Friends or Foes?

The Interrelationship between Angel and Venture Capital Markets

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— ONLINE APPENDIX —

Angel market: equilibrium equity shares and entrepreneur's outside option.

According to the Nash product, α^* is implicitly defined by

$$\frac{dD_1^E(e_1^*)}{d\alpha} D_1^A(e_1^*) + (D_1^E(e_1^*) - U_1^E) \frac{dD_1^A(e_1^*)}{d\alpha} = 0. \tag{A.1}$$

Applying the Envelope Theorem we find that $dD_1^E(e_1^*)/d\alpha < 0$. We can then infer from Eq. (A.1) that $dD_1^A(e_1^*)/d\alpha > 0$ must hold for $\alpha = \alpha^*$. Using Eq. (A.1) we can implicitly differentiate α^* w.r.t. U_1^E:

$$\frac{d\alpha^*}{dU_1^E} = \frac{\frac{dD_1^A}{d\alpha}}{\frac{d}{d\alpha}\left[\frac{dD_1^E}{d\alpha} D_1^A + (D_1^E - U_1^E)\frac{dD_1^A}{d\alpha}\right]}. \tag{A.2}$$

Note that the denominator is strictly negative due to the second-order condition for α^*. Moreover, recall that $dD_1^A/d\alpha > 0$. Thus, $d\alpha^*/dU_1^E < 0$.

Angel market: optimal transfer payment.

Suppose the angel makes the transfer T to the entrepreneur in exchange for an additional equity stake $\widehat{\alpha}(T)$. The angel's new equity share is then given by $\alpha(T) \equiv \alpha^* + \widehat{\alpha}(T)$, with $\alpha'(T) > 0$, $\alpha(T) \geq 0 \; \forall T \geq 0$, and $\alpha(T) < 0 \; \forall T < 0$. Note that any post bargaining transfers aimed at adjusting the equity allocation, must improve joint efficiency to be implementable. The joint utility at the deal stage is

$$D_1^A + D_1^E = \rho_1(e_1)\left[g\left[U_1^A + U_2^E\right] + (1-g)y_1\right] - k_1 - c(e_1), \tag{A.3}$$

where $e_1 \equiv e_1(\alpha(T))$. Thus, the marginal effect of a transfer T on joint utility is given by

$$\frac{d\left[D_1^A + D_1^E\right]}{dT} = \underbrace{\left[\rho_1'(e_1) g\left[U_2^A + U_2^E\right] - c'(e_1)\right]}_{\equiv X} \frac{de_1}{d\alpha}\frac{d\alpha(T)}{dT}. \tag{A.4}$$

Recall that $de_1/d\alpha < 0$, so that e_1 is maximized at $\alpha = 0$. Moreover, $X \geq 0$. Thus, $d\left[D_1^A + D_1^E\right]/dT > 0$ requires that $d\alpha(T)/dT < 0$, and therefore $T < 0$. However, because of his zero wealth, the entrepreneur cannot make a payment to the angel. Thus, $T^* = 0$.

Derivation of angel market equilibrium.

Using $q_1^A = x_1/M_1^A$ and $x_1 = \phi_1\left[M_1^E M_1^A\right]^{0.5}$, we can write Eq. (6) as

$$\phi_1 D_1^A \left[\frac{M_1^E}{M_1^A}\right]^{0.5} = \sigma_1^A. \tag{A.5}$$

Using $\theta_1 = M_1^A/M_1^E$ we then get the equilibrium degree of competition for the angel market: $\theta_1^* = \left[\phi_1 D_1^A/\sigma_1^A\right]^2$. Next, note that we can write Eq. (A.5) as

$$M_1^A = M_1^E\left[\frac{\phi_1 \widetilde{D}_1^A}{\sigma_1^A}\right]^2 = M_1^E\theta_1. \tag{A.6}$$

Solving Eq. (8) for M_1^E and using $q_1^E = \phi_1\left[M_1^E M_1^A\right]^{0.5}/M_1^E = \phi_1\left[M_1^A/M_1^E\right]^{0.5}$, we get the equilibrium stock of entrepreneurs in the early stage market:

$$M_1^{E*} = \frac{F(U_1^E)}{\delta_1 + q_1^E} = \frac{F(U_1^E)}{\delta_1 + \phi_1\left[M_1^A/M_1^E\right]^{0.5}} = \frac{F(U_1^E)}{\delta_1 + \phi_1\sqrt{\theta_1^*}}. \tag{A.7}$$

Thus, the equilibrium stock of angels is given by

$$M_1^{A*} = M_1^{E*}\theta_1^* = \frac{F(U_1^E)\theta_1^*}{\delta_1 + \phi_1\sqrt{\theta_1^*}}. \tag{A.8}$$

Using $M_1^{E*} = M_1^{A*}/\theta_1^*$ we can then write x_1^* as

$$x_1^* = \phi_1\left[M_1^{A*}M_1^{E*}\right]^{0.5} = \frac{\phi_1 M_1^{A*}}{\sqrt{\theta_1^*}} = F(U_1^E)\frac{\phi_1\sqrt{\theta_1^*}}{\delta_1 + \phi_1\sqrt{\theta_1^*}}. \tag{A.9}$$

Moreover, using Eq. (9) and $q_1^A = x_1/M_1^A$ we get $m_1^{A*} = q_1^A M_1^{A*} = x_1^*$.

Proof of Proposition 1.

Recall that the equilibrium of the angel market is determined by the deal values D_1^E and D_1^A, and therefore by the late stage utilities U_2^E and U_2^A, as well as by the entrepreneur's outside option U_1^E (through α^*). We will show in Proof of Proposition 4 that U_2^E and U_2^A do not depend on ϕ_1, δ_1, σ_1^E, σ_1^A, and k_1. Next we need to derive a condition which defines U_1^E. The equilibrium condition (5) can be written as

$$U_1^E\left[r + \delta_1\right] = -\sigma_1^E + q_1^E\left[D_1^E - U_1^E\right]. \tag{A.10}$$

Using $q_1^E = \phi_1 \left[M_1^{A*}/M_1^{E*}\right]^{0.5} = \phi_1\sqrt{\theta_1^*} = \phi_1^2 D_1^A/\sigma_1^A$ we get the following condition which defines U_1^E:

$$U_1^E\left[r+\delta_1\right] - \frac{\phi_1^2}{\sigma_1^A} D_1^A \left[D_1^E - U_1^E\right] + \sigma_1^E = 0. \quad \text{(A.11)}$$

Now consider the equilibrium degree of competition θ_1^*. Differentiating θ_1^* w.r.t. δ_1 yields

$$\frac{d\theta_1^*}{d\delta_1} = 2\frac{\phi_1^2 D_1^A}{\left[\sigma_1^A\right]^2}\frac{dD_1^A}{d\delta_1} = 2\frac{\phi_1^2 D_1^A}{\left[\sigma_1^A\right]^2}\frac{dD_1^A}{d\alpha}\frac{d\alpha^*}{dU_1^E}\frac{dU_1^E}{d\delta_1}. \quad \text{(A.12)}$$

Next we define $\Gamma \equiv D_1^A\left[D_1^E - U_1^E\right]$. We then get

$$\frac{dU_1^E}{d\delta_1} = -\frac{U_1^E}{r+\delta_1 - \frac{\phi_1^2}{\sigma_1^A}\left[\frac{d\Gamma}{d\alpha}\frac{d\alpha^*}{dU_1^E} + \frac{\partial\Gamma}{\partial U_1^E}\right]}. \quad \text{(A.13)}$$

Note that $d\Gamma/d\alpha = 0$ due to the first-order condition for α^*. Moreover, $\partial\Gamma/\partial U_1^E = -D_1^A$. Consequently,

$$\frac{dU_1^E}{d\delta_1} = -\frac{U_1^E}{r+\delta_1 + \frac{\phi_1^2}{\sigma_1^A}D_1^A} < 0. \quad \text{(A.14)}$$

This in turn implies that $d\theta_1^*/d\delta_1 > 0$. Likewise,

$$\frac{d\theta_1^*}{d\sigma_1^E} = 2\frac{\phi_1^2 D_1^A}{\left[\sigma_1^A\right]^2}\frac{dD_1^A}{d\alpha}\frac{d\alpha^*}{dU_1^E}\frac{dU_1^E}{d\sigma_1^E}, \quad \text{(A.15)}$$

with $dD_1^A/d\alpha > 0$, $d\alpha^*/dU_1^E < 0$, and

$$\frac{dU_1^E}{d\sigma_1^E} = -\frac{1}{r+\delta_1 + \frac{\phi_1^2}{\sigma_1^A}D_1^A} < 0. \quad \text{(A.16)}$$

Thus, $d\theta_1^*/d\sigma_1^E > 0$. Moreover, note that $dD_1^A/dk_1 < 0$. Consequently, $d\theta_1^*/dk_1 < 0$. For the remaining comparative statics it is useful to express the condition for U_1^E in terms of θ_1^*:

$$U_1^E\left[r+\delta_1\right] - \phi_1\sqrt{\theta_1^*}\left[D_1^E - U_1^E\right] + \sigma_1^E = 0, \quad \text{(A.17)}$$

so that

$$\frac{dU_1^E}{d\theta_1^*} = \frac{\phi_1 \frac{1}{2\sqrt{\theta_1^*}}\left[D_1^E - U_1^E\right]}{r+\delta_1+\phi_1\sqrt{\theta_1^*}} > 0. \quad \text{(A.18)}$$

