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Rigidity Theorems for Spherical Hyperexpansions

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Abstract The class of spherical hyperexpansions is a multi-variable analog of the class of hyperexpansive operators with spherical isometries and spherical 2-isometries being special subclasses. It is known that in dimension one, an invertible 2-hyperexpansion is unitary. This rigidity theorem allows one to prove a variant of the Berger–Shaw Theorem which states that a finitely multi-cyclic 2-hyperexpansion is essentially normal. In the present paper, we seek for multi-variable manifestations of this rigidity theorem. In particular, we provide several conditions on a spherical hyperexpansion which ensure it to be a spherical isometry. We further carry out the analysis of the rigidity theorems at the Calkin algebra level and obtain some conditions for essential normality of a spherical hyperexpansion. In the process, we construct several interesting examples of spherical hyperexpansions which are structurally different from the Drury-Arveson m-shift.

Keywords Subnormal · Spherical p-isometry · Drury-Arveson m-shift · Spherical Cauchy dual · Defect operators · Essentially normal

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1 Introduction

The investigation of the multi-variable scenario in the context of subnormal and completely hyperexpansive operators has led to several interesting problems. In this note, we take up rigidity theorems (Theorems 1.1 and 1.2 below) in the 1-variable case for studying their possible incarnations in the multi-variable situation.

The study of completely hyperexpansive operators was initiated in [4,10]. The class of completely hyperexpansive operators is closely associated with the negative definite functions on the semigroup $\mathbb{N}$ of natural numbers and is in some sense antithetical to the class of subnormal contractions. For a masterful exposition on subnormal operators, the reader is referred to [19]. Another class of operators which arises naturally in the study of hyperexpansions is the class of m-isometric operators. These operators are systematically studied in [1–3]. It is well known that every 2-isometry is completely hyperexpansive and thus the Dirichlet shift is an important example of a completely hyperexpansive operator. It was observed in [39, Example 2.3] that for $1 \leq \lambda \leq 2$, the one-variable weighted shift $T_\lambda : \sqrt{(n+\lambda)/(n+1)}$ is completely hyperexpansive. This weighted shift, which we shall refer to as λ-*shift* can be judiciously used to construct interesting examples of operator tuples which are important in the context of the rigidity theorems under consideration (see Sect. 3).

We shall discuss two analogs of hyperexpansivity in higher dimensions viz. *toral* and *spherical*. The toral case is studied in [11] while the spherical case is initiated in [16] and further studied in [14]. It turns out that there are quite a few structural similarities between the Drury-Arveson 2-shift [5,25] and the classical Dirichlet shift. In particular, the Drury-Arveson 2-shift is a spherical 2-isometry [29, Theorem 4.2], and hence a spherical complete hyperexpansion [16, Proposition 4.9]. One may think of interpreting the Drury-Arveson 2-shift as the ‘spherical’ analog of the Dirichlet shift.

We now turn our attention to the following rigidity theorem proved in [39, Remark 3.4] which provides a strong motivation for the results in this paper:

Theorem 1.1 *Every invertible 2-hyperexpansive operator is unitary. In particular, the spectrum $\sigma(T)$ of a 2-hyperexpansion T admits the following spectral dichotomy:*

$$\sigma(T) = \overline{\mathbb{D}} \text{ or } \sigma(T) \subseteq \partial\mathbb{D},$$

where $\mathbb{D}$ denotes the open unit disc in the complex plane $\mathbb{C}$, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

A special case of Theorem 1.1, where T is a 2-isometry, is independently proved in [1, Proposition 1.23].

The class of m-isometries also enjoys the spectral dichotomy of Theorem 1.1 [1, Lemma 1.21]. It is further known that there exist invertible non-unitary m-isometries in abundance [1–3], and hence in general the first half of Theorem 1.1 does not hold true for m-isometries. However, Theorem 1.2 stated below provides a counterpart of Theorem 1.1 for m-isometries.

Theorem 1.2 *Every invertible, expansive m-isometric operator is unitary. In particular, the spectrum $\sigma(T)$ of an m-isometry T admits the following spectral dichotomy:*

$$\sigma(T) = \overline{\mathbb{D}} \text{ or } \sigma(T) \subseteq \partial\mathbb{D},$$

where $\mathbb{D}$ denotes the open unit disc in the complex plane $\mathbb{C}$.

The first half of Theorem 1.2 follows from the fact that expansive m-isometries are operators close to isometries in the sense of [13, Example 2.3]. This may also be deduced from [Proposition 4.4, Sect. 4]. Although Theorems 1.1 and 1.2 admit simple proofs, they have some important consequences. For instance, Theorem 1.1 plays a vital role in the derivation of the Wold-type decomposition for 2-hyperexpansions [13,38]. One may also use this result (resp. Theorem 1.2) to obtain a weaker version of the Berger–Shaw Theorem, which states that any 2-hyperexpansion (resp. expansive m-isometry) with finite-dimensional co-kernel admits a compact self-commutator [15].

By the *Berger–Shaw phenomenon*, we understand a result which states that under some finiteness condition (e.g. finite cyclicity, finite-dimensional co-kernel), the self-commutator is small (e.g. trace-class, compact) (refer to [12,26,32]). We find it necessary here to bring out the relation between the Berger–Shaw phenomenon for spherical hyperexpansions and the multi-variable analogs of Theorems 1.1 and 1.2.

Proposition 1.3 *Let $\mathscr{F}$ denote a family of spherical 2-hyperexpansive (resp. spherical p-isometric) m-tuples such that $\mathscr{F}$ is invariant under unital $*$-representations. Assume further that if $T \in \mathscr{F}$ is Taylor invertible then T is a spherical unitary. Then every Fredholm member of $\mathscr{F}$ is essentially spherical unitary.*

Remark 1.4 In case $m = 1$, every finitely multi-cyclic member of $\mathscr{F}$ turns out to be Fredholm.

We do not include the proof of Proposition 1.3 in this paper. However, we note that it can be obtained using a technique of Agler and Stankus (see Sect. 5 for the details). It is a challenging problem to find a family $\mathcal{F}$ satisfying the hypothesis of Proposition 1.3 in dimension bigger than 1 if it exists. In fact, in the next section, we observe that the family of spherical 2-expansions does not admit the rigidity property stated in Proposition 1.3. This is nothing but a reflection of the failure of the Berger–Shaw phenomenon. This failure also demands a modification of Proposition 1.3. To state that we need some nomenclature.

Definition 1.5 Let $\mathscr{F}$ be a family of operator tuples. We say that $\mathscr{F}$ is a *Berger–Shaw family of type $\mathcal{U}$* (resp. a *Berger–Shaw family of type $\mathcal{I}$*) if

1. $\mathscr{F}$ is invariant under unital $*$-representations,
2. there exists a *rigidity condition* R_u *(resp. R_i)* such that the following happens: If $T \in \mathscr{F}$ satisfies R_u (resp. R_i) then T is a spherical unitary (resp. a spherical isometry).

Example 1.6 The family of 2-hyperexpansive operators (resp. expansive m-isometries) is a Berger–Shaw family of type $\mathcal{U}$. Indeed, in view of Theorem 1.1 (resp. Theorem 1.2), one may take the rigidity condition R_u as the usual invertibility of operators.

Proposition 1.7 *Suppose $\mathscr{F}$ is a Berger–Shaw family of type $\mathcal{U}$ (resp. $\mathcal{I}$) with the rigidity condition R_u (resp. R_i). If the image of T under the Calkin map satisfies R_u (resp. R_i) then T is essentially spherical unitary (resp. essentially spherical isometry).*

Although, we do not use Definition 1.5 or Proposition 1.7 in the remaining part of this paper, our ploy essentially is to find Berger–Shaw families of spherical hyperexpansions.

The paper is organized as follows. In Sect. 2, we introduce the notation and terminology. In Sect. 3, we present examples of spherical hyperexpansions which include two intrinsically different varieties of such operator tuples. The main results of the paper are discussed in Sects. 4 and 5. Section 4 deals with the rigidity theorems in higher dimensions providing several conditions on a spherical hyperexpansion which ensure it to be a spherical isometry. In Sect. 5, we prove some rigidity theorems at the Calkin algebra level obtaining conditions for essential normality of a spherical hyperexpansion. We conclude the paper with a brief discussion of possible lines of investigations.

2 Preliminaries

For a Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$. If $T := (T_1, \ldots, T_m)$ is a tuple of commuting bounded linear operators T_i $(1 \leq i \leq m)$ on $\mathcal{H}$ then we interpret T^* to be $(T_1^*, \ldots, T_m^*)$ and for $p = (p_1, p_2, \ldots, p_m) \in \mathbb{N}^m$, T^p to be $T_1^{p_1} \cdots T_m^{p_m}$.

