Fostering Entrepreneurship: Promoting Founding or Funding?

Thomas Hellmann and Veikko Thiele

— ONLINE APPENDIX —

Benchmark Model: Comparative Statics.

In this Online Appendix we consider a more general model specification where (i) all parties share the discount factor $\delta = 1 + r \in (0, 1]$, (ii) l has a uniform distribution over the interval $[0, \mu_E]$, and (iii) θ has a uniform distribution over the interval $[0, \mu_I]$.

Given the more general specification we can write the entry and market clearing conditions for the benchmark model as follows:

$$n_E = \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y \tag{A.1}$$

$$\frac{1}{\mu_E}\delta\gamma^2\rho(1-\alpha)y\phi = \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho y}{\phi} - 1\right]\widetilde{n}\widetilde{w}. \tag{A.2}$$

Implicitly differentiating (A.2) we find

$$\frac{d\alpha^*}{d\phi} = \frac{\frac{1}{\mu_E}\delta\gamma^2\rho(1-\alpha)y + \frac{1}{\mu_I}\alpha\frac{\delta\rho y}{\phi^2}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}} > 0 \qquad \frac{d\alpha^*}{d\left(\widetilde{n}\widetilde{w}\right)} = -\frac{\frac{1}{\mu_I}\left[\alpha\frac{\delta\rho y}{\phi} - 1\right]}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}} < 0$$

$$\frac{d\alpha^*}{d\gamma} = \frac{2\frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y\phi}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}} > 0.$$

Likewise,

$$\frac{d\alpha^*}{d\left(\rho y\right)} = \frac{\overbrace{\frac{1}{\mu_E}\delta\gamma^2(1-\alpha)\phi - \frac{1}{\mu_I}\alpha\frac{\delta}{\phi}\widetilde{n}\widetilde{w}}^{\equiv Z}}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}}.$$

Note that the market clearing condition (A.2) can be written as

$$\rho y \underbrace{\left[\frac{1}{\mu_E}\delta\gamma^2(1-\alpha)\phi - \frac{1}{\mu_I}\alpha\frac{\delta}{\phi}\widetilde{n}\widetilde{w}\right]}_{=Z} = -\frac{1}{\mu_I}\widetilde{n}\widetilde{w}.$$

Thus, $Z < 0$ in equilibrium. Consequently, $d\alpha^*/d(\rho y) < 0$.

Finally, using (A.1) we find

$$\frac{dn_E^*}{d\phi} = -\frac{1}{\mu_E}\delta\gamma\rho y \underbrace{\frac{d\alpha^*}{d\phi}}_{>0} < 0 \qquad \frac{dn_E^*}{d(\widetilde{n}\widetilde{w})} = -\frac{1}{\mu_E}\delta\gamma\rho y \underbrace{\frac{d\alpha^*}{d(\widetilde{n}\widetilde{w})}}_{<0} > 0$$

$$\frac{dn_E^*}{d(\rho y)} = \frac{1}{\mu_E}\delta\gamma\left[(1-\alpha) - \rho y \underbrace{\frac{d\alpha^*}{d(\rho y)}}_{<0}\right] > 0.$$

Proof of Proposition 1.

To show that the market equilibrium is efficient, we derive the socially optimal (i.e., first best) ownership stake for investors, denoted by α^{fb}, which then defines the socially optimal level of entrepreneur entry, denoted by n_E^{fb}.

We first derive the total expected utility of all investors, denoted by TU_I:

$$TU_I = \widetilde{n}\left[\int_0^{\widehat{\theta}}\left(\alpha\frac{\delta\rho y}{\phi}\widetilde{w} - \theta\widetilde{w}\right)\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}}^{\mu_I}\widetilde{w}\frac{1}{\mu_I}d\theta\right] = \widetilde{n}\widetilde{w}\frac{1}{\mu_I}\left[\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\widehat{\theta} - \frac{1}{2}\widehat{\theta}^2 + \mu_I\right].$$

Using $\widehat{\theta} = \alpha\frac{\delta\rho y}{\phi} - 1$ we get

$$TU_I = \widetilde{n}\widetilde{w}\left[1 + \frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)^2\right].$$

Next consider the total expected utility of all entrepreneurs prior to market entry, denoted by TU_E. Note that the entry cost for the marginal entrepreneur is given by $\widehat{l} = U_E = \delta\gamma\rho(1-\alpha)y$. Thus,

$$TU_E = n_E\int_0^{\widehat{l}}\left(\delta\gamma\rho(1-\alpha)y - l\right)\frac{1}{\widehat{l}}dl = \frac{1}{2}n_E\delta\gamma\rho(1-\alpha)y.$$