Moreover, using the definition of θ_1^* we define

$$G \equiv \theta_1^* - \left[\frac{\phi_1}{\sigma_1^A}D_1^A\right]^2 = 0 \quad \text{(A.19)}$$

where $D_1^A = D_1^A(\alpha^*(U_1^E(\theta_1^*)))$. We get

$$\frac{d\theta_1^*}{d\phi_1} = \frac{2\frac{\phi_1}{\left[\sigma_1^A\right]^2}\left[D_1^A\right]^2}{1-2\left[\frac{\phi_1}{\sigma_1^A}\right]^2 D_1^A \frac{dD_1^A}{d\alpha}\frac{d\alpha^*}{dU_1^E}\frac{dU_1^E}{d\theta_1^*}}. \tag{A.20}$$

Recall that $dD_1^A/d\alpha > 0$, $d\alpha^*/dU_1^E < 0$, and $dU_1^E/d\theta_1^* > 0$. Thus, the denominator is positive, which implies that $d\theta_1^*/d\phi_1 > 0$. Likewise, using Eq. (A.19), we get

$$\frac{d\theta_1^*}{d\sigma_1^A} = -\frac{2\frac{\phi_1^2}{\left[\sigma_1^A\right]^3}\left[D_1^A\right]^2}{1-2\left[\frac{\phi_1}{\sigma_1^A}\right]^2 D_1^A \frac{dD_1^A}{d\alpha}\frac{d\alpha^*}{dU_1^E}\frac{dU_1^E}{d\theta_1^*}}. \tag{A.21}$$

Again, the denominator is positive, which implies that $d\theta_1^*/d\sigma_1^A < 0$.

Next, note that $dm_1^{E*}/dU_1^E = F'(U_1^E) > 0$, and recall that $dU_1^E/d\delta_1$, $dU_1^E/d\sigma_1^E < 0$. Moreover, using Eq. (A.11) we find

$$\frac{dU_1^E}{d\phi_1} = \frac{2\frac{\phi_1}{\sigma_1^A}D_1^A\left[D_1^E - U_1^E\right]}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}D_1^A} > 0 \tag{A.22}$$

$$\frac{dU_1^E}{d\sigma_1^A} = -\frac{\frac{\phi_1^2}{\left[\sigma_1^A\right]^2}D_1^A\left[D_1^E - U_1^E\right]}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}D_1^A} < 0. \tag{A.23}$$

Likewise, using $\Gamma = D_1^A\left[D_1^E - U_1^E\right]$,

$$\frac{dU_1^E}{dk_1} = \frac{\frac{\phi_1^2}{\sigma_1^A}\frac{d\Gamma}{dk_1}}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}D_1^A}, \tag{A.24}$$

with

$$\frac{d\Gamma}{dk_1} = \underbrace{\frac{d\Gamma}{d\alpha}}_{=0}\frac{d\alpha^*}{dk_1} + \frac{\partial\Gamma}{\partial k_1} = -\left[D_1^E - U_1^E\right] < 0. \tag{A.25}$$

Thus, $dU_1^E/dk_1 < 0$. All this implies that m_1^{E*} is increasing in ϕ_1, and decreasing in δ_1, σ_1^E, σ_1^A, and k_1.

Next, recall that $m_1^{A*} = x_1^*$ is given by

$$m_1^{A*} = x_1^* = F(U_1^E)\underbrace{\frac{\phi_1\sqrt{\theta_1^*}}{\delta_1+\phi_1\sqrt{\theta_1^*}}}_{\equiv T}. \tag{A.26}$$

It is straightforward to show that $dT/d(\phi_1\sqrt{\theta_1^*}) > 0$. Because $dU_1^E/d\phi_1 > 0$ and $d\theta_1^*/d\phi_1 > 0$, we then have $dm_1^{A*}/d\phi_1 = dx_1^*/d\phi_1 > 0$. Likewise, we know that $dU_1^E/d\sigma_1^A$, $dU_1^E/dk_1 < 0$

and $d\theta_1^*/d\sigma_1^A$, $d\theta_1^*/dk_1 < 0$. Thus, $dm_1^{A*}/d\sigma_1^A = dx_1^*/d\sigma_1^A < 0$ and $dm_1^{A*}/dk_1 = dx_1^*/dk_1 < 0$. Moreover, we have shown that $dU_1^E/d\delta_1$, $dU_1^E/d\sigma_1^E < 0$, while $d\theta_1^*/d\delta_1$, $d\theta_1^*/d\sigma_1^E > 0$. Thus, the effects of δ_1 and σ_1^E on $m_1^{A*} = x_1^*$ are ambiguous.

Now consider the equilibrium valuation V_1^*. Note that V_1^* is decreasing in the angel's equilibrium equity share α^*, which is defined by Eq. (A.1). Recall that $d\alpha^*/dU_1^E < 0$, $dU_1^E/d\phi_1 > 0$ and $dU_1^E/d\delta_1$, $dU_1^E/d\sigma_1^E$, $dU_1^E/d\sigma_1^A < 0$. Consequently, $d\alpha^*/d\phi_1 < 0$ and $d\alpha^*/d\delta_1$, $d\alpha^*/d\sigma_1^E$, $d\alpha^*/d\sigma_1^A > 0$. All this implies that V_1^* is increasing in ϕ_1, and decreasing in δ_1, σ_1^E and σ_1^A. Furthermore, note that k_1 affects D_1^A and U_1^A. Using Eq. (A.1) we get

$$\frac{d\alpha^*}{dk_1} = -\frac{\frac{dD_1^E}{d\alpha}\frac{\partial D_1^A}{\partial k_1} - \frac{dU_1^E}{dk_1}\frac{dD_1^A}{d\alpha} + (D_1^E - U_1^E)\frac{d^2 D_1^A}{d\alpha dk_1}}{\frac{d}{d\alpha}\left[\frac{dD_1^E}{d\alpha}D_1^A + (D_1^E - U_1^E)\frac{dD_1^A}{d\alpha}\right]}, \tag{A.27}$$

where the denominator is strictly negative due to the second-order condition for α^*. Thus, to prove that $d\alpha^*/dk_1 > 0$, we need to show that the numerator is positive. We know that $dD_1^E/d\alpha < 0$, $dD_1^A/d\alpha > 0$, and $dU_1^E/dk_1 < 0$. Moreover, $\partial D_1^A/\partial k_1 = -1$ and $d^2 D_1^A/(d\alpha dk_1) = 0$. Thus, the numerator is strictly positive, so that $d\alpha^*/dk_1 > 0$. This in turn implies that the effect of k_1 on $V_1^* = k_1/\alpha^*$ is ambiguous.

Finally consider the equilibrium success probability $\rho_1(e_1^*)$, with $\rho_1'(e_1^*) > 0$. Using Eq. (3) we get

$$\frac{de_1^*}{d\alpha} = \frac{\rho_1'(e_1)(1-q)y_1}{\frac{d}{de_1}\left[\rho_1'(e_1)\left[qU_2^A + (1-q)(1-\alpha)y_1\right] - c'(e_1)\right]}, \tag{A.28}$$

where the denominator is strictly negative due to the second-order condition for e_1^*. Thus, $de_1^*/d\alpha < 0$. Our comparative statics results for α^* then imply that $d\rho_1(e_1^*)/d\phi_1 > 0$ and $d\rho_1(e_1^*)/d\delta_1$, $d\rho_1(e_1^*)/d\sigma_1^E$, $d\rho_1(e_1^*)/d\sigma_1^A$, $d\rho_1(e_1^*)/dk_1 < 0$. □

Early Stage Investment and Valuation.

Consider first our base model with endogenous effort. Differentiating V_1^* w.r.t. k_1 yields

$$\frac{dV_1^*}{dk_1} = \frac{d}{dk_1}\left(\frac{k_1}{\alpha^*}\right) = \frac{\alpha^* - k_1\frac{d\alpha^*}{dk_1}}{\left[\alpha^*\right]^2}. \tag{A.29}$$

Note that $dV_1^*/dk_1 > 0$ when $k_1 \to 0$. Thus, the equilibrium valuation V_1^* is decreasing in k_1 when k_1 is sufficiently small.