Definition 2.1 A commuting m-tuple $Q := (Q_1, \ldots, Q_m)$ of positive, bounded, linear operators $Q_1, \ldots, Q_m$ acting on $B(\mathcal{H})$ is called as the *generating m-tuple* on $\mathcal{H}$.

The study of the hyperexpansive and hypercontractive generating m-tuples is initiated in [16]. In the present paper, we are mainly interested in the spherical generating 1-tuples. We recall a few definitions for ready reference.

Definition 2.2 Given a commuting m-tuple $T = (T_1, \ldots, T_m)$ of operators on $\mathcal{H}$, the *spherical generating* 1*-tuple associated with T* is given by

$$Q_s(X) := \sum_{i=1}^{m} T_i^* X T_i \ (X \in B(\mathcal{H})).$$

Definition 2.3 Fix an integer $p \geq 1$. We say that T is a *spherical p-expansion (resp. spherical p-contraction*) if

$$B_p(Q_s) := \sum_{q \in \mathbb{N}, 0 \leq q \leq p} (-1)^q \binom{p}{q} Q_s^q(I) \leq 0 \text{ (resp. } \geq 0), \tag{2.1}$$

where $Q_s^0(I) = I$.

We say that T is a *spherical p-hyperexpansion* (resp. *spherical p-hypercontraction*) if T is a spherical k-expansion (resp. spherical k-contraction) for all $k = 1, \ldots, p$. If equality occurs in (2.1), then T is a *spherical p-isometry*. We say that T is a *spherical complete hyperexpansion* (resp. *spherical complete hypercontraction*) if T is a spherical p-expansion (resp. spherical p-contraction) for all positive integers p. In all the above definitions, if $p = 1$ then we drop the prefix 1- and if $m = 1$ then we drop the term spherical.

Let Q_s be the spherical generating 1-tuple associated with T. Then T is *jointly left-invertible* if there exists $\alpha > 0$ such that $Q_s(I) \geq \alpha I$.

Definition 2.4 Let T be a jointly left-invertible m-tuple of bounded linear operators on $\mathcal{H}$. The *spherical Cauchy dual* of T is defined to be the m-tuple $T^{\mathfrak{s}} := (T_1^{\mathfrak{s}}, \ldots, T_m^{\mathfrak{s}})$, where $T_i^{\mathfrak{s}} := T_i(Q_s(I))^{-1}$ $(i = 1, \ldots, m)$.

For the basic theory of spherical Cauchy dual tuples, the reader is referred to [14, 16]. In particular, we note that a commuting spherical Cauchy dual of a spherical completely hyperexpansive multi-variable weighted shift is always subnormal [14, Proposition 3.4]. Thus the notion of the spherical Cauchy dual tuple allows one to think of the theory of spherical complete hyperexpansions as an antithesis of that of the subnormal tuples.

Recall that an m-tuple $S = (S_1, \ldots, S_m)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an m-tuple $N = (N_1, \ldots, N_m)$ of commuting normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that $N_i h = S_i h$ for every $h \in \mathcal{H}$ and $1 \leq i \leq m$.

A commuting m-tuple $T = (T_1, \ldots, T_m)$ is called a *spherical unitary* if both T and T^* are spherical isometries. Since a spherical isometry is subnormal [8, Proposition 2], T is a spherical unitary if and only if T is a normal spherical isometry. By a similar argument, it can be seen that a spherical isometry on a finite-dimensional Hilbert space is necessarily a spherical unitary.

We record here some elementary but useful facts pertaining to spherical hyperexpansions.

1. Every spherical 2-expansion is a spherical 2-hyperexpansion [16, Proposition 4.1(i)]. This is a spherical analog of the Richter's Lemma [36, Lemma 1(a)].
2. The spherical Cauchy dual of a spherical 2-expansion is well-defined.
3. A normal spherical 2-expansion (resp. spherical p-isometry) is a spherical unitary.
4. The restriction $T|_{\mathcal{M}}$ of a spherical 2-expansion (resp. a spherical p-isometry) $T = (T_1, \ldots, T_m)$ to an invariant subspace $\mathcal{M}$ (that is, $T_i\mathcal{M} \subseteq \mathcal{M}$ for all $i = 1, \ldots, m$) is again a spherical 2-expansion (resp. a spherical p-isometry).

Definition 2.5 Given a commuting m-tuple $T = (T_1, \ldots, T_m)$ on $\mathcal{H}$, a *toral generating m-tuple* is given by

$$Q_t := (Q_1, \ldots, Q_m), \;\; Q_i(X) := T_i^* X T_i \;\; (X \in B(\mathcal{H})).$$

We say that T is *toral complete hyperexpansion* if

$$B_n(Q_t) := \sum_{p \in \mathbb{N}^m, 0 \leq p \leq n} (-1)^{|p|} \binom{n}{p} Q_t^p(I) \leq 0 \quad \text{for all } n \in \mathbb{N}^m \setminus \{0\}, \tag{2.2}$$

where $Q_t^q(I) = (Q_1^{q_1} \circ \cdots \circ Q_m^{q_m})(I)$ for $q = (q_1, \ldots, q_m) \in \mathbb{N}^m$.

Similarly, one may define *toral* p*-isometry* and *toral* p*-expansion*.

For the basic theory of toral complete hyperexpansions, refer to [11].

Throughout this paper, we will frequently deal with the multi-sequence $\{Q^n(I)\}_{n \in \mathbb{N}^m}$ associated with a generating m-tuple Q. Note that the multi-sequence associated with the toral generating m-tuple Q_t is given by $\{T^{*n}T^n\}_{n \in \mathbb{N}^m}$. On the other hand, the multi-sequence associated with the spherical generating 1-tuple Q_s is given by

$$\left\{ \sum_{p \in \mathbb{N}^m, |p| = n} \frac{n!}{p!} T^{*p} T^p \right\}_{n \in \mathbb{N}}.$$

3 Examples

An m-variable weighted shift $T = (T_1, \ldots, T_m)$ with respect to an orthonormal basis $\{e_n\}_{n \in \mathbb{N}^m}$ of a Hilbert space $\mathcal{H}$ is defined by

$$T_i e_n := w_n^{(i)} e_{n+\epsilon_i} \ (1 \leq i \leq m),$$

where ϵ_i is the m-tuple with 1 in the ith place and zeros elsewhere. We indicate the m-variable weighted shift operator T with weight sequence

$$\left\{ w_n^{(i)} : 1 \leq i \leq m, n \in \mathbb{N}^m \right\}$$

by $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$. We always assume that $\left\{ w_n^{(i)} : 1 \leq i \leq m, n \in \mathbb{N}^m \right\}$ is a bounded subset of the positive real line.

Notice that T_i commutes with T_j if and only if $w_n^{(i)} w_{n+\epsilon_i}^{(j)} = w_n^{(j)} w_{n+\epsilon_j}^{(i)}$ for all $n \in \mathbb{N}^m$.

A rather special example of a spherical m-isometry is the *Drury-Arveson* m*-shift* ([5,25], [29, Theorem 4.2]). The Drury-Arveson m-shift is the operator m-tuple $M_{z,m}$ of multiplication by the co-ordinate functions $z_1, \ldots, z_m$ in the reproducing kernel Hilbert space associated with the positive definite kernel

$$\frac{1}{1 - z_1 \overline{w}_1 - \cdots - z_m \overline{w}_m} \ (z, w \in \mathbb{B}),$$

where $\overline{u}$ denotes the complex conjugate of the complex number u, and $\mathbb{B}$ denotes the open unit ball in the m-dimensional hermitian space $\mathbb{C}^m$. The operator m-tuple $M_{z,m}$

can also be realized as the weighted shift with weight-sequence

$$\left\{\sqrt{\frac{n_i+1}{|n|+1}} : 1 \le i \le m,\ n \in \mathbb{N}^m\right\},$$

where $|n| := n_1 + \cdots + n_m$ ($n \in \mathbb{N}^m$). It may be concluded from [16, Proposition 4.9] that the Drury-Arveson m-shift is a spherical complete hyperexpansion if and only if $m = 2$.

Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ be an m-variable weighted shift. Let $\beta_n(T)$ denote

$$\left(\sum_{i=1}^{m}\left(w_n^{(i)}\right)^2\right)^{\frac{1}{2}} \quad (n \in \mathbb{N}^m).$$

We refer to the one-variable weighted shift $T_\beta : \{\beta_{k\epsilon_1}\}_{k\in\mathbb{N}}$ as the *shift associated with* T.

The following lemma is borrowed from [14, Lemmas 3.1 and 3.3] for ready reference.