Using the entry condition (A.1), we get $TU_E = \frac{1}{2\mu_E}\left[\delta\gamma\rho(1-\alpha)y\right]^2$. Thus, the total welfare W is given by

$$W = TU_E + TU_I = \frac{1}{2\mu_E}\left[\delta\gamma\rho(1-\alpha)y\right]^2 + \widetilde{n}\widetilde{w}\left[1 + \frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)^2\right].$$

The first best equity share for investors, α^{fb}, is then defined by the (simplified) first-order condition:

$$\frac{1}{\mu_E}\delta\gamma^2\rho\left(1-\alpha\right)y\phi=\frac{1}{\mu_I}\left[\alpha\frac{\delta\rho y}{\phi}-1\right]\widetilde{n}\widetilde{w}.$$

Note that this condition, which defines α^{fb}, is identical to the market clearing condition (2). Thus, the equilibrium ownership stake for investors is socially efficient, i.e., $\alpha^*=\alpha^{fb}$. The entry condition for entrepreneurs, $n_E=\frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y$, then implies that the equilibrium number of new ventures is also socially efficient, i.e., $n_E^*=n_E^{fb}$. □

Proof of Proposition 2.

We first analyze the effect of a founding subsidy S_E. With $\delta\in(0,1]$, $l\sim\mathcal{U}(0,\mu_E)$, and $\theta\sim\mathcal{U}(0,\mu_I)$, we can write the entry and market clearing conditions as follows:

$$n_E = \frac{1}{\mu_E}\left[\delta\gamma\rho(1-\alpha)y+S_E\right] \tag{A.3}$$

$$\frac{1}{\mu_E}\gamma\left[\delta\gamma\rho(1-\alpha)y+S_E\right]\phi = \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho y}{\phi}-1\right]\widetilde{n}\widetilde{w}. \tag{A.4}$$

Using (A.4) we get

$$\frac{d\alpha^*(S_E)}{dS_E}=\frac{\frac{1}{\mu_E}\gamma\phi}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi+\frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}}>0.$$

Moreover, using (A.3),

$$\frac{dn_E^*(S_E)}{dS_E}=\frac{1}{\mu_E}\left[1-\delta\gamma\rho y\frac{d\alpha^*}{dS_E}\right]=\frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\gamma^2\phi+\frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}>0.$$

Now consider the effect of a funding subsidy $S_I=\phi s_I$. With S_I the entry and market clearing conditions can be written as

$$n_E = \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y \tag{A.5}$$

$$\frac{1}{\mu_E}\delta\gamma^2\rho\left(1-\alpha\right)y\phi = \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho y}{\phi}-1+\frac{1}{\phi}\frac{1}{\gamma}S_I\right]\widetilde{n}\widetilde{w}. \tag{A.6}$$

Implicitly differentiating (A.6) yields

$$\frac{d\alpha^*(S_I)}{dS_I} = -\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}} < 0.$$

Moreover, using (A.5),

$$\frac{dn_E^*(S_I)}{dS_I} = -\frac{1}{\mu_E}\delta\gamma\rho y\frac{d\alpha^*(S_I)}{dS_I} = \frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\gamma^2\phi + \frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}} > 0.$$

Note that $n_E^*(S_E = 0) = n_E^*(S_I = 0) = n_E^*$ and $dn_E^*(S_E)/dS_E = dn_E^*(S_I)/dS_I$. Thus, $n_E^*(S_E) = n_E^*(S_I) > n_E^*$ for all $S_E = S_I$. Moreover, note that $\alpha^*(S_E = 0) = \alpha^*(S_I = 0) = \alpha^*$. The fact that $d\alpha^*(S_E)/dS_E > 0$ and $d\alpha^*(S_I)/dS_I < 0$ then implies that $\alpha^*(S_E) > \alpha^* > \alpha^*(S_I)$ for all $S_E = S_I$. Moreover, $\alpha^*(S_E) > \alpha^* > \alpha^*(S_I)$ implies that entrepreneurs with good projects get a higher expected wealth ($\rho(1-\alpha)y$) under a funding subsidy (S_I) than under a founding subsidy (S_E).