Next, suppose the entrepreneur's effort e_1 is exogenous, and define $\rho_1 \equiv \rho_1(e_1)$. The early stage deal values are then given by

$$D_1^E = \rho_1\left[gU_2^E + (1-g)(1-\alpha)y_1\right] - c \tag{A.30}$$

$$D_1^A = \rho_1\left[gU_2^A + (1-g)\alpha y_1\right] - k_1, \tag{A.31}$$

where c is the entrepreneurs disutility of providing effort e_1. The optimal equity share for the angel, α^*, then satisfies the symmetric Nash bargaining solution, which accounts for the outside option of each party (U_1^E for the entrepreneur, and 0 for the angel because of free entry). Let $\widetilde{D}_1^E$ and $\widetilde{D}_1^A$ denote the deal values reflecting the Nash bargaining solution, which are given by

$$\widetilde{D}_1^E = \frac{1}{2}\left[\rho_1\left[g\left(U_2^E + U_2^A\right) + (1-g)y_1\right] - k_1 - c + U_1^E\right] \tag{A.32}$$

$$\widetilde{D}_1^A = \frac{1}{2}\left[\rho_1\left[g\left(U_2^E + U_2^A\right) + (1-g)y_1\right] - k_1 - c - U_1^E\right]. \tag{A.33}$$

The equilibrium equity share for the angel, α^*, then satisfies $D_1^E(\alpha^*) = \widetilde{D}_1^E$ and $D_1^A(\alpha^*) = \widetilde{D}_1^A$. Recall that $U_2^A = U_2^E$ in equilibrium. Thus,

$$\alpha^* = \frac{1}{2} + \frac{k_1 - c - U_1^E}{2\rho_1(1-g)y_1}. \tag{A.34}$$

The equilibrium early stage valuation is $V_1^* = k_1/\alpha^*$. We get

$$\frac{dV_1^*}{dk_1} = \frac{\overbrace{\alpha^* - k_1\frac{d\alpha^*}{dk_1}}^{\equiv N}}{[\alpha^*]^2}. \tag{A.35}$$

The denominator is always non-negative. Moreover, note that $N \geq 0$ for $k_1 \to 0$, which implies that $dV_1^*/dk_1 \geq 0$ for $k_1 \to 0$. To show that $dV_1^*/dk_1 > 0$ for all $k_1 > 0$, it is thus sufficient to verify that $dN/dk_1 > 0$:

$$\frac{dN}{dk_1} = \frac{d\alpha^*}{dk_1} - \left(\frac{d\alpha^*}{dk_1} + k_1\frac{d^2\alpha^*}{dk_1^2}\right) = -k_1\frac{d^2\alpha^*}{dk_1^2}. \tag{A.36}$$

We need to find the sign of $d^2\alpha^*/dk_1^*$. We start by taking the first derivative of α^* w.r.t. k_1:

$$\frac{d\alpha^*}{dk_1} = \frac{1}{2\rho_1(1-g)y_1}\left[1 - \frac{dU_1^E}{dk_1}\right]. \tag{A.37}$$

It is easy to see that $\widetilde{D}_1^E - U_1^E = \widetilde{D}_1^A$. Thus, the condition defining U_1^E simplifies to

$$U_1^E\left[r + \delta_1\right] - \frac{\phi_1^2}{\sigma_1^A}\left[\widetilde{D}_1^A\right]^2 + \sigma_1^E = 0. \tag{A.38}$$

Thus,

$$\frac{dU_1^E}{dk_1} = -\frac{a_1\widetilde{D}_1^A}{r + \delta_1 + a_1\widetilde{D}_1^A}, \tag{A.39}$$

where $a_1 = \phi_1^2/\sigma_1^A$. Consequently,

$$\frac{d\alpha^*}{dk_1} = \frac{1}{2\rho_1(1-g)y_1}\left[1 + \frac{1}{(r+\delta_1)\left[a_1\widetilde{D}_1^A\right]^{-1} + 1}\right]. \tag{A.40}$$

We then get

$$\frac{d^2\alpha^*}{dk_1^2} = \frac{1}{2\rho_1(1-g)y_1} \frac{-\frac{1}{2}a_1 (r+\delta_1) \left[a_1 \widetilde{D}_1^A\right]^{-2} \left[1 + \frac{dU_1^E}{dk_1}\right]}{\left[(r+\delta_1)\left[a_1 \widetilde{D}_1^A\right]^{-1} + 1\right]^2}. \tag{A.41}$$

Note that

$$1 + \frac{dU_1^E}{dk_1} = 1 - \frac{a_1 \widetilde{D}_1^A}{r+\delta_1 + a_1 \widetilde{D}_1^A} = \frac{r+\delta_1}{r+\delta_1 + a_1 \widetilde{D}_1^A} > 0. \tag{A.42}$$

Thus, $d^2\alpha^*/dk_1^2 < 0$. This implies that $dN/dk_1 > 0$, and therefore $dV_1^*/dk_1 > 0$.

Proof of Proposition 2.

In equilibrium, $U_2^E = U_2^A$. Moreover, we will show in Proof of Proposition 3 that $dU_2^E/d\phi_2 > 0$ and $dU_2^E/d\delta_2$, $dU_2^E/d\sigma_2$, $dU_2^E/d\sigma_2^V$, $dU_2^E/dk_2 < 0$. Consider the equilibrium degree of competition θ_1^*. With $U_2^E = U_2^A$ note that

$$\frac{d\theta_1^*}{dU_2^E} = 2\frac{\phi_1^2}{\left[\sigma_1^A\right]^2} D_1^A \frac{dD_1^A}{dU_2^A}. \tag{A.43}$$

For a given α we find that

$$\frac{dD_1^A}{dU_2^E} = \rho_1'(e_1^*)\frac{de_1^*}{dU_2^E}\left[gU_2^E + (1-g)\alpha y_1\right] + \rho_1(e_1^*)g > 0. \tag{A.44}$$

Moreover, applying the Envelope Theorem we get $dD_1^E/dU_2^E = g\rho_1(e_1^*) > 0$. Thus, the bargaining frontier shifts outwards, so that $dD_1^E/dU_2^E > 0$ and $dD_1^A/dU_2^E > 0$ at the equilibrium equity share α^*. This implies that $d\theta_1^*/dU_2^E > 0$, and consequently, $d\theta_1^*/d\phi_2 > 0$ and $d\theta_1^*/d\delta_2$, $d\theta_1^*/d\sigma_2$, $d\theta_1^*/d\sigma_2^V$, $d\theta_1^*/dk_2 < 0$.

Now consider the equilibrium inflow of entrepreneurs $m_1^{E*} = F(U_1^E)$, with $F'(U_1^E) > 0$. Using Eq. (A.11) we get

$$\frac{dU_1^E}{dU_2^E} = \frac{\frac{\phi_1^2}{\sigma_1^A}\frac{d\Gamma}{dU_2^E}}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}D_1^A}, \tag{A.45}$$

where $\Gamma = D_1^A\left[D_1^E - U_1^E\right]$. Note that

$$\frac{d\Gamma}{dU_2^E} = \frac{d\Gamma}{de_1}\frac{de_1^*}{U_2^E} + \frac{d\Gamma}{d\alpha}\frac{d\alpha^*}{U_2^E} + \frac{\partial\Gamma}{\partial U_2^E}, \tag{A.46}$$

where $de_1^*/dU_2^E > 0$ and $d\Gamma/d\alpha = 0$ (see Eq. (A.1)). Moreover,

$$\begin{aligned} \frac{d\Gamma}{de_1} &= \frac{dD_1^A}{de_1}\left[D_1^E - U_1^E\right] + D_1^A \underbrace{\frac{D_1^E}{de_1}}_{=0} \\ &= \rho_1'(e_1^*)\underbrace{\left[gU_2^A + (1-g)\alpha y_1\right]}_{>0}\underbrace{\left[D_1^E - U_1^E\right]}_{>0} > 0 \end{aligned} \tag{A.47}$$

$$\frac{\partial\Gamma}{\partial U_2^E} = \underbrace{\frac{\partial D_1^A}{\partial U_2^E}}_{>0}\underbrace{\left[D_1^E - U_1^E\right]}_{>0} + D_1^A \underbrace{\frac{\partial D_1^E}{\partial U_2^E}}_{>0} > 0 \tag{A.48}$$

This implies that $dU_1^E/dU_2^E > 0$, and therefore, $dF(U_1^E)/dU_2^E > 0$. Our comparative statics results for U_2^E (see Proof of Proposition 3) then imply that $dm_1^{E*}/d\phi_2 > 0$ and $dm_1^{E*}/d\delta_2$, $dm_1^{E*}/d\sigma_2$, $dm_1^{E*}/d\sigma_2^V$, $dm_1^{E*}/dk_2 < 0$.

Next consider the equilibrium inflow of angels, m_1^{A*}, which is defined by

$$m_1^{A*} = x_1^* = \underbrace{F(U_1^E)}_{=m_1^{E*}} \underbrace{\frac{\phi_1\sqrt{\theta_1^*}}{\delta_1 + \phi_1\sqrt{\theta_1^*}}}_{\equiv T}. \tag{A.49}$$

One can show that $dT/d\sqrt{\theta_1^*}) > 0$. Our comparative statics results for m_1^{E*} and θ_1^* then imply that $dm_1^{A*}/d\phi_2 > 0$ and $dm_1^{A*}/d\delta_2$, $dm_1^{A*}/d\sigma_2$, $dm_1^{A*}/d\sigma_2^V$, $dm_1^{A*}/dk_2 < 0$.

Now consider the equilibrium equity share α^* for angels. Recall that $dD_1^E/dU_2^E > 0$ and $dD_1^A/dU_2^E > 0$ at the equilibrium equity share α^*. Moreover, using the Envelope Theorem it is straightforward to show that $dD_1^A/dU_2^E > dD_1^E/dU_2^E$. The Nash bargaining solution then implies that $d\alpha^*/dU_2^E < 0$. Thus, $d\alpha^*/d\phi_2 < 0$ and $d\alpha^*/d\delta_2$, $d\alpha^*/d\sigma_2$, $d\alpha^*/d\sigma_2^V$, $d\alpha^*/dk_2 > 0$. For the equilibrium valuation $V_1^* = k_1/\alpha^*$ we can then infer that $dV_1^*/d\phi_2 > 0$ and $dV_1^*/d\delta_2$, $dV_1^*/d\sigma_2$, $dV_1^*/d\sigma_2^V$, $dV_1^*/dk_2 < 0$.