Lemma 3.1 *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ denote a commuting m-variable weighted shift and let $T^{\mathfrak{s}}$ denote its spherical Cauchy dual m-tuple. Then $T^{\mathfrak{s}}$ is commuting if and only if $\{\beta_n(T)\}_{n\in\mathbb{N}^m}$ satisfies*

$$\beta_n(T) = \beta_{|n|\epsilon_1}(T) \quad \text{for all } n \in \mathbb{N}^m,$$

where $|n| = n_1 + \cdots + n_m$. If this condition holds then T is a spherical p-expansion (resp. spherical p-isometry) if and only if T_β is a p-expansion (resp. p-isometry), where T_β stand for the shift associated with T.

We now present two examples of m-variable weighted shifts, each of which include the Drury-Arveson m-shift as a special case.

Example 3.2 For $\lambda \ge 1$, the m-variable weighted shift with weight sequence

$$\sqrt{\frac{n_i+1}{|n|+m}}\sqrt{\frac{|n|+\lambda}{|n|+1}} \quad (n \in \mathbb{N}^m, 1 \le i \le m)$$

is denoted by $T_{\lambda,m}$. The following special cases are noteworthy:

1. $\lambda = 1$: Notice that $T_{1,m}$ is nothing but the $Szeg$ö $m-shift$ with weight sequence

$$\left\{\sqrt{\frac{n_i+1}{|n|+m}} : 1 \le i \le m, n \in \mathbb{N}^m\right\}.$$

2. $\lambda = 2$: Notice that $T_{2,m}$ is the m-variable weighted shift with weight sequence

$$\left\{ \sqrt{\frac{n_i + 1}{|n| + m}} \sqrt{\frac{|n| + 2}{|n| + 1}} : 1 \leq i \leq m, n \in \mathbb{N}^m \right\}.$$

Further if $m = 2$, then note that $T_{2,m}$ is nothing but the Drury-Arveson 2-shift.

3. $\lambda = m$: Notice that $T_{m,m}$ is the Drury-Arveson m-shift.

Notice that the shift associated with $T_{\lambda,m}$ is precisely the λ-shift T_λ as discussed in Sect. 1. By Lemma 3.1, $T_{\lambda,m}$ is a spherical complete hyperexpansion if and only if T_λ is a complete hyperexpansion. The latter one is true if and only if $1 \leq \lambda \leq 2$ [39, Example 2.3]. It is easy to conclude from Lemma 3.1 and [9, Proposition 8] that $T_{\lambda,m}$ is a spherical p-isometry if and only if $\lambda = p$ is a positive integer.

Example 3.3 We use the notation $T_{\lambda \cdot m}$ for the m-variable weighted shift with weight sequence

$$\sqrt{\frac{n_i + \lambda}{|n| + 1}} \ (n \in \mathbb{N}^m, 1 \leq i \leq m),$$

where λ is a positive real number. Notice that $T_{1 \cdot m}$ is the Drury-Arveson m-shift. Observe that the shift associated with $T_{\lambda \cdot m}$ is precisely the (λm)-shift $T_{\lambda m}$. Again by Lemma 3.1, $T_{\lambda \cdot m}$ is a spherical complete hyperexpansion if and only if $T_{\lambda m}$ is a complete hyperexpansion. The latter one is true if and only if $1/m \leq \lambda \leq 2/m$ [39, Example 2.3]. Similarly, one may conclude from [9, Proposition 8] that $T_{\lambda \cdot m}$ is a spherical p-isometry if and only if $\lambda m = p$ is a positive integer. In particular, $T_{(p/m) \cdot m}$ is a spherical p-isometry.

For an excellent account on various notions of invertibility, Fredholmness and multiparameter spectral theory, the reader is referred to [21]. For $T \in B(\mathcal{H})$, we reserve the symbols $\sigma(T), \sigma_p(T), \sigma_{ap}(T), \sigma_e(T)$ for the Taylor spectrum, point-spectrum, approximate-point spectrum, essential spectrum of T respectively.

The following result records some elementary spectral properties of spherical 2-hyperexpansions [16, Corollary 4.2] and spherical p-isometries [29, Proposition 3.1 and Lemma 3.2]:

Proposition 3.4 *Let T be a spherical 2-expansive (resp. a spherical p-isometric) m-tuple. Then the following statements are true:*

(i) The approximate-point spectrum of T is contained in the boundary of the closed unit ball in $\mathbb{C}^m$. In particular, the Taylor spectrum of T is contained in the closed unit ball in $\mathbb{C}^m$.

(ii) The spectral radius $\sup\{\|z\| : z \in \sigma(T)\}$ *of T is* 1, *where* $\|z\|^2 := |z_1|^2 + \cdots + |z_m|^2$ *for* $z := (z_1, \ldots, z_m) \in \mathbb{C}^m$.

Corollary 3.5 *Let* $T : \{w_n^{(i)}\}$ *be a spherical* 2*-expansive (resp. a spherical* p*-isometric)* m*-variable weighted shift. If*

$$\inf_{n\in\mathbb{N}^m} \sqrt{\left(w^{(1)}_{n-\epsilon_1}\right)^2 + \cdots + \left(w^{(m)}_{n-\epsilon_m}\right)^2} = 1, \tag{3.3}$$

with the interpretation that $w^{(i)}_{n-\epsilon_i} = 0$ *if* $n_i = 0$, *then the Taylor spectrum of* T *is the closed unit ball in* $\mathbb{C}^m$.

Proof In view of (3.3), one may conclude from [22, Corollary 4.3] that $\mathbb{B} \subseteq \sigma_p(T^*)$, where $\mathbb{B}$ denotes the open unit ball in $\mathbb{C}^m$. However, since $\sigma_p(T^*) \subseteq \sigma(T^*)$ and $\sigma(T^*) = \{\overline{z} : z \in \sigma(T)\}$, we have $\mathbb{B} \subseteq \sigma(T)$, where $\overline{z}$ denotes the componentwise complex conjugate of $z \in \mathbb{C}^m$. Also, since $\sigma(T)$ is closed, we must have $\overline{\mathbb{B}} \subseteq \sigma(T)$. By the preceding proposition, $\sigma(T) = \overline{\mathbb{B}}$. □

Remark 3.6 Let $T_{\lambda,m}$ and $T_{\lambda\cdot m}$ be as in Examples 3.2 and 3.3 respectively. It is easy to see that $T_{\lambda,m}$ and $T_{\lambda.m}$ satisfy the condition (3.3) above. Consequently, we obtain the following:

$\lambda \geq m$: If $1 \leq \lambda \leq 2$ or λ is a positive integer then $\sigma(T_{\lambda,m}) = \overline{\mathbb{B}}$.
$\lambda \geq 1$: If $1/m \leq \lambda \leq 2/m$ or λm is a positive integer then $\sigma(T_{\lambda\cdot m}) = \overline{\mathbb{B}}$.

In particular, the Taylor spectrum of the Drury-Arveson m-shift is the closed unit ball in $\mathbb{C}^m$.

The following proposition describes a method of constructing a spherical hyperexpansion by using a toral hyperexpansion:

Proposition 3.7 *Let* $T := (T_1, \ldots, T_m)$ *be an* m*-tuple on* $\mathcal{H}$ *and set* $S := (T_1/\sqrt{m}, \ldots, T_m/\sqrt{m})$. *If* T *is a toral complete hyperexpansion (resp. toral* p*-expansion resp. toral* p*-isometry) then* S *is a spherical complete hyperexpansion (resp. spherical* p*-expansion resp. spherical* p*-isometry).*

Proof Let $Q_t := (Q_1, \ldots, Q_m)$ denote the toral generating m-tuple associated with T. Let Q (resp. P) denote the spherical generating 1-tuple associated with T (resp. S). Note that for $X \in B(\mathcal{H})$, $P(X) = \frac{1}{m}Q(X)$ and $Q(X) = \sum_{i=1}^m Q_i(X)$. We contend that

$$B_n(P) = \frac{1}{m^n} \sum_{|p|=n} B_p(Q_t) \ (n \in \mathbb{N}),$$

where $B_n(\cdot)$ and $B_p(\cdot)$ be as defined in (2.2) of Sect. 2. We prove this by induction on $n \geq 1$. If $n = 1$ then

$$B_1(P) = I - P(I) = \frac{1}{m}(mI - Q(I)) = \frac{1}{m}\sum_{i=1}^m [I - Q_i(I)] = \frac{1}{m^1}\sum_{|p|=1} B_p(Q_t).$$

Next suppose that the desired identity is true for some $n \geq 1$. Observe that

$$\begin{aligned} B_{n+1}(P) &= B_n(P) - P(B_n(P)) \\ &= \frac{1}{m^n} \sum_{|p|=n} B_p(Q_t) - P\left(\frac{1}{m^n} \sum_{|p|=n} B_p(Q_t)\right) \\ &= \frac{1}{m^n} \sum_{|p|=n} \left[B_p(Q_t) - \frac{1}{m} Q(B_p(Q_t))\right] \\ &= \frac{1}{m^{n+1}} \sum_{|p|=n} \sum_{i=1}^{m} B_{p+\epsilon_i}(Q_t), \end{aligned}$$

where ϵ_i is the m-tuple with 1 in the i-th entry and zeros elsewhere. The required result is now immediate. □

Note that the converse of Proposition 3.7 is false as seen by considering the Drury-Arveson 2-shift.