Finally, the expected wealth of an entrepreneur under a founding vs. funding subsidy, is given by

$$E[w_E(S_E)] = \delta\gamma\rho(1-\alpha^*(S_E))y \qquad E[w_E(S_I)] = \delta\gamma\rho(1-\alpha^*(S_I))y.$$

Recall that $\alpha^*(S_E) > \alpha^*(S_I)$ for all $S_E = S_I$. Thus, $E[w_E(S_E)] < E[w_E(S_I)]$. Likewise, the expected wealth of an angel investor under a founding vs. funding subsidy, is given by

$$E[w_A(S_E)] = \delta\rho\alpha^*(S_E)y\frac{\widetilde{w}}{\phi} \qquad E[w_A(S_I)] = \delta\rho\alpha^*(S_I)y\frac{\widetilde{w}}{\phi} + \frac{1}{\phi}\frac{1}{\gamma}S_I\widetilde{w}.$$

Note that

$$\begin{aligned}
\frac{dE[w_A(S_E)]}{dS_E} &= \delta\rho y\frac{\widetilde{w}}{\phi}\frac{d\alpha^*(S_E)}{dS_E} = \frac{\frac{1}{\mu_E}\gamma\widetilde{w}}{\frac{1}{\mu_E}\gamma^2\phi + \frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}} \\
\frac{dE[w_A(S_I)]}{dS_I} &= \left(\delta\rho y\frac{d\alpha^*(S_I)}{dS_I} + \frac{1}{\gamma}\right)\frac{\widetilde{w}}{\phi} = \frac{\frac{1}{\mu_E}\gamma\widetilde{w}}{\frac{1}{\mu_E}\gamma^2\phi + \frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}.
\end{aligned}$$

Clearly, $dE[w_A(S_E)]/dS_E = dE[w_A(S_I)]/dS_I$, which implies that $E[w_A(S_E)] = E[w_A(S_I)]$ for all $S_E = S_I$. □

Model with Intergenerational Dynamics: Equilibria.

Consider the more general model with $\delta \in (0,1]$, $l \sim \mathcal{U}(0,\mu_E)$, and $\theta \sim \mathcal{U}(0,\mu_I)$. Using $\widehat{\theta}_{t+1} = \alpha_{t+1}\frac{\delta\rho y}{\phi} - 1$ we can write the expected utility of an entrepreneur in period t as

$$\begin{aligned} U_{E,t} &= \delta\gamma\rho\left[\int_0^{\widehat{\theta}_{t+1}}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - \theta\right)(1-\alpha_t)\, y\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}_{t+1}}^{\mu_I}(1-\alpha_t)\, y\frac{1}{\mu_I}d\theta\right] \\ &= \delta\gamma\rho\,(1-\alpha_t)\, y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right]. \end{aligned}$$

The market equilibrium is then defined by the following entry and market clearing conditions:

$$\begin{aligned} J &\equiv \frac{1}{\mu_E}\delta\gamma\rho\,(1-\alpha_t)\, y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right] - n_{E,t} = 0 \\ H &\equiv \frac{1}{\mu_I}\left[\alpha_t\frac{\delta\rho y}{\phi} - 1\right]\rho n_{E,t-1}(1-\alpha_{t-1})y - n_{E,t}\phi = 0. \end{aligned}$$

Using J we can implicitly differentiate α_t w.r.t. $n_{E,t}$:

$$\frac{d\alpha_t}{dn_{E,t}} = -\frac{1}{\frac{1}{\mu_E}\delta\gamma\rho y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right]} < 0.$$

Thus, the demand curve in Figure 4 (E) is monotone and decreasing in $n_{E,t}$.

Next, using Cramer's rule we get

$$\frac{dn^*_{E,t}(n_{E,t-1})}{dn_{E,t-1}} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial n_{E,t-1}} & \frac{\partial J}{\partial \alpha_t} \\ -\frac{\partial H}{\partial n_{E,t-1}} & \frac{\partial H}{\partial \alpha_t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_{E,t}} & \frac{\partial J}{\partial \alpha_t} \\ \frac{\partial H}{\partial n_{E,t}} & \frac{\partial H}{\partial \alpha_t} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial n_{E,t-1}}\frac{\partial H}{\partial \alpha_t} + \frac{\partial H}{\partial n_{E,t-1}}\frac{\partial J}{\partial \alpha_t}}{\frac{\partial J}{\partial n_{E,t}}\frac{\partial H}{\partial \alpha_t} - \frac{\partial H}{\partial n_{E,t}}\frac{\partial J}{\partial \alpha_t}},$$

where $\partial J/\partial n_{E,t-1} = 0$, $\partial J/\partial n_{E,t} = -1$, $\partial H/\partial n_{E,t} = -\phi$, and

$$\frac{\partial H}{\partial n_{E,t-1}} = \frac{1}{\mu_I}\left[\alpha_t\frac{\delta\rho y}{\phi} - 1\right]\rho(1-\alpha_{t-1})y > 0 \qquad \frac{\partial H}{\partial \alpha_t} = \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\rho n_{E,t-1}(1-\alpha_{t-1})y > 0$$

$$\frac{\partial J}{\partial \alpha_t} = -\frac{1}{\mu_E}\delta\gamma\rho y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right] < 0.$$

