(\sum_{|\alpha|=n-1} \frac{(n-1)!}{|\alpha|!} \|Q_T(I)T^\alpha h\|^2 \right)^{1/2} \langle Q_T^{n-1}(I)h,\, h\rangle^{\frac{1}{2}},
\end{aligned}$$

by Cauchy–Schwarz inequality. To see (3.11), it now suffices to check that

$$\sum_{|\alpha|=n-1} \frac{(n-1)!}{|\alpha|!} \|Q_T(I)T^\alpha h\|^2 \leq \langle Q_T^{n+1}(I)h,\, h\rangle \ (h \in \mathcal{H}, n \geq 1). \tag{3.12}$$

Since $Q_T(I)^2 \leq Q_T^2(I)$, we obtain the inequality

$$\|Q_T(I)T^\alpha h\|^2 = \langle Q_T(I)^2 T^\alpha h,\, T^\alpha h\rangle \leq \langle Q_T^2(I)T^\alpha h,\, T^\alpha h\rangle.$$

It is now easy to deduce (3.12) from the identity

$$\sum_{|\alpha|=n-1} \frac{(n-1)!}{|\alpha|!} \langle Q_T^2(I)T^\alpha h,\, T^\alpha h\rangle = \langle Q_T^{n+1}(I)h,\, h\rangle.$$

This completes the proof of the claim. One may now argue as in the proof of [9, Proposition 4.9(i)] to see that

$$\|Q_T(I)\|^n \leq \|Q^n(I)\| \ (n \in \mathbb{N}).$$

By the spectral radius formula (3.10), we obtain $\|Q_T(I)\|^{1/2} \leq r(T)$. It follows that $Q_T(I) \leq r(T)^2 I$. Also, since the convex envelopes of $\sigma_l(T)$ and $\sigma(T)$ coincide [12, Theorem 1], we have $r_l(T) = r(T)$. The desired conclusion is now immediate from Lemma 3.9. □

Remark 3.11. A careful examination of the proof of Lemma 3.10 shows that its conclusion holds true for any commuting m-tuple T for which $Q_T(I)^2 \leq Q_T^2(I)$. Further, since any doubly commuting tuple of hyponormal operators is joint hyponormal, the main theorem of [11] can be recovered from the above result.

Proof of Proposition 3.6. The implication (1) implies (2) is well-known [3]. Also, the implication (3) implies (1) is precisely [4, Proposition 2]. Thus it suffices to check the implication (2) implies (3). So suppose that T is joint hyponormal. By Lemma 3.10, the spectral radius of T coincides with $\sqrt{\|Q_T(I)\|}$. It follows from Lemma 3.7 that $Q_T(I) \leq I$. One may now imitate the proof of [9, Lemma 4.3] to see that $Q_T(I) \geq I$. Thus T is necessarily a joint isometry. □

Corollary 3.12. *Any joint hyponormal, q-isometry tuple is necessarily a joint isometry.*

We note that the following lemma in dimension one has been obtained independently in [17, Theorem 2.5] by completely different means (cf. [16, Proposition 2.3]).

Lemma 3.13. *Let k be an even integer and let T be a commuting m-tuple of operators in $B(\mathcal{H})$. If $B_k(Q_T) \leq 0$ then $B_{k-1}(Q_T) \leq 0$.*

Proof. Note that $B_k(Q_T) \leq 0$ if and only if for every $h \in \mathcal{H}$ and for every $n \in \mathbb{N}$, $\nabla^k \psi_h(n) \leq 0$, where $\psi_h(n) := \langle Q_T^n(I)h,\, h\rangle$. The desired conclusion now follows from Remark 2.18. □

Remark 3.14. The case $m = 1$ and $k = 2$, usually known as Richter's Lemma, is precisely [21, Lemma 1].

Proposition 3.15. *Let k be an even integer. Then any complete hyperexpansion of order k is a complete hyperexpansion of order $k-1$. In particular, any joint k-isometry is a complete hyperexpansion of order $k-1$.*

Proof. We already recorded that T is a joint complete hyperexpansion of order k if and only if for every $h \in \mathcal{H}$, the sequence $\{\langle Q_T^n(I)h, h\rangle\}$ is completely alternating of order $k-1$. The desired conclusion is now immediate from the preceding lemma. □

We now discuss two applications of the preceding proposition to single variable operator theory. The first of which generalizes [1, Proposition 1.23].

Corollary 3.16. *Let k be an even integer and let T be a complete hyperexpansive operator of order k. If T is invertible then T is a $(k-1)$-isometry.*

Proof. Assume that T is invertible. Since $B_k(Q_T) \le 0$ and k is even, after applying $(T^*)^{-1}$ and T^{-1} from left and right respectively k times, we obtain $B_k(Q_{T^{-1}}) \le 0$. By Lemma 3.13, $B_{k-1}(Q_{T^{-1}}) \le 0$. Also, $B_{k-1}(Q_T) \le 0$ once again by Proposition 3.15. Since $k-1$ is odd, $B_{k-1}(Q_{T^{-1}}) \ge 0$. Thus we obtain $B_{k-1}(Q_{T^{-1}}) = 0$, and hence $B_{k-1}(Q_T) = 0$. □

Applying the preceding result to the image of T under the Calkin map, we immediately obtain the following Berger–Shaw-type result (cf. [1, Proposition 10.6] and [17, Theorem 4.6(a)]).