Finally consider the equilibrium success rate $\rho_1(e_1^*)$, with $\rho_1'(e_1^*) > 0$. Using Eq. (3) it is straightforward to show that $\partial e_1^*/\partial U_2^E > 0$ and $\partial e_1^*/\partial\alpha < 0$. Using our comparative statics results for U_2^E and α^* we can then infer that $de_1^*/d\phi_2 > 0$ and $de_1^*/d\delta_2$, $de_1^*/d\sigma_2$, $de_1^*/d\sigma_2^V$, $de_1^*/dk_2 < 0$. Consequently, $d\rho_1(e_1^*)/d\phi_2 > 0$ and $d\rho_1(e_1^*)/d\delta_2$, $d\rho_1(e_1^*)/d\sigma_2$, $d\rho_1(e_1^*)/d\sigma_2^V$, $d\rho_1(e_1^*)/dk_2 < 0$. □

VC market: derivation of deal values and equity shares.

Let CV_i denote the value generated by the coalition $i = EAV, EV, EA, AV, E, A, V$. Using the Shapley value we get the following general deal values from the tripartite bargaining game:

$$D_2^E = \frac{1}{3}[CV_{EAV} - CV_{AV}] + \frac{1}{6}[CV_{EA} - CV_A] + \frac{1}{6}[CV_{EV} - CV_V] + \frac{1}{3}CV_E \quad (A.50)$$

$$D_2^A = \frac{1}{3}[CV_{EAV} - CV_{EV}] + \frac{1}{6}[CV_{EA} - CV_E] + \frac{1}{6}[CV_{AV} - CV_V] + \frac{1}{3}CV_A \quad (A.51)$$

$$D_2^V = \frac{1}{3}[CV_{EAV} - CV_{EA}] + \frac{1}{6}[CV_{EV} - CV_E] + \frac{1}{6}[CV_{AV} - CV_A] + \frac{1}{3}CV_V \quad (A.52)$$

We note that $CV_{EAV} = \pi$ and $CV_{AV} = CV_{EV} = CV_E = CV_A = CV_V = 0$. Moreover, by assumption we have $U_2^E + U_2^A > y_1$, so that $CV_{EA} = U_2^E + U_2^A$. Thus,

$$D_2^E = \frac{1}{3}\pi + \frac{1}{6}\left[U_2^E + U_2^A\right] \quad (A.53)$$

$$D_2^A = \frac{1}{3}\pi + \frac{1}{6}\left[U_2^E + U_2^A\right] \quad (A.54)$$

$$D_2^V = \frac{1}{3}\pi - \frac{1}{3}\left[U_2^E + U_2^A\right] \quad (A.55)$$

The deal values then allow us to derive the equilibrium equity shares β^{E*}, β^{A*}, and β^{V*}. The equilibrium equity share for entrepreneurs, β^{E*}, ensures that their actual net payoff equals their deal value from the bargaining game: $\beta^{E*}y_2 = D_2^E$. Solving this for β^{E*} yields

$$\beta^{E*} = \frac{D_2^E}{y_2} = \frac{1}{6y_2}\left[2\pi + U_2^E + U_2^A\right]. \quad (A.56)$$

Likewise we get

$$\beta^{A*} = \frac{D_2^A}{y_2} = \frac{1}{6y_2}\left[2\pi + U_2^E + U_2^A\right] \quad (A.57)$$

$$\beta^{V*} = \frac{k_2 + D_2^V}{y_2} = \frac{1}{3y_2}\left[3k_2 + \pi - \left(U_2^E + U_2^A\right)\right]. \quad (A.58)$$

Derivation of VC market equilibrium.

The first part of the derivation follows along the lines of the derivation of the angel market equilibrium: Using Eq. (13) we get $\theta_2^* = \left[\phi_2 D_2^V / \sigma_2^V\right]^2$. Moreover, using Eq. (14) and the relationship $M_2^{V*} = M_2^{E*}\theta_2^*$ we find

$$M_2^{V*} = g\rho_1(e_1^*)x_1^* \frac{\theta_2^*}{\delta_2 + \phi_2\sqrt{\theta_2^*}}. \quad (A.59)$$

Using $M_2^{E*} = M_2^{V*}/\theta_2^*$ and the definition of M_2^{V*}, we can write x_2^* as

$$x_2^* = \phi_2 \left[M_2^{V*} M_2^{E*}\right]^{0.5} = \frac{\phi_2 M_2^{V*}}{\sqrt{\theta_2^*}} = m_2^{E*} \frac{\phi_2 \sqrt{\theta_2^*}}{\delta_2 + \phi_2 \sqrt{\theta_2^*}}, \tag{A.60}$$

where $m_2^{E*} = g\rho_1(e_1^*)x_1^*$. Furthermore, using Eq. (15) and $q_2^V = x_2/M_2^V$ we find that $m_2^{V*} = q_2^V M_2^{V*} = x_2^*$.

Finally, using the equilibrium equity share β^{V*} for VCs we can write V_2^* as follows:

$$V_2^* = \frac{k_2}{\beta^{V*}} = \frac{k_2 y_2}{k_2 + D_2^V} = \left(\frac{3k_2}{3k_2 + \pi - (U_2^E + U_2^A)}\right) y_2. \tag{A.61}$$

Proof of Proposition 3.

First we need to derive a condition which defines U_2^E. We can write Eq. (12) as

$$U_2^E [r + \delta_2] = -\sigma_2 + q_2^E \left[D_2^E - U_2^E\right]. \tag{A.62}$$

Note that $D_2^E - U_2^E = \pi/3 - 2U_2^E/3 = D_2^V$. Using $q_2^E = \phi_2 \left[M_2^{V*}/M_2^{E*}\right]^{0.5} = \phi_2\sqrt{\theta_2^*} = \phi_2^2 D_2^V/\sigma_2^V$, we get the following condition which defines U_2^E:

$$U_2^E [r + \delta_2] - \frac{\phi_2^2}{\sigma_2^V}\left[D_2^V\right]^2 + \sigma_2 = 0. \tag{A.63}$$

Consider the equilibrium degree of competition θ_2^*. Recall that $U_2^A = U_2^E$ in equilibrium; thus,

$$\frac{d\theta_2^*}{dU_2^A} = \frac{d\theta_2^*}{dU_2^E} = 2\frac{\phi_2^2 D_2^V}{[\sigma_2^V]^2}\frac{dD_2^V}{dU_2^E} = -\frac{4}{3}\frac{\phi_2^2 D_2^V}{[\sigma_2^V]^2} < 0. \tag{A.64}$$

Note that δ_2 only affects U_2^E in the definition of θ_2^*. Implicitly differentiating U_2^E w.r.t. δ_2 yields

$$\frac{dU_2^E}{d\delta_2} = -\frac{U_2^E}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0, \tag{A.65}$$

which implies that $d\theta_2^*/d\delta_2 > 0$. Likewise, σ_2 only affects U_2^E in the definition of θ_2^*. We get

$$\frac{dU_2^E}{d\sigma_2} = -\frac{1}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \tag{A.66}$$

Thus, $d\theta_2^*/d\sigma_2 > 0$. Next, differentiating U_2^E w.r.t. ϕ_2 yields

$$\frac{d\theta_2^*}{d\phi_2} = 2\frac{\phi_2 D_2^V}{[\sigma_2^V]^2}\left[D_2^V + \phi_2\frac{dD_1^V}{dU_2^E}\frac{dU_2^E}{d\phi_2}\right] = 2\frac{\phi_2 D_2^V}{[\sigma_2^V]^2}\left[D_2^V - \frac{2}{3}\phi_2\frac{dU_2^E}{d\phi_2}\right], \tag{A.67}$$

with

$$\frac{dU_2^E}{d\phi_2} = \frac{2\frac{\phi_2}{\sigma_2^V}\left[D_2^V\right]^2}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} > 0. \tag{A.68}$$

Therefore,

$$\frac{d\theta_2^*}{d\phi_2} = 2\frac{\phi_2 D_2^V}{\left[\sigma_2^V\right]^2}\frac{(r+\delta_2)D_2^V}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} > 0. \tag{A.69}$$

Likewise,

$$\frac{d\theta_2^*}{d\sigma_2^V} = 2\frac{\phi_2^2 D_2^V}{\sigma_2^V}\frac{1}{\left[\sigma_2^V\right]^2}\left[-\frac{2}{3}\frac{dU_2^E}{d\sigma_2^V}\sigma_2^V - D_2^V\right], \quad \text{with} \quad \frac{dU_2^E}{d\sigma_2^V} = -\frac{\frac{\phi_2^2 D_2^V D_2^V}{\left[\sigma_2^V\right]^2}}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \tag{A.70}$$

Consequently,

$$\frac{d\theta_2^*}{d\sigma_2^V} = -2\frac{\phi_2^2 D_2^V}{\sigma_2^V}\frac{1}{\left[\sigma_2^V\right]^2}\frac{\frac{2}{3}\frac{\phi_2^2}{\sigma_2^V}\left[D_2^V\right]^2 + (r+\delta_2)D_2^V}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \tag{A.71}$$