Example 3.8 Let T be a complete hyperexpansion (resp. p-isometry). It is easy to see that the m-tuple $(T, \ldots, T)$ is a toral complete hyperexpansion (resp. toral p-isometry). Now by Proposition3.7, $(T/\sqrt{m}, \ldots, T/\sqrt{m})$ is a spherical complete hyperexpansion (resp. spherical p-isometry). In particular if $1 \leq \lambda \leq 2$ (resp. $\lambda = p$ a positive integer), then the m-tuple $(T_\lambda/\sqrt{m}, \ldots, T_\lambda/\sqrt{m})$ is a spherical complete hyperexpansion (resp. spherical p-isometry). For instance if D denotes the Dirichlet shift T_2, then $(D/\sqrt{2}, D/\sqrt{2})$ is a spherical 2-isometry.

The following example is crucial in the context of possible generalization of Theorems 1.1 and 1.2 in higher dimensions. In fact, it shows that their verbatim analogs fail in dimension bigger than one.

Example 3.9 By Proposition 3.7, $S = (T_\lambda/\sqrt{2}, U/\sqrt{2})$ is spherical complete hyperexpansion (resp. spherical p-isometry) for $1 \leq \lambda \leq 2$ (resp. $\lambda = p$ a positive integer), where U a unitary operator which commutes with the λ-shift T_λ. Since the weight-sequence of T_λ converges to 1, the spectrum of T_λ is contained in the closed unit disc $\bar{\mathbb{D}}$ [37]. Now by the projection property for the Taylor spectrum [21, Theorem 4.9], $\sigma(S) \subseteq \frac{\bar{\mathbb{D}}}{\sqrt{2}} \times \frac{\partial\mathbb{D}}{\sqrt{2}}$. Consequently, S is Taylor invertible.

It can be easily checked that for $\lambda > 1$, S is not a spherical isometry.

We would like to point out some distinctive features of the examples discussed above.

Remark 3.10 1. The Taylor spectrum of $(D/\sqrt{2}, D/\sqrt{2})$ is contained in the bidisc

$$\{(z, w) \in \mathbb{C}^2 : |z| \leq 1/\sqrt{2}, |w| \leq 1/\sqrt{2}\},$$

whereas the Taylor spectrum of the Drury-Arveson 2-shift is the entire closed unit ball (see Remark 3.6 above).

2. The components of the Drury-Arveson 2-shift are non-isometric subnormal contractions [31, Proposition 4]. On the contrary, the components of a spherical complete hyperexpansion constructed via Proposition 3.7 above are subnormal if and only if they are scalar multiples of isometries. This follows from the fact that a subnormal 2-hyperexpansion is necessarily an isometry. In the next section, we shall prove however that the subnormality and joint hyponormality of a spherical 2-hyperexpansion are equivalent to it being a spherical isometry.
3. For the subclass of spherical 2-hyperexpansions S, as discussed in Proposition 3.7, there is an easy analog of Theorems 1.1: If the Taylor spectrum of S is contained in the ball-shell

$$\{(z, w) \in \mathbb{C}^2 : 0 < |z| \leq 1/\sqrt{2}, 0 < |w| \leq 1/\sqrt{2}\}$$

then S is a spherical unitary. This is immediate from the projection property of the Taylor spectrum [21] and Theorem 1.1. Similar assertion holds for the class of spherically expansive spherical p-isometries discussed in Proposition 3.7.
4. It was noted in [9, Proposition 8] that for $\lambda = p$, an integer, the λ-shift T_λ is a p-isometry but not a $(p-1)$-isometry. Thus for every integer $p \geq 2$, the operator $S = (T_p/\sqrt{2}, U/\sqrt{2})$ of Example 3.9 is a Taylor invertible spherical p-isometry which is not a spherical $(p-1)$ isometry.

   Note that in dimension 1, it is well known that if T is an invertible m-isometry and m is even, then T is an $(m-1)$-isometry [1, Proposition 1.23]. Thus the spherical analog of this result is no longer true in higher dimensions.

4 Some Rigidity Theorems

A multi-variable analog of Theorem 1.1 (resp. Theorem 1.2) would require an invertible spherical 2-hyperexpansion (resp. spherically expansive spherical p-isometry) to be a spherical unitary. The notion of invertibility of an operator tuple has different manifestations; Taylor invertibility being most profound.

As already pointed out, with Taylor invertibility, the verbatim analog of Theorem 1.1 (resp. Theorem 1.2) fails in higher dimensions. In fact, it has been well-known that there are examples of Taylor invertible spherical isometries which are not spherical unitaries [28, Theorem 3.1]. Further, Example 3.9 shows that an invertible spherical 2-hyperexpansion may not even be a spherical isometry and as a special case, if D denotes the Dirichlet shift, then $S = (D/\sqrt{2}, I/\sqrt{2})$ is a Taylor-invertible spherical 2-isometry, which is not a spherical isometry. Note that this example also reveals that the spectral dichotomy of Theorems 1.1 and 1.2 does not hold as $\sigma_p(S^*) \cap \mathbb{B}$ (and hence $\sigma(S) \cap \mathbb{B}$) is a non-empty set that excluds the origin.

At this stage we introduce a notion of structural invertibility of an operator tuple, suitable for our investigations, and prove a multi-variable analog of Theorem 1.1 with invertibility interpreted as the structural invertibility.

Definition 4.1 Let $T = (T_1, \ldots, T_m)$ be a commuting m-tuple and Q_s be the spherical generating 1-tuple associated with T. We say that T is *structurally invertible* if there exists a commuting m-tuple $S = (S_1, \ldots, S_m)$ with associated spherical generating 1-tuple P_s such that

$$Q_s^k \circ P_s^k(I) = I = P_s^k \circ Q_s^k(I) \quad \text{for every positive integer } k.$$

We refer to S as a *structural inverse* of T.

Clearly, a spherical isometry is structurally invertible. Further, its structural inverse is again a spherical isometry. Furthermore, if a bounded linear operator is invertible then it is structurally invertible.

Proposition 4.2 *Let T be a spherical 2-expansive structurally invertible m-tuple on $\mathcal{H}$. Then T is a spherical isometry.*

Proof Let S denote a structural inverse of T. Let P_s and Q_s denote the spherical generating 1-tuples associated with the commuting m-tuples S and T respectively.

By the linearity of P_s and the definition of the structural inverse,

$$P_s^2(I - 2Q_s(I) + Q_s^2(I)) = P_s^2(I) - 2P_s(I) + I.$$

Now the fact that P_s sends negative elements to negative elements implies that S is a spherical 2-expansion, and hence a spherical expansion (see the discussion prior to Definition 2.5). Consequently, $I = Q_s \circ P_s(I) \geq Q_s(I)$, so that $Q_s(I) = I$. Hence T is a spherical isometry as desired. □

To obtain a multi-variable counterpart of Theorem 1.2, we need a lemma.