For $\alpha_t^* > \phi/(\delta\rho y)$ we then get

$$\frac{dn_{E,t}^*(n_{E,t-1})}{dn_{E,t-1}} = \frac{\overbrace{\frac{\partial H}{\partial n_{E,t-1}}}^{>0}\overbrace{\frac{\partial J}{\partial \alpha_t}}^{<0}}{-\underbrace{\frac{\partial H}{\partial \alpha_t}}_{>0}+\phi\underbrace{\frac{\partial J}{\partial \alpha_t}}_{<0}} > 0.$$

Moreover, note that there is no capital supply when $n_{E,t-1} = 0$. Thus, $n_{E,t}^*(0) = 0$.

Next we identify and characterize the steady state equilibria. In the steady state we have $n_{E,t} = n_{E,t-1}$ and $\alpha_t = \alpha_{t-1}$. Adjusting the market clearing condition for the steady state with $n_E \equiv n_{E,t} = n_{E,t-1}$ and $\alpha \equiv \alpha_t = \alpha_{t-1}$, we define

$$\underbrace{n_E\phi}_{\equiv E(\alpha)} = \underbrace{\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\rho n_E\left(1-\alpha\right)y,}_{\equiv I(\alpha)} \tag{A.7}$$

where $E(\alpha)$ is the total capital demand, and $I(\alpha)$ is the total capital supply. Let $\Psi(\alpha) \equiv I(\alpha)/E(\alpha)$ denote the excess supply function. The steady state market clearing condition (A.7) implies that

$$\Psi(\alpha) = \frac{1}{\phi}\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\rho\left(1-\alpha\right)y = 1$$

in the steady state equilibrium, where $\alpha\frac{\delta\rho y}{\phi} - 1 = \widehat{\theta} \geq 0$. Next we analyze the shape of $\Psi(\alpha)$. We get

$$\frac{d\Psi(\alpha)}{d\alpha} = \frac{1}{\phi}\frac{1}{\mu_I}\rho y\left[\underbrace{\frac{\delta\rho y}{\phi}\left(1-\alpha\right)}_{\equiv Z_1} - \underbrace{\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)}_{\equiv Z_2}\right].$$

Clearly, Z_1 is positive and decreasing in α, while $Z_2 = \widehat{\theta}$ is also positive but decreasing in α. Note that $Z_2 = \widehat{\theta} = \alpha\frac{\delta\rho y}{\phi} - 1 = 0$ for all $\alpha > \phi/(\delta\rho y)$. Thus, $\Psi(\alpha) = 0$ for all $\alpha \leq \phi/\left(\delta\rho y\right)$. Moreover, $\Psi(1) = 0$. This implies that $\Psi(\alpha)$ has an inverted U-shape, with $\Psi(\alpha) = 0$ for all $\alpha \leq \phi/(\delta\rho y)$ and $\Psi(1) = 0$. This also implies that there exits a unique α, denoted by $\overline{\alpha}$, which maximizes $\Psi(\alpha)$. Using the first-order condition we find that $\overline{\alpha} = \left(1 + \frac{\delta\rho y}{\phi}\right)/\left(2\frac{\delta\rho y}{\phi}\right)$. Evaluating $\Psi(\alpha)$ at $\alpha = \overline{\alpha}$ yields

$$\Psi(\overline{\alpha}) = \frac{1}{4\delta}\frac{1}{\mu_I}\left(\frac{\delta\rho y}{\phi} - 1\right)^2.$$

Note that $\partial\Psi\left(\overline{\alpha}\right)/\partial\phi < 0$, with $\lim_{\phi\to 0}\Psi\left(\overline{\alpha}\right) = \infty$ and $\Psi\left(\overline{\alpha}\right) = 0$ for all $\phi \geq \delta\rho y$.[1] Thus, there exists a threshold $\widehat{\phi} \in (0, \delta\rho y)$ such that

(i) for $\phi > \widehat{\phi}$ there exists no value of α which satisfies $\Psi\left(\alpha\right) = 1$,

(ii) for $\phi = \widehat{\phi}$ there exists a unique α, namely $\overline{\alpha}$, so that $\Psi\left(\alpha = \overline{\alpha}\right) = 1$, and

(iii) for $\phi < \widehat{\phi}$ there exist two values of α, denoted by α' and α'', with $\alpha' < \alpha''$, which both satisfy $\Psi\left(\alpha\right) = 1$.