Moreover, we get

$$\frac{d\theta_2^*}{dk_2} = 2\frac{\phi_2^2 D_2^V}{\left[\sigma_2^V\right]^2}\frac{dD_2^V}{dk_2} = 2\frac{\phi_2^2 D_2^V}{\left[\sigma_2^V\right]^2}\left[-\frac{1}{3} - \frac{2}{3}\frac{dU_2^E}{dk_2}\right], \quad \text{with} \quad \frac{dU_2^E}{dk_2} = -\frac{\frac{2}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \tag{A.72}$$

Thus,

$$\frac{d\theta_2^*}{dk_2} = -\frac{2}{3}\frac{\phi_2^2 D_2^V}{\left[\sigma_2^V\right]^2}\frac{r+\delta_2}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \tag{A.73}$$

Next, recall that $m_2^{V*} = x_2^*$ is given by

$$m_2^{V*} = x_2^* = \underbrace{g\rho_1(e_1^*)x_1^*}_{=m_2^{E*}}\underbrace{\frac{\phi_2\sqrt{\theta_2^*}}{\delta_2+\phi_2\sqrt{\theta_2^*}}}_{\equiv T}. \tag{A.74}$$

We have shown in Proof of Proposition 2 that $dx_1^*/d\phi_2 > 0$ and $dx_1^*/d\delta_2$, $dx_1^*/d\sigma_2$, $dx_1^*/d\sigma_2^V$, $dx_1^*/dk_2 < 0$. Likewise, we have shown that $d\rho_1(e_1^*)/d\phi_2 > 0$ and $d\rho_1(e_1^*)/d\delta_2$, $d\rho_1(e_1^*)/d\sigma_2$, $d\rho_1(e_1^*)/d\sigma_2^V$, $d\rho_1(e_1^*)/dk_2 < 0$. Moreover, it is straightforward to verify that $dT/d(\phi_2\sqrt{\theta_2^*}) > 0$. Using our comparative statics results for θ_2^*, we can infer that $dT/d\phi_2$, $dT/d\delta_2$, $dT/d\sigma_2 > 0$, and $dT/d\sigma_2^V$, $dT/dk_2 < 0$. All this implies that $dm_2^{V*}/d\phi_2 > 0$ and $dm_2^{V*}/d\sigma_2^V$, $dm_2^{V*}/dk_2 < 0$, while the effects of δ_2 and σ_2 on m_2^{V*} are ambiguous.

Now consider the equilibrium late stage valuation V_2^*:

$$V_2^* = \left(\frac{3k_2}{3k_2+\pi-2U_2^E}\right)y_2. \tag{A.75}$$

Recall that $dU_2^E/d\phi_2 > 0$, and $dU_2^E/d\sigma_2$, $dU_2^E/d\sigma_2^V$, $dU_2^E/d\delta_2 < 0$. Thus, $dV_2^*/d\phi_2 > 0$ and $dV_2^*/d\sigma_2$, $dV_2^*/d\sigma_2^V$, $dV_2^*/d\delta_2 < 0$. Furthermore, recall that V_2^* can also be written as $V_2^* = k_2 y_2/(k_2 + D_2^V)$. Taking the first derivative of V_2^* w.r.t. k_2 yields

$$\frac{dV_2^*}{dk_2} = \frac{k_2 + D_2^V - k_2\left[1 - \frac{1}{3} - \frac{2}{3}\frac{dU_2^E}{dk_2}\right]}{\left[k_2 + D_2^V\right]^2} y_2 = \frac{\overbrace{\frac{1}{3}k_2 + D_2^V + \frac{2}{3}k_2\frac{dU_2^E}{dk_2}}^{\equiv N}}{\left[k_2 + D_2^V\right]^2} y_2. \tag{A.76}$$

Note that the denominator is always positive. Moreover, we have $N > 0$ for $k_2 \to 0$. Thus, $dV_2^*/dk_2 > 0$ for $k_2 \to 0$. To verify that $dV_2^*/dk_2 > 0$ for all $k_2 > 0$, it is sufficient to show that $dN/dk_2 > 0$:

$$\frac{dN}{dk_2} = \frac{1}{3} - \frac{1}{3} - \frac{2}{3}\frac{dU_2^E}{dk_2} + \frac{2}{3}\left[\frac{dU_2^E}{dk_2} + k_2\frac{d^2U_2^E}{dk_2^2}\right] = \frac{2}{3}k_2\frac{d^2U_2^E}{dk_2^2}. \tag{A.77}$$

It remains to identify the sign of $d^2U_2^E/dk_2^2$. Using $a_2 \equiv \phi_2^2/\sigma_2^V$ we can write dU_2^E/dk_2 as

$$\frac{dU_2^E}{dk_2} = -\frac{\frac{2}{3}a_2 D_2^V}{r + \delta_2 + \frac{4}{3}a_2 D_2^V} = -\frac{\frac{2}{3}}{(r + \delta_2)\left[a_2 D_2^V\right]^{-1} + \frac{4}{3}}. \tag{A.78}$$

Thus,

$$\frac{d^2U_2^E}{dk_2^2} = \frac{\frac{2}{9}a_2(r + \delta_2)\left[a_2 D_2^V\right]^{-2}\left[1 + 2\frac{dU_2^E}{dk_2}\right]}{\left[(r + \delta_2)\left[a_2 D_2^V\right]^{-1} + \frac{4}{3}\right]^2}. \tag{A.79}$$

Note that

$$1 + 2\frac{dU_2^E}{dk_2} = 1 - \frac{\frac{4}{3}a_2 D_2^V}{r + \delta_2 + \frac{4}{3}a_2 D_2^V} = \frac{r + \delta_2}{r + \delta_2 + \frac{4}{3}a_2 D_2^V} > 0. \tag{A.80}$$

Hence, $d^2U_2^E/dk_2^2 > 0$, so that $dN/dk_2 > 0$. Consequently, $dV_2^*/dk_2 > 0$. □

Proof of Proposition 4.

We can see from Eq. (A.63) that U_2^E (and therefore U_2^A) does not depend on the early stage parameters ϕ_1, δ_1, σ_1^E, σ_1^A, and k_1. This also implies that D_2^V, and therefore θ_2^* and V_2^*, do not depend on these parameters.

Now consider the equilibrium inflow of start-ups $m_2^{E*} = g\rho_1(e_1^*)x_1^*$. Recall from Proposition 1 that $dx_1^*/d\phi_1 > 0$ and $dx_1^*/d\sigma_1^A$, $dx_1^*/dk_1 < 0$, while the effects of δ_1 and σ_1^E are ambiguous. Moreover, we know from Proposition 1 that $d\rho_1(e_1^*)/d\phi_1 > 0$ and $d\rho_1(e_1^*)/d\delta_1$, $d\rho_1(e_1^*)/d\sigma_1^E$, $d\rho_1(e_1^*)/d\sigma_1^A$, $d\rho_1(e_1^*)/dk_1 < 0$. This implies that $dm_2^{E*}/d\phi_1 > 0$ and $dm_2^{E*}/d\sigma_1^A$, $dm_2^{E*}/dk_1 < 0$, while the effects of δ_1 and σ_1^E are ambiguous.

Finally consider the equilibrium inflow of VCs m_2^{V*}, as defined by

$$m_2^{V*} = x_2^* = m_2^{E*}\frac{\phi_2\sqrt{\theta_2^*}}{\delta_2 + \phi_2\sqrt{\theta_2^*}}. \tag{A.81}$$

Recall that θ_2^* does not depend on the early stage parameters. Our comparative statics results for m_2^{E*} then imply that $dm_2^{V*}/d\phi_1 > 0$ and $dm_2^{V*}/d\sigma_1^A$, $dm_2^{V*}/dk_1 < 0$, while the effects of δ_1 and σ_1^E are ambiguous. □

Angel protection: derivation of deal values and equity shares.

The new coalition values are given by $CV_{EAV} = \pi$, $CV_{EA} = U_2^E + U_2^A$, $CV_{EV} = \lambda\pi$, and $CV_{AV} = CV_E = CV_A = CV_V = 0$. Using the general deal values (A.50), (A.51), and (A.52), we get

$$D_2^E = \frac{1}{6}[2+\lambda]\pi + \frac{1}{6}\left[U_2^E + U_2^A\right] \tag{A.82}$$

$$D_2^A = \frac{1}{3}[1-\lambda]\pi + \frac{1}{6}\left[U_2^E + U_2^A\right] \tag{A.83}$$

$$D_2^V = \frac{1}{6}[2+\lambda]\pi - \frac{1}{3}\left[U_2^E + U_2^A\right] \tag{A.84}$$

The new equilibrium equity share for entrepreneurs, β^{E*}, ensures that their actual net payoff equals their deal value from the bargaining game: $\beta^{E*}y_2 = D_2^E$. Solving this for β^{E*} yields

$$\beta^{E*} = \frac{D_2^E}{y_2} = \frac{1}{6y_2}\left[(2+\lambda)\pi + U_2^E + U_2^A\right]. \tag{A.85}$$

Likewise we get

$$\beta^{A*} = \frac{D_2^A}{y_2} = \frac{1}{6y_2}\left[2(1-\lambda)\pi + U_2^E + U_2^A\right] \tag{A.86}$$

$$\beta^{V*} = \frac{k_2 + D_2^V}{y_2} = \frac{1}{6y_2}\left[6k_2 + (2+\lambda)\pi - 2\left(U_2^E + U_2^A\right)\right]. \tag{A.87}$$

Proof of Proposition 5.