Lemma 4.3 *Let T be a spherical p-isometric m-tuple on $\mathcal{H}$. Then T is a spherical contraction if and only if it is a spherical isometry.*

Proof Let T be a spherical contraction and a spherical p-isometry for $p \geq 2$. We claim that T is a spherical $(p-1)$-isometry. Let Q_s denote the spherical generating 1-tuple associated with T. Since T is a spherical contraction, the sequence $\{Q_s^k(I)\}_{k \in \mathbb{N}}$ of positive contractions is monotonically non-increasing. It follows that $\{Q_s^k(I)\}_{k \in \mathbb{N}}$ converges in the strong operator topology to a positive operator, say $A \in B(\mathcal{H})$. Let $B_n(\cdot)$ be as in (2.1) of Sect. 2. Since

$$0 = B_p(Q_s) = B_{p-1}(Q_s) - Q_s(B_{p-1}(Q_s)),$$

by an inductive argument,

$$B_{p-1}(Q_s) = Q_s^k(B_{p-1}(Q_s)), \ k \in \mathbb{N}.$$

It suffices now to check that $\{Q_s^k(B_{p-1}(Q_s))\}_{k \in \mathbb{N}}$ converges in the strong operator topology to the zero operator. Note that

$$Q_s^k(B_{p-1}(Q_s)) = \sum_{q \in \mathbb{N}, 0 \leq q \leq p-1} (-1)^q \binom{p}{q} Q_s^{q+k}(I) \longrightarrow \sum_{q \in \mathbb{N}, 0 \leq q \leq p-1} (-1)^q \binom{p}{q} A$$

in the strong operator topology. Since $p \geq 2$, the claim stands verified. By a finite inductive argument, T is a spherical isometry. □

Proposition 4.4 *Let T be a spherical p-isometric structurally invertible m-tuple on $\mathcal{H}$. If T is a spherical expansion then it is a spherical isometry.*

Proof Assume that T is a spherical expansion. Let S denote a (structural) inverse of T and let $B_n(\cdot)$ be as in (2.1) of Sect. 2. It is easy to see that $P_s^p(B_p(Q_s)) = (-1)^p B_p(P_s)$, where Q_s (resp P_s) denotes the spherical generating 1-tuple associated with T (resp. S). In particular, the structural inverse S of T is a spherical p-isometry. Since T is a spherical expansion, S is a spherical contraction. By the previous lemma, S must be a spherical isometry. It follows that T is also a spherical isometry. □

We now introduce a notion of defect operators which provides a suitable language for the discussion that follows in the sequel, and also measures in some sense the deviation of a tuple from it being a spherical isometry.

Definition 4.5 Let T be an m-tuple and let Q_s be the spherical generating 1-tuple associated with T. The *defect operator* $D_{T,n}$ is given by

$$D_{T,n} := Q_s^n(I) - Q_s(I)^n \ (n \geq 2).$$

Remark 4.6 Clearly, the defect operator $D_{T,n}$ are unitary invariants of T. Note that for a spherical isometry (or a normal tuple) T, all the defect operators $D_{T,n}$ are 0. Conversely, if all the defect operators $D_{T,n}$ are 0, then by the Spectral Theorem, there exists a spectral measure $E(\cdot)$ such that

$$Q_s^n(I) = \int_{[0,\|Q_s(I)\|]} t^n dE(t) \ (n \geq 1).$$

If, in addition, T is a spherical contraction, then T is a spherical complete hypercontraction.

Lemma 4.7 *Let T be a spherical 2-expansive m-tuple. Then we have:*

(i) $D_{T,2} \leq 0$.
(ii) $D_{T,3} \leq 0$.

Proof Let Q_s be the spherical generating 1-tuple associated with T.

The first part follows from

$$(I - Q_s(I))^2 + Q_s^2(I) - (Q_s(I))^2 = I - 2Q_s(I) + Q_s^2(I) \leq 0.$$

The proof of (ii) capitalizes on the notion of the spherical Cauchy dual tuple. Let P_s denote the spherical generating 1-tuple associated with the spherical Cauchy dual $T^{\mathfrak{s}}$

of T. Notice that the first half of [16, Theorem 6.6] may be rephrased as $P_s \circ Q_s(I) \leq I$. It follows that

$$\begin{aligned}(P_s(I)^{\frac{1}{2}} - P_s(I)^{-\frac{1}{2}})^2 + P_s \circ Q_s^2(I) - P_s(I)^{-1} \\ = P_s(I) - 2I + P_s \circ Q_s^2(I) \\ \leq P_s(I) - 2P_s \circ Q_s(I) + P_s \circ Q_s^2(I) \\ = P_s(I - 2Q_s(I) + Q_s^2(I)).\end{aligned}$$

By the spherical 2-expansivity of T and the positivity of P_s, one has

$$P_s \circ Q_s^2(I) \leq P_s(I)^{-1} = Q_s(I). \tag{4.4}$$

But $P_s \circ Q_s^2(I) = Q_s(I)^{-1} Q_s^3(I) Q_s(I)^{-1}$, which yields the desired result. □

Proposition 4.8 *Let T be a spherical 2-expansive m-tuple. Then T is a spherical isometry if and only if T satisfies any one of the following:*

(i) $D_{T,2} \geq 0$.
(ii) $D_{T,3} \geq 0$.

Proof (i) By Lemma 4.7, $Q_s^2(I) = Q_s(I)^2$. As T is a spherical 2-expansion, $I - 2Q_s(I) + Q_s^2(I) \leq 0$. Now $Q_s^2(I) = Q_s(I)^2$ gives $I - 2Q_s(I) + Q_s(I)^2 = (I - Q_s(I))^2 \leq 0$ forcing T to be a spherical isometry.

(ii) Let Q_s and P_s be as in the proof of Lemma 4.7. By Lemma 4.7, $Q_s^3(I) = Q_s(I)^3$. Then (4.4) implies that $P_s \circ Q_s^2(I) = Q_s(I)$. Then arguing as in the proof of Lemma 4.7, one obtains

$$P_s(I) - 2I + Q_s(I) \leq 0.$$

Since $P_s(I) = Q_s(I)^{-1}$, we must have $Q_s(I) = I$.

□

Proposition 4.9 *Let T be a spherical 2-expansive (resp. spherical p-isometric) m-tuple on $\mathcal{H}$. Then T is a spherical isometry if and only if T satisfies any one of the following:*

(i) There is an integer $N \geq 2$ such that $D_{T,n} \geq 0$ for all $n \geq N$.
(ii) Let Q_s be the spherical generating 1-tuple associated with T. Then

$$\langle Q_s^n(I)h,\, h\rangle \leq \langle Q_s^{n-1}(I)h,\, h\rangle^{\frac{1}{2}} \langle Q_s^{n+1}(I)h,\, h\rangle^{\frac{1}{2}} \tag{4.5}$$

for all $h \in \mathcal{H}$ and for all integers $n \geq 1$.

Proof Let $n \geq N$. We claim that the inequality $\|Q_s^n(I)\| \geq \|Q_s(I)\|^n$ holds under the assumptions of both (i) and (ii). The argument is fairly standard.

(i) Since $D_{T,n} \geq 0$, one has $\|Q_s^n(I)\| \geq \|Q_s(I)^n\| = \|Q_s(I)\|^n$.

(ii) Since the norm of a positive operator P is given by $\sup_{\|h\|=1}\langle Ph, h\rangle$, by (4.5),

$$\|Q_s^n(I)\| \leq \|Q_s^{n-1}(I)\|^{\frac{1}{2}} \|Q_s^{n+1}(I)\|^{\frac{1}{2}}.$$

It then follows that for $k = 0, \ldots, n-1$,

$$\frac{\|Q_s^{n-k}(I)\|^{k+2}}{\|Q_s^{n-k-1}(I)\|^{k+1}} \geq \frac{\|Q_s^{n-k-1}(I)\|^{2(k+2)}/\|Q_s^{n-k-2}(I)\|^{k+2}}{\|Q_s^{n-k-1}(I)\|^{k+1}} = \frac{\|Q_s^{n-k-1}(I)\|^{k+3}}{\|Q_s^{n-k-2}(I)\|^{k+2}}.$$

In particular, $A_k := \frac{\|Q_s^{n-k}(I)\|^{k+2}}{\|Q_s^{n-k-1}(I)\|^{k+1}}$ is decreasing in k. Thus

$$\|Q_s^{n+1}(I)\| \geq \frac{\|Q_s^n(I)\|^2}{\|Q_s^{n-1}(I)\|} = A_0 \geq A_1 \geq \cdots \geq A_{n-1} = \|Q_s(I)\|^{n+1}$$

for every positive integer n. This completes the proof of the claim.

To complete the proof, we need the following spectral radius formula of Müller and Soltysiak [34, Theorem 1] (see also [17, Theorem 1]): For a commuting m-tuple T of bounded linear operators on a Hilbert space, the spectral radius $r(T)$ of T is given by

$$r(T) = \lim_{n\to\infty} \|Q_s^n(I)\|^{\frac{1}{2n}}.$$

It follows from the claim and the spectral radius formula that $r(T)$ is at least $\|Q_s(I)\|^{\frac{1}{2}}$. However, by Proposition 3.4(ii), $r(T) = 1$. Thus T is a spherical contraction. Since a spherical 2-expansion is a spherical expansion (see the discussion prior to Definition 2.5), T must be a spherical isometry in case T is a spherical 2-expansion. In case T is a spherical p-isometry, the desired conclusion follows from Lemma 4.3. □

An m-tuple $S = (S_1, \ldots, S_m)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is *jointly hyponormal* if the $m \times m$ matrix $([S_j^*, S_i])_{1\leq i,j\leq m}$ is positive definite, where $[A, B]$ stands for the commutator $AB - BA$ of A and B.

It is not difficult to see that a subnormal tuple is jointly hyponormal [7, Proposition 2].