Corollary 3.17. *Let k be an even integer and let T be a complete hyperexpansive operator of order k. If T is Fredholm then T is essentially $(k-1)$-isometry, that is, the image of T under the Calkin map is a $(k-1)$-isometry.*

We next obtain an integral representation for the spherical generating tuples associated with completely hyperexpansive tuples of finite order.

Theorem 3.18. *Let T be a commuting m-tuple of bounded linear operators $T_1, \cdots, T_m$ on $\mathcal{H}$, and let Q_T denote the spherical generating 1-tuple associated with T. Then the following statements are equivalent:*

(1) *T is a complete hyperexpansion of order k.*
(2) *There exist a polynomial p_k of degree $k-1$ with coefficients in $B(\mathcal{H})$ and a semi-spectral measure E on $[0,1]$ such that for all non-negative integers n,*

$$Q_T^n(I) = p_k(n) + (-1)^{k-1}\binom{n}{k}E(\{1\}) + \int\limits_{[0,1)} \frac{\left(\sum_{j=0}^{k-1}\frac{(x-1)^j}{j!}(n)_j - x^n\right)}{(1-x)^k}dE$$

$$= p_k(n) + \int\limits_{[0,1]} \sum_{j=0}^{n-k}\left((-1)^{k-1}\frac{(x-1)^j}{(j+k)!}(n)_{j+k}\right)dE,$$

where the integral in the last expression is absent if $n < k$.

If this holds then the integral representation in (2) is unique in the sense that the coefficients of p_k and the semi-spectral measure $E(\cdot)$ are completely determined by T.

Proof. The desired conclusions are immediate from Theorem 2.5 and the polarization technique employed in [18, Theorem 4.2] (see the proof of [10, Theorem 4.11] for details). □

We will not discuss applications of Theorem 3.18 as most of the applications are straight-forward modifications of their counterparts as presented in [10] (for instance, see [10, Theorem 4.20]). In the remaining part of this section, we obtain an analogue of Theorem 1.2 for completely hyperexpansive tuples of finite order. The discussion to follow relies on Section 2.3.

Let $T : \{w_p^{(i)}\}_{p\in\mathbb{N}^m}$ be a joint left invertible (that is, $\sum_{j=1}^m T_j^* T_j$ is bounded from below) m-variable weighted shift with respect to the orthonormal basis $\{e_p\}_{p\in\mathbb{N}^m}$ of $\mathcal{H}$. Let β_p denote

$$\Big(\sum_{i=1}^m \Big(w_p^{(i)}\Big)^2\Big)^{\frac{1}{2}} \ (p \in \mathbb{N}^m).$$

Assume that the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting. It has been proved in the proof of [7, Lemma 3.3] that

$$\langle Q_s^l(I)e_q, e_q\rangle = \beta^2_{|q|\epsilon_1}\beta^2_{(|q|+1)\epsilon_1}\cdots\beta^2_{(|q|+l-1)\epsilon_1} \ (l \geq 1,\ q \in \mathbb{N}^m), \tag{3.13}$$

Remark 3.19. It may be concluded from the preceding discussion and Example 2.19 that the transform $T \longmapsto T^{\mathfrak{s}}$ need not send a complete hyperexpansive multishift of order k to complete hypercontractive multishift of order k unless $k = 1$.

Assume that T is completely hyperexpansive of order k such that $B_k(Q_T)$ is a positive invertible operator (see (3.9)). Let $n \in \mathbb{N}^m$ be fixed. Then ψ_n given by $\psi_n(0) = 1$ and

$$\psi_n(l) := \langle Q_T^l(I)e_n, e_n\rangle \ (l \geq 1)$$

is a completely alternating function of order k. Since $\langle B_k(Q_T)e_n, e_n\rangle > 0$, we have $\nabla^k\psi_n(l) > 0$. Hence, by Theorem 2.22, ψ_n admits a completely hypercontractive Cauchy dual ϕ_n of order k such that $\phi_n(0) = 1$. We define a *spherical Cauchy dual of order* k of T as the m-variable weighted shift $T^{\mathfrak{s},k} : \{\eta_p^{(i)}\}_{p\in\mathbb{N}^m}$ with weight sequence given by

$$\eta_p^{(i)} = \frac{w_p^{(i)}}{\beta_p}\sqrt{\frac{\phi_n(|p|+1)}{\phi_n(|p|)}} \ (1 \leq i \leq m,\ p \in \mathbb{N}^m).$$

It is easy to see from (3.13) that

$$\langle Q^l_{T^{\mathfrak{s},k}}(I)e_q, e_q\rangle = \frac{\phi_n(|q|+l)}{\phi_n(|q|)} \ (l \geq 1,\ q \in \mathbb{N}^m).$$

Also, since ϕ_n is completely monotone of order k, $T^{\mathfrak{s},k}$ is completely hypercontractive of order k. Thus we proved the following analogue of Theorem 1.2 for completely hyperexpansive shifts of finite order:

Proposition 3.20. *Let T be a joint left invertible, m-variable weighted shift and assume that the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting. Assume further that T is completely hyperexpansive of order k. If $B_k(Q_T)$ is a positive invertible operator then a spherical Cauchy dual $T^{\mathfrak{s},k}$ of order k is completely hypercontractive of order k.*

In view of the preceding result, it would be interesting to know whether there exists an operator transform (analogous to Shimorin's Cauchy dual as introduced in [24]), which will send completely hyperexpansive tuples of order k to completely hypercontractive tuples of order k. Also, since isometries are the only fixed points of this Cauchy dual transform, it is natural to expect that such a transform, if it exists, would send a k-isometry to itself.

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