We first show that $dU_2^A/d\lambda < 0$. Note that $D_2^A \neq D_2^E$ for $\lambda > 0$, and recall that $q_2^E = \phi_2\left[M_2^{V*}/M_2^{E*}\right]^{0.5} = \phi_2^2 D_2^V/\sigma_2^V$. Thus, using Eq. (12) we define

$$F \equiv U_2^E(r+\delta_2) + \sigma - a_2 D_2^V\left[D_2^E - U_2^E\right] = 0 \tag{A.88}$$

$$G \equiv U_2^A(r+\delta_2) + \sigma - a_2 D_2^V\left[D_2^A - U_2^A\right] = 0, \tag{A.89}$$

where $a_2 = \phi_2^2/\sigma_2^V$. Using Cramer's rule we get

$$\frac{dU_2^A}{d\lambda} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial\lambda} & \frac{\partial F}{\partial U_2^E} \\ -\frac{\partial G}{\partial\lambda} & \frac{\partial G}{\partial U_2^E}\end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial U_2^A} & \frac{\partial F}{\partial U_2^E} \\ \frac{\partial G}{\partial U_2^A} & \frac{\partial G}{\partial U_2^E}\end{vmatrix}} = \frac{-\frac{\partial F}{\partial\lambda}\frac{\partial G}{\partial U_2^E} + \frac{\partial G}{\partial\lambda}\frac{\partial F}{\partial U_2^E}}{\frac{\partial F}{\partial U_2^A}\frac{\partial G}{\partial U_2^E} - \frac{\partial G}{\partial U_2^A}\frac{\partial F}{\partial U_2^E}}. \tag{A.90}$$

The denominator is negative if

$$\frac{\partial G}{\partial U_2^A}\frac{\partial F}{\partial U_2^E} > \frac{\partial F}{\partial U_2^A}\frac{\partial G}{\partial U_2^E}, \tag{A.91}$$

which is equivalent to

$$\begin{aligned} & \left[r+\delta_2+\frac{1}{6}a_2\left[2\left[D_2^A-U_2^A\right]+5D_2^V\right]\right]\left[r+\delta_2+\frac{1}{6}a_2\left[2\left[D_2^E-U_2^E\right]+5D_2^V\right]\right] \\ > \; & \frac{1}{6}a_2\left[2\left[D_2^E-U_2^E\right]-D_2^V\right]\frac{1}{6}a_2\left[2\left[D_2^A-U_2^A\right]-D_2^V\right]. \end{aligned} \tag{A.92}$$

If this condition holds for $r+\delta_2=0$, then it also holds for all $r+\delta_2>0$. Setting $r+\delta_2=0$ we get

$$10\left[D_2^A-U_2^A\right]D_2^V+10D_2^V\left[D_2^E-U_2^E\right]+24\left[D_2^V\right]^2 > -2\left[D_2^E-U_2^E\right]D_2^V-2\left[D_2^A-U_2^A\right]D_2^V. \tag{A.93}$$

This condition is satisfied as $D_2^E>U_2^E$ and $D_2^A>U_2^A$. Thus, the denominator of $dU_2^A/d\lambda$ is strictly negative. Likewise, the numerator is positive if

$$\frac{\partial G}{\partial \lambda}\frac{\partial F}{\partial U_2^E} > \frac{\partial F}{\partial \lambda}\frac{\partial G}{\partial U_2^E}, \tag{A.94}$$

which is equivalent to

$$\begin{aligned} & \frac{1}{6}\pi a_2\left[\left[D_2^A-U_2^A\right]-2D_2^V\right]\left[r+\delta_2+\frac{1}{6}a_2\left[2\left[D_2^E-U_2^E\right]+5D_2^V\right]\right] \\ < \; & \frac{1}{6}\pi a_2\left[\left[D_2^E-U_2^E\right]+D_2^V\right]\frac{1}{6}a_2\left[2\left[D_2^A-U_2^A\right]-D_2^V\right]. \end{aligned} \tag{A.95}$$

This condition can be written as

$$\frac{2}{a_2}(r+\delta_2)\left[\left[D_2^A-U_2^A\right]-2D_2^V\right]+\left[D_2^A-U_2^A\right]D_2^V-D_2^V\left[D_2^E-U_2^E\right]<3\left[D_2^V\right]^2. \tag{A.96}$$

From F and G we know that

$$D_2^V\left[D_2^E-U_2^E\right]=\frac{U_2^E(r+\delta_2)+\sigma}{a_2} \quad \text{and} \quad D_2^V\left[D_2^A-U_2^A\right]=\frac{U_2^A(r+\delta_2)+\sigma}{a_2}, \tag{A.97}$$

so that we can write condition (A.96) as follows:

$$\frac{2}{a_2}(r+\delta_2)\left[\left[D_2^A-U_2^A\right]-2D_2^V\right]+\frac{U_2^A(r+\delta_2)+\sigma}{a_2}-\frac{U_2^E(r+\delta_2)+\sigma}{a_2}<3\left[D_2^V\right]^2 \tag{A.98}$$

$$\Leftrightarrow \quad (r+\delta_2)\underbrace{\left[2D_2^A-U_2^A-4D_2^V-U_2^E\right]}_{\equiv T}<3\left[D_2^V\right]^2 a_2. \tag{A.99}$$

We now show that $T < 0$. Using the definitions of D_2^A and D_2^V we can write $T < 0$ as

$$\frac{2}{3}[1-\lambda]\pi + \frac{1}{3}\left[U_2^E + U_2^A\right] - U_2^A - \frac{2}{3}[2+\lambda]\pi + \frac{4}{3}\left[U_2^E + U_2^A\right] - U_2^E < 0 \quad \text{(A.100)}$$

$$\Leftrightarrow \qquad U_2^E + U_2^A < [1+2\lambda]\pi. \quad \text{(A.101)}$$

This condition is satisfied for all $\lambda \geq 0$ because $\pi > U_2^E + U_2^A$. Thus, the numerator of $dU_2^A/d\lambda$ is strictly positive. Consequently, $dU_2^A/d\lambda < 0$. Finally note that $\partial D_2^E/\partial\lambda = \pi/6 < \left|\partial D_2^A/\partial\lambda\right| = \pi/3$. Thus, $d\left[U_2^E + U_2^A\right]/d\lambda < 0$, which implies that $dD_2^V/d\lambda > 0$.

Next we analyze the effects of λ on the early stage equilibrium variables. Consider the equilibrium degree of competition θ_1^*. We get

$$\frac{d\theta_1^*}{d\lambda} = 2\frac{\phi_1^2}{\left[\sigma_1^A\right]^2} D_1^A \frac{dD_1^A}{d\lambda}. \quad \text{(A.102)}$$

Recall that

$$\frac{d}{d\lambda}\left(U_2^A + U_2^E\right) = \underbrace{\frac{dU_2^A}{d\lambda}}_{<0} + \underbrace{\frac{dU_2^E}{d\lambda}}_{>0} < 0. \quad \text{(A.103)}$$

This implies

$$\frac{dD_1^A}{d\lambda} + \frac{dD_1^E}{d\lambda} < 0 \quad \Rightarrow \quad \frac{dD_1^A}{d\lambda} < 0. \quad \text{(A.104)}$$

Thus, $d\theta_1^*/d\lambda < 0$.

Now consider the equilibrium entry of entrepreneurs m_1^{E*}. Using Eq. (A.11), we get

$$\frac{dU_1^E}{d\lambda} = \frac{\frac{\phi_1^2}{\sigma_1^A}\left[\frac{dD_1^A}{d\lambda}\left[D_1^E - U_1^E\right] + D_1^A\frac{dD_1^E}{d\lambda}\right]}{r + \delta_1 - \frac{\phi_1^2}{\sigma_1^A}\left[\frac{d\Gamma}{d\alpha}\frac{d\alpha^*}{dU_1^E} + \frac{\partial\Gamma}{\partial U_1^E}\right]}, \quad \text{(A.105)}$$

where $\Gamma = D_1^A\left[D_1^E - U_1^E\right]$. Note that $d\Gamma/d\alpha = 0$; see Eq. (A.1). Thus,

$$\frac{dU_1^E}{d\lambda} = \frac{\frac{\phi_1^2}{\sigma_1^A}\left[\frac{dD_1^A}{d\lambda}\left[D_1^E - U_1^E\right] + D_1^A\frac{dD_1^E}{d\lambda}\right]}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A}D_1^A}, \quad \text{(A.106)}$$

where the denominator is positive. Consequently, $dU_1^E/d\lambda < 0$ if

$$\frac{dD_1^A}{d\lambda}\left[D_1^E - U_1^E\right] + D_1^A\frac{dD_1^E}{d\lambda} < 0. \quad \text{(A.107)}$$

Using Eq. (A.1) we can derive the following expression for $D_1^E - U_1^E$:

$$D_1^E - U_1^E = -\frac{\frac{dD_1^E}{d\alpha}}{\frac{dD_1^A}{d\alpha}} D_1^A, \quad \text{(A.108)}$$

so that Eq. (A.107) can be written as

$$\frac{dD_1^A}{d\lambda}\underbrace{\frac{\left(-\frac{dD_1^E}{d\alpha}\right)}{\frac{dD_1^A}{d\alpha}}}_{\equiv X}+\frac{dD_1^E}{d\lambda}<0. \tag{A.109}$$

Recall that $d(D_1^A+D_1^E)/d\lambda<0$, with $dD_1^A/d\lambda<0$; thus, this condition is satisfied when $X\geq 1$. Note that $dD_1^E/d\alpha<0$ and $dD_1^A/d\alpha>0$. Hence, $X\geq 1$ if

$$0\geq\frac{dD_1^A}{d\alpha}+\frac{dD_1^E}{d\alpha}=\frac{d}{d\alpha}\left[D_1^A+D_1^E\right]. \tag{A.110}$$

It is easy to show that the joint surplus is maximized when $\alpha=0$ (which maximizes effort incentives for the entrepreneur); thus

$$\left.\frac{d\left[D_1^A+D_1^E\right]}{d\alpha}\right|_{\alpha=\alpha^*>0}<0, \tag{A.111}$$

so that $X\geq 1$. Consequently, $dU_1^E/d\lambda<0$, and therefore $dm_1^{E*}/d\lambda=dF(U_1^E)/d\lambda<0$.