Lemma 4.10 *Let T be a jointly hyponormal m-tuple. Then $D_{T,2} \geq 0$.*

Proof Observe that

$$\begin{aligned} Q_s^2(I) - Q_s(I)^2 &= \sum_{i,j=1}^m \left(T_i^*T_j^*T_iT_j - T_i^*T_iT_j^*T_j\right) \\ &= \sum_{i,j=1}^m T_i^*[T_j^*, T_i]T_j. \end{aligned}$$

It is now easy to see that

$$\langle ([T_j^*, T_i])_{1\le i,j\le m}(T_i x)_{1\le i\le m},\ (T_i x)_{1\le i\le m}\rangle \ge 0$$

for all $x \in \mathcal{H}$ if and only if $Q_s^2(I) \ge Q_s(I)^2$. In particular, $D_{T,2} \ge 0$. □

Proposition 4.11 *For any spherical* 2*-expansive* m*-tuple* T*, the following are equivalent:*

(i) T *is subnormal.*
(ii) T *is jointly hyponormal.*
(iii) T *is a spherical isometry.*

Proof We already recorded the implication (i) $\Longrightarrow$ (ii) while (iii) $\Longrightarrow$ (i) is a well-known result of A. Athavale [8, Proposition 2]. To see (ii) $\Longrightarrow$ (iii), note that $D_{T,2} \ge 0$ in view of Lemma 4.10. The desired conclusion now follows from Proposition 4.8(i). □

Next we present a counterpart of the previous proposition for spherical p-isometries (cf. [16], Proposition 3.16).

Proposition 4.12 *Let* T *be a spherical* p*-isometric* m*-tuple on* $\mathcal{H}$*. Then* T *is subnormal if and only* T *is a spherical isometry.*

Proof The result generalizes [39, Proposition 4.5] and can be obtained along the same lines. For completeness, we give here a direct proof. Suppose T is a subnormal m-tuple with normal extension N. Then

$$\langle Q_s^n(I)h,\ h\rangle = \langle P_s^n(I)h,\ h\rangle = \langle P_s(I)^n h,\ h\rangle$$

for any $h \in \mathcal{H}$, where P_s denotes the spherical generating 1-tuple associated with N. Now a simple application of the Cauchy-Schwarz inequality yields (4.5). By Proposition 4.9, T is necessarily a spherical isometry. □

The question whether there are non-trivial jointly hyponormal spherical p-isometries in higher dimensions remains unanswered. In particular, it is interesting to know whether a jointly hyponormal tuple always satisfies (4.5) of Proposition 4.9.

As an immediate consequence of Theorem 1.1 (resp. Theorem 1.2), note that a 2-hyperexpansion (resp. expansive m-isometry) on a finite-dimensional complex Hilbert space must be unitary. It is interesting to know what happens in the multivariable situation. Although the verbatim analog of Theorem 1.1 (resp. Theorem 1.2) is false in higher dimensions, we are able to provide a partial answer to this. First a lemma (cf. [12, Proposition 2.7]).

Lemma 4.13 *Let* $T = (T_1, \ldots, T_m)$ *be a spherical expansive* m*-tuple. Then the following statements are true:*

(i) T *admits the decomposition*

$$T_i = N_i \oplus C_i\ (i = 1, \ldots, m),$$

where $N = (N_1, \ldots, N_m)$ *is a normal spherical expansion and* $C = (C_1, \ldots, C_m)$ *is a completely non-normal spherical expansion.*

(ii) Let $T^{\mathfrak{s}}$ *be the spherical Cauchy dual of* T. *Suppose* $T^{\mathfrak{s}*} := ((T_1^{\mathfrak{s}})^*, \ldots, (T_m^{\mathfrak{s}})^*)$ *is a spherical contraction. If there exists* $\lambda \in \partial\mathbb{B}$ *such that* λ *is an eigenvalue of* T *then the restriction of* T *to* $\ker(T - \lambda I)$ *is normal.*

Proof (i) One may decompose any arbitrary m-tuple of bounded linear operators on a Hilbert space into a normal part and a completely non-normal part [28, Corollary 4.2]. Write $T_i = N_i \oplus C_i$ $(i = 1, \ldots, m)$, where $N = (N_1, \ldots, N_m)$ is a normal spherical expansion and $C = (C_1, \ldots, C_m)$ is a completely non-normal spherical expansion. Here we used the fact that the restriction of a spherical expansion to an invariant subspace is again a spherical expansion.

(ii) Let $\lambda \in \partial\mathbb{B}$ belongs to the point-spectrum $\sigma_p(T)$ of T. Thus there exists a non-zero vector $h \in \mathcal{H}$ such that $T_i h = \lambda_i h$ for all $i = 1, \ldots, m$. We first verify that

$$Q_s(I)h = h, \tag{4.6}$$

where $Q_s(I) = \sum_{i=1}^m T_i^* T_i$. Since $\langle Q_s(I)h,\, h\rangle = \|h\|^2$ and since $Q_s(I) \geq I$, one has $Q_s(I)h = h$. This completes the verification of (4.6).

Suppose $T^{\mathfrak{s}*}$ is a spherical contraction. It suffices to prove that

$$T_i^* h = \overline{\lambda}_i h \text{ for all } i = 1, \ldots, m.$$

Fix $1 \leq i \leq m$. It follows from (4.5) that $T_i^{\mathfrak{s}} h = T_i Q_s(I)^{-1} h = T_i h = \lambda_i h$. Now consider

$$\begin{aligned}
\sum_{i=1}^m \|(T_i^{\mathfrak{s}})^* h - \overline{\lambda}_i h\|^2 &= \sum_{i=1}^m \Big(\|(T_i^{\mathfrak{s}})^* h\|^2 - [\langle \lambda_i h,\, T_i^{\mathfrak{s}} h\rangle + \langle T_i^{\mathfrak{s}} h,\, \lambda_i h\rangle] + \|\lambda_i h\|^2\Big) \\
&= \sum_{i=1}^m \Big(\|(T_i^{\mathfrak{s}})^* h\|^2 - |\lambda_i|^2 \|h\|^2\Big) \\
&= \sum_{i=1}^m \|(T_i^{\mathfrak{s}})^* h\|^2 - \|h\|^2.
\end{aligned}$$

Since $T^{\mathfrak{s}*}$ is a spherical contraction, it follows that $(T_i^{\mathfrak{s}})^* h = \overline{\lambda}_i h$. Since $Q_s(I)h = h$ in view of (4.6), one has $T_i^* h = \overline{\lambda}_i h$. □

Remark 4.14 The additional assumption of Lemma 4.13(ii) that $T^{\mathfrak{s}*}$ is a spherical contraction is redundant in case $m = 1$.

Proposition 4.15 *Let* $T = (T_1, \ldots, T_m)$ *be a spherical* 2*-expansive (resp. spherically expansive spherical* p*-isometric)* m*-tuple on a finite-dimensional Hilbert space and let* $T^{\mathfrak{s}}$ *denote the spherical Cauchy dual of* T. *If* $T^{\mathfrak{s}*} := ((T_1^{\mathfrak{s}})^*, \ldots, (T_m^{\mathfrak{s}})^*)$ *is a spherical contraction, then* T *is a spherical unitary.*

Proof Suppose $T^{\mathfrak{s}*}$ is a spherical contraction. We prove the result only for spherical 2-expansions as the same argument works for spherical p-isometries. By Lemma 4.13 (i), T can be decomposed into a normal part N and a completely non-normal part C. Since the restriction of a spherical 2-expansion to an invariant subspace is again a spherical 2-expansion, N and C both are spherical 2-expansions. Also, since a normal spherical 2-expansion is a spherical unitary, N is a spherical unitary. Observe next that in view of $\sigma_p(T) \subseteq \sigma_{ap}(T)$, $\sigma_p(T) \subseteq \partial\mathbb{B}$ by Proposition 3.4. Since C is completely non-normal, by Lemma 4.13(ii), its point-spectrum is empty. It then follows that in case $\mathcal{H}$ is finite-dimensional, the completely non-normal part of T must be absent, and hence T is a spherical unitary. □

Remark 4.16 Let $T = (T_1, \ldots, T_m)$ be a spherical 2-expansive (resp. spherical p-isometric) m-tuple on $\mathcal{H}$. If $\mathcal{H}$ is infinite-dimensional then T_i is not compact for some $1 \leq i \leq m$. The proof of this relies on the basic fact that the Calkin algebra is non-trivial if and only if $\mathcal{H}$ is infinite-dimensional.

We remark that the proof of Proposition 4.15 could be done without using part(ii) of Lemma 4.13. The previous results raise the following natural question: Is the theory of spherical 2-expansions (in particular, the theory of spherical 2-isometries) strictly infinite-dimensional? After the communication of the present paper, it was revealed from the electronic version of the notes by Prof. S. Richter that this is indeed not the case.

5 Rigidity Theorems at the Calkin Algebra Level

In this section, capitalizing on the ideas of Agler and Stankus [3, Proposition 10.6], we obtain some rigidity theorems at the Calkin algebra level. These results rely heavily on the results of Sect. 4.

Let $\mathcal{C}(\mathcal{H})$ denote the norm-closed ideal of compact operators on $\mathcal{H}$. Since $B(\mathcal{H})/\mathcal{C}(\mathcal{H})$ is a unital C^*-algebra, the *Calkin algebra*, there exist a Hilbert space $\mathcal{K}$ and an injective unital $*$-representation $\pi : B(\mathcal{H})/\mathcal{C}(\mathcal{H}) \to B(\mathcal{K})$ [18, Chapter VIII]. In particular, $\pi \circ q : B(\mathcal{H}) \to B(\mathcal{K})$ is a unital $*$-representation, where $q : B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{C}(\mathcal{H})$ is the *quotient* (*Calkin*) *map*. Set $\pi \circ q(T) := (\pi \circ q(T_1), \ldots, \pi \circ q(T_m))$. Finally, let D_T denote the defect operator $I - \sum_{i=1}^{m} T_i^* T_i$.

Recall that $T = (T_1, \ldots, T_m)$ is *essentially normal* (resp. *essentially spherical isometry* resp. *essentially spherical unitary*) if $\pi \circ q(T)$ is normal (resp. spherical isometry resp. spherical unitary). Note that T is essentially spherical isometry (resp. essentially spherical unitary) if and only if $D_T \in \mathcal{C}(\mathcal{H})$ (resp. $D_T, D_{T^*} \in \mathcal{C}(\mathcal{H})$).

Remark 5.1 Note that if $D_T = 0$ (resp. $D_T \in \mathcal{C}(\mathcal{H})$) then the defect operator $D_{T,n} = 0$ (resp. $D_{T,n} \in \mathcal{C}(\mathcal{H})$) for all $n \geq 2$ (see Definition 4.5).

Here is a partial converse to Remark 5.1.

Proposition 5.2 *Let T be a spherical 2-expansive m-tuple on $\mathcal{H}$ such that $D_{T,2} \in \mathcal{C}(\mathcal{H})$ or $D_{T,3} \in \mathcal{C}(\mathcal{H})$. Then T is essentially spherical isometry.*

Proof We use the notations introduced in the beginning of this section. Note that $D_{\pi\circ q(T),n} = \pi \circ q(D_{T,n}) = 0$ for $n = 2$ or 3. Since unital $*$-representations send self-adjoint (resp. positive) elements to self-adjoint (resp. positive) ones, $\pi \circ q(T)$ is a spherical 2-expansion such that either $D_{\pi\circ q(T),2} = 0$ or $D_{\pi\circ q(T),3} = 0$. Now apply Proposition 4.8 to the m-tuple $\pi \circ q(T)$. □

Remark 5.3 A spherical 2-hyperexpansion T is *essentially jointly hyponormal* (that is, $\pi \circ q(T)$ is jointly hyponormal) if and only if T is essentially spherical isometry.

In case $m = 1$, the following captures the well-known fact that any finitely multi-cyclic isometry is essentially unitary.

Lemma 5.4 *Let T be a spherical expansive m-tuple. Then T is essentially spherical unitary iff T^* is essentially spherical isometry.*

Proof Suppose that T^* is essentially spherical isometry. Since unital $*$-representations send self-adjoint (resp. positive) elements to self-adjoint (resp. positive) ones, $D_{\pi\circ q(T)^*} = \pi \circ q(D_{T^*}) = 0$, so that $\pi \circ q(T)^*$ is a spherical isometry, and $\pi \circ q(T)$ is a spherical expansion. By [8, Proposition 2], the components of $\pi \circ q(T)^*$ are hyponormal operators. Thus

$$0 = D_{\pi\circ q(T)^*} \leq D_{\pi\circ q(T)} \leq 0.$$

That is, $\pi \circ q(T)$ is a spherical unitary. Equivalently, T is essentially spherical unitary. □

For a subspace $\mathcal{M}$ of $\mathcal{H}$, $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$ and $E_{\mathcal{M}}$ denotes the embedding of $\mathcal{M}$ into $\mathcal{H}$.

Proposition 5.5 *Let T be a spherical expansive m-tuple on $\mathcal{H}$ such that T^* is essentially spherical isometry. If $\mathcal{M}$ is an invariant subspace of T such that $P_{\mathcal{M}} \sum_{i=1}^m T_i (I - P_{\mathcal{M}})T_i^* E_{\mathcal{M}}$ is compact then $T|_{\mathcal{M}}$ is essentially spherical unitary.*

Proof Suppose $P_{\mathcal{M}} \sum_{i=1}^m T_i(I - P_{\mathcal{M}})T_i^* E_{\mathcal{M}}$ is compact. Since T is a spherical expansion, so is $S := T|_{\mathcal{M}}$. By Lemma 5.4, it now suffices to check that $S^* = P_{\mathcal{M}} T^* E_{\mathcal{M}}$ is essentially spherical isometry. Note that

$$\begin{aligned} D_{S^*} &= P_{\mathcal{M}} E_{\mathcal{M}} - P_{\mathcal{M}} \sum_{i=1}^m T_i P_{\mathcal{M}} T_i^* E_{\mathcal{M}} \\ &= P_{\mathcal{M}} E_{\mathcal{M}} - P_{\mathcal{M}} \sum_{i=1}^m T_i T_i^* E_{\mathcal{M}} + P_{\mathcal{M}} \sum_{i=1}^m T_i (I - P_{\mathcal{M}}) T_i^* E_{\mathcal{M}} \\ &= P_{\mathcal{M}} D_{T^*} E_{\mathcal{M}} + \text{ a compact operator.} \end{aligned}$$

Thus D_{S^*} is compact if so is D_{T^*}. □

Corollary 5.6 *Let T be a spherical expansive m-tuple on $\mathcal{H}$ such that T^* is essentially spherical isometry. Then for any invariant subspace $\mathcal{M}$ of T of finite co-dimension, $T|_{\mathcal{M}}$ is essentially spherical unitary.*

Example 5.7 Let $T_{\lambda,m}$ (resp. $T_{\lambda \cdot m}$) be the m-variable weighted shifts as discussed in Example 3.2 (resp. Example 3.3), where $\lambda \geq 1$ (resp. $\lambda m \geq 1$). Then $T^*_{\lambda,m}$ (resp. $T^*_{\lambda \cdot m}$) turns out to be spherically expansive essentially spherical isometry, and hence essentially spherical unitary (Lemma 5.4). This can be seen by direct computations as well.

Let T stand for $T_{\lambda,m}$ or $T_{\lambda \cdot m}$. Then T can be represented as the operator tuple of multiplication by the co-ordinate functions $z_1, \ldots, z_m$ on a reproducing kernel Hilbert space $\mathscr{H}_m$ with reproducing kernel $\kappa(\cdot, \cdot)$.

1. For $k \in \mathbb{N}^m$, consider the subspace $\mathcal{N}_k$ of $\mathscr{H}_m$ given by

$$\mathcal{N}_k = \{f \in \mathscr{H}_m : \langle f, z^l \rangle = 0 \text{ if } l_i < k_i \text{ for some } 1 \leq i \leq m\}.$$

   Note that $\mathcal{N}_k$ is an invariant subspace of T. Also, $\mathcal{N}_k$ (resp. $\mathcal{N}_k^{\perp}$) is spanned by $\{z^l : l \geq k\}$ (resp. $\{z^l : l_i < k_i \text{ for some } 1 \leq i \leq m\}$.). It is easy to check that

$$\sum_{i=1}^{m} \left\| (I - P_{\mathcal{N}_k}) T_i^* P_{\mathcal{N}_k} \frac{z^l}{\|z^l\|} \right\|^2 \to 0 \quad \text{as } |l| \to \infty.$$

   By Proposition 5.5, $T|_{\mathcal{N}_k}$ is essentially spherical unitary.
2. For positive integer n and for $w_1, \ldots w_n \in \mathbb{B}$, consider the subspace $\mathcal{M}_n$ of $\mathscr{H}_m$ given by

$$\mathcal{M}_n = \{f \in \mathscr{H}_m : f(w_i) = 0 \text{ for } i = 1, \ldots, n\}.$$

   Clearly, $\mathcal{M}_n$ is an invariant subspace of T. Also, the subspace orthogonal to $\mathcal{M}_n$ is spanned by $\kappa(\cdot, w_i)$ $(i = 1, \ldots, m)$, and hence $\mathcal{M}_n$ is of codimension n. By Corollary 5.6, $T|_{\mathcal{M}_n}$ is essentially spherical unitary.