Next consider the equilibrium inflow of angels, m_1^{A*}, which is defined by

$$m_1^{A*}=x_1^*=\underbrace{F(U_1^E)}_{=m_1^{E*}}\underbrace{\frac{\phi_1\sqrt{\theta_1^*}}{\delta_1+\phi_1\sqrt{\theta_1^*}}}_{\equiv T}. \tag{A.112}$$

Note that $dT/d\sqrt{\theta_1^*})>0$. Our comparative statics results for m_1^{E*} and θ_1^* then imply that $dm_1^{A*}/d\lambda=dx_1^*/d\lambda<0$.

Now consider the angel's equilibrium equity share α^*, which is defined by Eq. (A.1). We get

$$\frac{d\alpha^*}{d\lambda}=\frac{d\alpha^*}{dU_2^E}\frac{dU_2^E}{d\lambda}+\frac{d\alpha^*}{dU_2^A}\frac{dU_2^A}{d\lambda}, \tag{A.113}$$

where $dU_2^E/d\lambda>0$ and $dU_2^A/d\lambda<0$. Moreover, the Nash bargaining solution implies that $d\alpha^*/dU_2^E>0$ and $d\alpha^*/dU_2^A<0$. Thus, $d\alpha^*/d\lambda>0$. For the equilibrium valuation $V_1^*=k_1/\alpha^*$ this concurrently implies that $dV_1^*/d\lambda<0$. Finally we know that $dD_1^E/d\lambda>0$ in equilibrium. Using the Envelope Theorem we get

$$\frac{dD_1^E}{d\lambda}=\rho_1(e_1)\underbrace{\frac{d}{d\lambda}\left[gU_2^E+(1-g)(1-\alpha^*)y_1\right]}_{\equiv T}>0, \tag{A.114}$$

which implies that $T>0$. Using Eq. (3) we find

$$\frac{de_1^*}{d\lambda}=-\frac{\rho_1'(e_1)\overbrace{\frac{d}{d\lambda}\left[gU_2^E+(1-g)(1-\alpha)y_1\right]}^{=T}}{\frac{d}{de_1}\left[\rho_1'(e_1)\left[gU_2^E+(1-g)(1-\alpha)y_1\right]-c'(e_1)\right]}, \tag{A.115}$$

where $T > 0$, and the denominator is negative due to the second-order condition for e_1^*. Thus, $de_1^*/d\lambda > 0$. This in turn implies that $d\rho_1(e_1^*)/d\lambda > 0$.

Finally we analyze the effects of λ on the late stage equilibrium variables. Note that $d\left(U_2^E + U_2^A\right)/d\lambda < 0$ also implies that $dD_2^V/d\lambda > 0$. Using the definitions of θ_2^*, β^{V*} and V_2^*, we can then infer that $d\theta_2^*/d\lambda > 0$, $d\beta^{V*}/d\lambda > 0$ and $dV_2^*/d\lambda < 0$. Moreover,

$$\frac{dm_2^{E*}}{d\lambda} = \frac{d}{d\lambda}\left[g\rho_1(e_1^*)x_1^*\right] = g\left[\rho_1'(e_1^*)\frac{de_1^*}{d\lambda}x_1^* + \rho_1(e_1^*)\frac{dx_1^*}{d\lambda}\right]. \tag{A.116}$$

In general, the effect on m_2^{E*} is ambiguous as $de_1^*/d\lambda > 0$ and $dx_1^*/d\lambda < 0$. However, we can see that $dm_2^{E*}/d\lambda < 0$ when $\rho_1'(e_1^*) \to 0$. Moreover, for $\delta_1 \to 0$ we have $m_1^{A*} = m_1^{E*}$; with m_1^{E*} being sufficiently inelastic, we have $dx_1^*/d\lambda \to 0$, so that $dm_2^{E*}/d\lambda > 0$. Next, recall that m_2^{V*} is defined by

$$m_2^{V*} = x_2^* = m_2^{E*}\underbrace{\frac{\phi_2\sqrt{\theta_2^*}}{\delta_2 + \phi_2\sqrt{\theta_2^*}}}_{\equiv T}. \tag{A.117}$$

One can show that $dT/d\sqrt{\theta_2^*} > 0$, so that $dT/d\lambda > 0$. Recall, however, that the sign of $dm_2^{E*}/d\lambda$ is ambiguous. Thus, the effect of λ on $m_2^{V*} = x_2^*$ is also ambiguous. □

Angel protection – angel not required for VC search.

Suppose the entrepreneur can search for a VC without the angel. The entrepreneur then incurs the search cost $\gamma\sigma$, with $\gamma > 2$. Using Nash bargaining, the deal values for the VC ($\widehat{D}_2^V$) and the entrepreneur ($\widehat{D}_2^E$) are then given by

$$\widehat{D}_2^V = \frac{1}{2}\left[\lambda\pi - \widehat{U}_2^E\right] \qquad \widehat{D}_2^E = \frac{1}{2}\left[\lambda\pi + \widehat{U}_2^E\right], \tag{A.118}$$

where $\widehat{U}_2^E$ denotes the entrepreneur's outside option.

Now consider the bargaining problem at the late stage between entrepreneur, angel, and VC. The new coalition values are given by $CV_{EAV} = \pi$, $CV_{EA} = U_2^E + U_2^A$, $CV_{EV} = \lambda\pi$, $CV_E = \widehat{U}_2^E$, and $CV_{AV} = CV_A = CV_V = 0$. Using the Shapley value we then get the following deal values:

$$D_2^E = \frac{1}{6}\left[2+\lambda\right]\pi + \frac{1}{6}\left[U_2^E + U_2^A\right] + \frac{1}{3}\widehat{U}_2^E \tag{A.119}$$

$$D_2^A = \frac{1}{3}\left[1-\lambda\right]\pi + \frac{1}{6}\left[U_2^E + U_2^A\right] - \frac{1}{6}\widehat{U}_2^E \tag{A.120}$$

$$D_2^V = \frac{1}{6}\left[2+\lambda\right]\pi - \frac{1}{3}\left[U_2^E + U_2^A\right] - \frac{1}{6}\widehat{U}_2^E \tag{A.121}$$

The expected utilities from search, U_2^A, U_2^E, and $\widehat{U}_2^E$, are then defined by

$$F \equiv U_2^A (r + \delta_2) + \sigma - a_2 D_2^V \left[D_2^A - U_2^A\right] = 0 \qquad \text{(A.122)}$$

$$G \equiv U_2^E (r + \delta_2) + \sigma - a_2 D_2^V \left[D_2^E - U_2^E\right] = 0 \qquad \text{(A.123)}$$

$$H \equiv \widehat{U}_2^E (r + \delta_2) + \gamma\sigma - a_2 \widehat{D}_2^V \left[\widehat{D}_2^E - \widehat{U}_2^E\right] = 0, \qquad \text{(A.124)}$$

where $a_2 = \phi_2^2/\sigma_2^V$. Using H we find that

$$\frac{d\widehat{U}_2^E}{d\lambda} = \frac{\frac{1}{2}a_2\pi\left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V\right]}{r + \delta_2 + \frac{1}{2}a_2\left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V\right]} > 0 \qquad \text{(A.125)}$$

$$\frac{d\widehat{U}_2^E}{d\gamma} = -\frac{\sigma}{r + \delta_2 + \frac{1}{2}a_2\left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V\right]} < 0 \qquad \text{(A.126)}$$