On the other hand, a straightforward computation reveals that the 2-variable weighted shift with weight sequence

$$w_n^{(1)} = w_n^{(2)} = \frac{1}{\sqrt{2}} \sqrt{\frac{|n| + 2}{|n| + 1}}$$

is a spherical 2-isometry, which is not essentially spherical unitary.

Remark 5.8 Note that in dimension 1, if T is a finitely cyclic 2-isometry, then T is essentially unitary [12, Corollary 2.29]. The example discussed in the last paragraph of Example 5.7 shows that the spherical analog of this result is not true in higher dimensions.

In the context of Proposition 5.5 and Example 5.7, we cannot resist referring the reader to the Arveson's Conjecture as discussed in [6] (refer also to [27] for its current status).

The next proposition generalizes substantially the fact that if S is direct sum of k copies of Drury-Arveson m-shift then S is essentially unitary if and only if k is finite.

Proposition 5.9 *Let T denote a spherical expansive m-tuple such that $\sum_{i=1}^m T_i T_i^*$ is an orthogonal projection. Let $ker(T^*)$ stand for the common null-space $\bigcap_{i=1}^m ker(T_i^*)$.*

(i) If $ker(T^*)$ is finite-dimensional then T and $T|_{ker(T^*)^\perp}$ are essentially spherical unitary.

(ii) If $ker(T^*)$ is infinite-dimensional then T^* is not essentially spherical isometry. Moreover, the essential spectrum of T is the entire closed unit ball if in addition the following holds: Whenever $x_1, \ldots, x_m \in \mathcal{H}$ with $\sum_{i=1}^m \pi \circ q(T_i)x_i = 0$, then there exists an antisymmetric matrix $\{y_{ij}\}_{1 \le i,j \le m}$ with entries $y_{ij} \in \mathcal{H}$ such that $x_i = \sum_{j=1}^m \pi \circ q(T_j)y_{ij}$ for $i = 1, \ldots, m$.

Proof (i) Suppose $\ker(T^*)$ is finite-dimensional. One may check that $D_{\pi \circ q(T)^*} = \pi \circ q(D_{T^*})$ is an orthogonal projection if so is $\sum_{i=1}^m T_i T_i^*$, and that $\pi \circ q(T)$ is a spherical expansion if so is T. Now the assumption $\sum_{i=1}^m T_i T_i^*$ is an orthogonal projection onto $\ker(T^*)^\perp$ implies that T^* is essentially spherical isometry. By Lemma 5.4, T is essentially spherical unitary. Since $\ker(T^*)$ is finite-dimensional, it now follows from Corollary 5.6 that $T|_{\ker(T^*)^\perp}$ is essentially spherical unitary.

(ii) Suppose $\ker(T^*)$ is infinite-dimensional. The assumption $\sum_{i=1}^m T_i T_i^*$ is an orthogonal projection implies that $I - \sum_{i=1}^m T_i T_i^*$ is identity on an infinite dimensional space, and hence cannot be compact. Thus T^* is not essentially spherical isometry.

A decomposition theorem of Richter and Sundberg [35] is crucially used in this part. The additional hypothesis on the Koszul complex for $\pi \circ q(T)$ allows us to apply [35, Corollary 1.5] implying that $\pi \circ q(T)$ is unitarily equivalent to $S \oplus V$, where S is a non-trivial direct sum of the Drury-Arveson m-shifts and V is a spherical unitary. Recall that the Taylor spectrum of the Drury-Arveson m-shift is the closed unit ball in $\mathbb{C}^m$ (Remark 3.6). One may now conclude from [20, Lemma 4.4] that

$$\sigma(\pi \circ q(T)) = \sigma(S \oplus V) = \sigma(S) = \overline{\mathbb{B}}.$$

It follows that $\sigma_e(T) = \sigma(\pi \circ q(T)) = \overline{\mathbb{B}}$. □

6 Epilogue

A study of multi-variable analog of a special type of operator is a rewarding enterprise and has led to interesting constructions and challenging problems. In particular, the spherical analog of an isometry turned out to be extremely fascinating. The subnormality of a spherical isometry, a remarkable result by Athavale [8], has made it possible to apply the theory of commuting subnormals to spherical isometries. The class of spherical isometries has been extensively studied in the recent past and a wide range of examples have been constructed with a view to attempt a possible classification scheme (see, for instance, [28]). Motivated by the success story of spherical isometries, one is naturally led to studying the spherical analogs of the other related classes of operators. Two classes of operators that include isometries as special examples are of m-isometries [1–3] and complete hyperexpansions [4,10]. While these operators have been studied in the multi-variable setting [11,23,24,29], the language of generating tuples [16] provides a unified approach.

The context, as described above, provides a framework for posing and scrutinizing a myriad range of problems. In particular, the intricacies of invertibility in higher dimensions has a vital bearing which deserves a substantial exploration. While the invertibility can be easily implemented in one variable, the same turns out to be lot more subtle in higher dimensions, the present paper being a testimony. While we have exemplified the failure of some rigidity results with Taylor invertibility, it is interesting to note that the notion of structural invertibility comes in handy to rescue the situation. However, we would like to draw the attention of the reader to the disparity in the rigidity results viz. Theorem 1.1 and Proposition 4.2 (resp. Theorem 1.2 and Proposition 4.4), in one and multi-variable cases respectively. An invertible 2-hyperexpansive operator is *unitary* whereas the structural invertibility of a spherical hyperexpansion yields *spherical isometry*.

The examples in this paper as well as those in [28] confirm that a Taylor invertible spherical isometry need not be a spherical unitary. The situation boils down to a few natural questions :

Question 6.1 *If $T = (T_1, \ldots, T_m)$ is a spherical isometry, what are conditions on the Taylor spectrum of T which ensure it to be a spherical unitary?*

Question 6.2 *If $T = (T_1, \ldots, T_m)$ is a spherical 2-hyperexpansion such that $\sigma(T) \subseteq \partial\mathbb{B}$, is T a spherical isometry?*

With the aid of variety of examples, we have pointed out that the spectral picture in the multi-variable case is lot more complicated than that in the one variable, where we have the spectral dichotomy as stated in Theorems 1.1 and 1.2. In the multi-variable set-up, we observe that if T is a spherical 2-hyperexpansive 2-tuple (resp. spherical p-isometry) such that T^* is a spherical expansion, then

$$\sigma(T) = \overline{\mathbb{B}} \quad \text{or} \quad \sigma(T) \subseteq \partial\mathbb{B},$$

where $\mathbb{B}$ denotes the open unit ball in $\mathbb{C}^2$. This can be seen as follows:

By Proposition 3.4(i), $\sigma_{ap}(T) \subseteq \partial\mathbb{B}$. Suppose now that T^* is a spherical expansive 2-tuple. Since

$$\sum_{i=1}^{2} \|T_i^* x\|^2 \geq 1$$

for unit vector x, it is easy to see that the approximate-point spectrum $\sigma_{ap}(T^*)$ of T^* is contained in the complement of the open unit $\mathbb{B}$ in $\mathbb{C}^2$. Moreover, $\sigma(T) \subseteq \overline{\mathbb{B}}$ (Proposition 3.4(i)). It is known that for 2-tuples, the boundary of the Taylor spectrum $\sigma(T)$ is contained in the union of $\sigma_{ap}(T)$ and the complex conjugate of $\sigma_{ap}(T^*)$ [33]. It follows that $\partial\sigma(T) \subseteq \partial\mathbb{B}$. The desired spectral dichotomy is now immediate.

The observation in the preceding paragraph coupled with the spectral picture in one variable case naturally leads to the following question :

Question 6.3 *If T be a spherical 2-hyperexpansive 2-tuple such that T^* is a spherical expansion, then is it true that $\sigma(T) \subseteq \partial\mathbb{B}$?*

Let T be a spherical 2-hyperexpansive m-variable weighted shift such that its spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting. Then, by Lemma 3.1, the shift $T_\beta : \{\beta_{k\epsilon_1}\}_{k\in\mathbb{N}}$ associated with T is a 2-hyperexpansion, where

$$\beta_n = \left(\sum_{i=1}^{m} \left(w_n^{(i)}\right)^2\right)^{\frac{1}{2}} \quad (n \in \mathbb{N}^m).$$

It is well-known that the weight-sequence of a 2-hyperexpansion converges to 1 [30, Proposition]. This observation along with the first half of Lemma 3.1 allows us to conclude that T is essentially spherical isometry.

The general case, where $T^{\mathfrak{s}}$ is not necessarily commuting, remains unanswered.

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