Next we show that $dU_2^A/d\lambda < 0$. Using Cramer's rule we get $dU_2^A/d\lambda = A/B$, where

$$A = \begin{vmatrix} -\frac{\partial F}{\partial\lambda} & \frac{\partial F}{\partial U_2^E} & \frac{\partial F}{\partial \widehat{U}_2^E} \\ -\frac{\partial G}{\partial\lambda} & \frac{\partial G}{\partial U_2^E} & \frac{\partial G}{\partial \widehat{U}_2^E} \\ -\frac{\partial H}{\partial\lambda} & \frac{\partial H}{\partial U_2^E} & \frac{\partial H}{\partial \widehat{U}_2^E} \end{vmatrix} \qquad B = \begin{vmatrix} \frac{\partial F}{\partial U_2^A} & \frac{\partial F}{\partial U_2^E} & \frac{\partial F}{\partial \widehat{U}_2^E} \\ \frac{\partial G}{\partial U_2^A} & \frac{\partial G}{\partial U_2^E} & \frac{\partial G}{\partial \widehat{U}_2^E} \\ \frac{\partial H}{\partial U_2^A} & \frac{\partial H}{\partial U_2^E} & \frac{\partial H}{\partial \widehat{U}_2^E} \end{vmatrix} \qquad \text{(A.127)}$$

Consider first the denominator B. Since $\partial H/\partial U_2^A = 0$ and $\partial H/\partial U_2^E = 0$, we can write B as

$$B = \frac{\partial H}{\partial \widehat{U}_2^E}\left[\frac{\partial F}{\partial U_2^A}\frac{\partial G}{\partial U_2^E} - \frac{\partial F}{\partial U_2^E}\frac{\partial G}{\partial U_2^A}\right], \qquad \text{(A.128)}$$

where

$$\frac{\partial H}{\partial \widehat{U}_2^E} = r + \delta_2 + \frac{1}{2}a_2\left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V\right] > 0. \qquad \text{(A.129)}$$

Thus, $B > 0$ if

$$\frac{\partial F}{\partial U_2^A}\frac{\partial G}{\partial U_2^E} > \frac{\partial F}{\partial U_2^E}\frac{\partial G}{\partial U_2^A}, \qquad \text{(A.130)}$$

which can be written as

$$\left[r + \delta_2 + \frac{1}{6}a_2\left[2\left[D_2^A - U_2^A\right] + 5D_2^V\right]\right]\left[r + \delta_2 + \frac{1}{6}a_2\left[2\left[D_2^E - U_2^E\right] + 5D_2^V\right]\right]$$
$$> \frac{1}{6}a_2\left[2\left[D_2^A - U_2^A\right] - D_2^V\right]\frac{1}{6}a_2\left[2\left[D_2^E - U_2^E\right] - D_2^V\right]. \qquad \text{(A.131)}$$

Note that this condition holds for all $r + \delta_2 > 0$ if it holds for $r + \delta_2 = 0$. Setting $r + \delta_2 = 0$ we get

$$12\left[D_2^A - U_2^A\right]D_2^V + 12D_2^V\left[D_2^E - U_2^E\right] + 24\left[D_2^V\right]^2 > 0. \qquad \text{(A.132)}$$

Note that $D_2^E > U_2^E$ and $D_2^A > U_2^A$. Thus, this condition is satisfied, so that $B > 0$. Next consider the numerator A. With $\partial H/\partial U_2^E = 0$ we can write A as

$$A = \underbrace{\left[-\frac{\partial F}{\partial \lambda}\frac{\partial G}{\partial U_2^E} + \frac{\partial F}{\partial U_2^E}\frac{\partial G}{\partial \lambda}\right]}_{\equiv X_1}\frac{\partial H}{\partial \widehat{U}_2^E} - \frac{\partial H}{\partial \lambda}\underbrace{\left[\frac{\partial F}{\partial U_2^E}\frac{\partial G}{\partial \widehat{U}_2^E} - \frac{\partial F}{\partial \widehat{U}_2^E}\frac{\partial G}{\partial U_2^E}\right]}_{\equiv X_2}. \tag{A.133}$$

Recall that $\partial H/\partial \widehat{U}_2^E > 0$. Moreover,

$$\frac{\partial H}{\partial \lambda} = -\frac{1}{2}\pi a_2\left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V\right] < 0. \tag{A.134}$$

Thus, $A < 0$ when $X_1 < 0$ and $X_2 < 0$. Note that $X_1 < 0$ if

$$\frac{\partial F}{\partial U_2^E}\frac{\partial G}{\partial \lambda} < \frac{\partial F}{\partial \lambda}\frac{\partial G}{\partial U_2^E}, \tag{A.135}$$

which can be written as

$$\begin{aligned} & \frac{1}{6}a_2\left[2\left[D_2^A - U_2^A\right] - D_2^V\right]\frac{1}{6}\pi a_2\left[D_2^E - U_2^E + D_2^V\right] \\ > \quad & \frac{1}{6}\pi a_2\left[D_2^A - U_2^A - 2D_2^V\right]\left[r + \delta_2 + \frac{1}{6}a_2\left[2\left[D_2^E - U_2^E\right] + 5D_2^V\right]\right]. \end{aligned} \tag{A.136}$$

Simplifying yields

$$\frac{2}{a_2}(r+\delta_2)\left[\left[D_2^A - U_2^A\right] - 2D_2^V\right] + \left[D_2^A - U_2^A\right]D_2^V - D_2^V\left[D_2^E - U_2^E\right] < 3\left[D_2^V\right]^2. \tag{A.137}$$

From F and G we know that

$$D_2^V\left[D_2^A - U_2^A\right] = \frac{U_2^A(r+\delta_2) + \sigma}{a_2} \quad \text{and} \quad D_2^V\left[D_2^E - U_2^E\right] = \frac{U_2^E(r+\delta_2) + \sigma}{a_2}, \tag{A.138}$$

so that condition (A.137) can be written as

$$(r+\delta_2)\underbrace{\left[2D_2^A - U_2^A - 4D_2^V - U_2^E\right]}_{\equiv T} \;<\; 3\left[D_2^V\right]^2 a_2. \tag{A.139}$$

It remains to prove that $T < 0$. Using the definitions of D_2^A and D_2^V we can write $T < 0$ as

$$U_2^E + U_2^A \;<\; [1 + 2\lambda]\,\pi. \tag{A.140}$$

This condition is satisfied for all $\lambda \geq 0$ as $\pi > U_2^E + U_2^A$. Thus, $X_1 < 0$. Moreover, $X_2 < 0$ if

$$\frac{\partial F}{\partial U_2^E}\frac{\partial G}{\partial \widehat{U}_2^E} < \frac{\partial F}{\partial \widehat{U}_2^E}\frac{\partial G}{\partial U_2^E}, \tag{A.141}$$

which is equivalent to

$$\begin{aligned} & \frac{1}{6}a_2\left[2\left[D_2^A-U_2^A\right]-D_2^V\right]\frac{1}{6}a_2\left[D_2^E-U_2^E-2D_2^V\right] \\ < \; & \frac{1}{6}a_2\left[D_2^A-U_2^A+D_2^V\right]\left[r+\delta_2+\frac{1}{6}a_2\left[2\left[D_2^E-U_2^E\right]+5D_2^V\right]\right]. \end{aligned} \tag{A.142}$$

Again, $D_2^A > U_2^A$ and $D_2^E > U_2^E$. Thus, if this condition holds for $r+\delta_2 = 0$, then it also holds for all $r+\delta_2 > 0$. Setting $r+\delta_2 = 0$ we get

$$0 < 3D_2^V \underbrace{\left[D_2^E-U_2^E\right]}_{>0} + 9D_2^V \underbrace{\left[D_2^A-U_2^A\right]}_{>0} + 3D_2^V D_2^V. \tag{A.143}$$

Hence, $X_2 < 0$, so that $A < 0$. Consequently, $dU_2^A/d\lambda < 0$. Moreover, note that $\partial D_2^E/\partial\lambda = \pi/6 < \left|\partial D_2^A/\partial\lambda\right| = \pi/3$. Thus, $d\left[U_2^E+U_2^A\right]/d\lambda < 0$. Finally, using H we get

$$\frac{d\widehat{U}_2^E}{d\lambda} = \pi \underbrace{\frac{\frac{1}{2}a_2\left[\widehat{D}_2^E-\widehat{U}_2^E+\widehat{D}_2^V\right]}{r+\delta_2+\frac{1}{2}a_2\left[\widehat{D}_2^E-\widehat{U}_2^E+\widehat{D}_2^V\right]}}_{\equiv Z}, \tag{A.144}$$

where $Z \in (0,1)$. Thus,

$$\frac{dD_2^V}{d\lambda} = \frac{1}{6}\pi\underbrace{[1-Z]}_{>0} - \frac{1}{3}\underbrace{\frac{d}{d\lambda}\left[U_2^E+U_2^A\right]}_{<0}. \tag{A.145}$$

Consequently, $dD_2^V/d\lambda > 0$.

All this implies that the results from Proposition 5 continue to hold when the entrepreneur can search for a VC without the angel.

Proof of Proposition 6.

Recall that $U_2^E = U_2^A$ in equilibrium. Moreover, as shown in Proof of Proposition 3, $dU_2^E/d\phi_2 > 0$, and $dU_2^E/d\sigma_2$, $dU_2^E/d\delta_2$, $dU_2^E/d\sigma_2^V$, $dU_2^E/dk_2 < 0$. Consequently, $d\gamma^*/d\phi_2 < 0$, and $d\gamma^*/d\sigma_2$, $d\gamma^*/d\delta_2$, $d\gamma^*/d\sigma_2^V$, $d\gamma^*/dk_2 > 0$. □

Proof of Proposition 7.

Recall from Proof of Proposition 5 that $d\left[U_2^E+U_2^A\right]/d\lambda < 0$. Thus, $d\gamma^*/d\lambda > 0$. □