Moreover, given the inverted U-shape of $\Psi(\alpha)$ we can infer that

$$\left.\frac{d\Psi\left(\alpha\right)}{d\alpha}\right|_{\alpha=\alpha'} > 0 \quad \text{and} \quad \left.\frac{d\Psi\left(\alpha\right)}{d\alpha}\right|_{\alpha=\alpha''} < 0.$$

We can now characterize the steady state equilibria in terms of entrepreneur entry $n_{E,t}$. Recall that the market is competitive, so when entering the market entrepreneurs take their future equity share α_{t+1} as given. From the entry condition we can see that α_t uniquely defines the equilibrium number of entrepreneurs, $n_{E,t}$. Moreover, $dn_{E,t}/d\alpha_t < 0$. We already know that $n_{E,t} = n_{E,t-1} = 0$ always constitutes a steady state equilibrium. Moreover, for $\phi < \widehat{\phi}$ we know that there exist two values of α, α' and α'', which satisfy the steady state market clearing condition (A.7). For $\phi < \widehat{\phi}$ we thus get two additional steady state equilibria, which we define as $n_E^M(\alpha^M)$ and $n_E^H(\alpha^H)$, where 'H' stands for 'high' and 'M' stands for 'medium' (where, using our original notation, $\alpha^M = \alpha''$ and $\alpha^H = \alpha'$). For $\phi = \widehat{\phi}$ we know that the steady state market clearing condition (A.7) is satisfied for $\alpha = \overline{\alpha}$. Consequently, for $\phi = \widehat{\phi}$ we have only one additional steady state equilibrium (in addition to $n_{E,t} = n_{E,t-1} = 0$): $n_E^M(\alpha^M)$.

It remains to verify which steady state equilibria are stable vs. unstable. A steady state equilibrium is stable if

$$\left.\frac{dn^*_{E,t}(n_{E,t-1})}{dn_{E,t-1}}\right|_{\substack{n_{E,t}=n_{E,t-1}\\ \alpha_t=\alpha_{t-1}}} < 1.$$

Using the partial derivatives we get

$$\frac{dn^*_{E,t}(n_{E,t-1})}{dn_{E,t-1}} = \frac{\frac{1}{\mu_I}\left[\alpha_t\frac{\delta\rho y}{\phi} - 1\right]\rho(1-\alpha_{t-1})y\frac{1}{\mu_E}\gamma\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right]}{\frac{1}{\mu_I}\frac{1}{\phi}\rho n_{E,t-1}(1-\alpha_{t-1})y + \phi\frac{1}{\mu_E}\gamma\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right]}.$$

[1]Note that the expected net payoff from a venture is negative when $\phi > \delta\rho y$. In this case it can never be optimal for investors to invest ϕ.

First consider the zero steady state equilibrium $n_{E,t} = n_{E,t-1} = 0$. Note that $n_{E,t-1} = 0$ corresponds to $\alpha_t = \phi/(\delta\rho y)$, in which case there is no capital supply (and therefore $n_{E,t} = 0$). Evaluating the derivative at $n_{E,t} = n_{E,t-1} = 0$ and $\alpha_t = \alpha_{t-1} = \phi/(\delta\rho y)$ we get

$$\left.\frac{dn_E^*}{dn_{E-1}}\right|_{\substack{n_{E,t}=n_{E,t-1}=0 \\ \alpha_t=\alpha_{t-1}=\phi/(\delta\rho y)}} = 0.$$

Thus, the zero steady state equilibrium $n_{E,t} = n_{E,t-1} = 0$ is stable. Next consider the two additional steady state equilibria for $\phi < \widehat{\phi}$. For the high steady state equilibrium $n_{E,t}^H(\alpha^H) = n_{E,t-1}^H(\alpha^H)$ we get

$$\left.\frac{dn_E^*}{dn_{E-1}}\right|_{\substack{n_{E,t}=n_{E,t-1}=n_E^H \\ \alpha_t=\alpha_{t-1}=\alpha^H}} = \frac{\frac{1}{\mu_I}\left[\alpha^H\frac{\delta\rho y}{\phi}-1\right]\rho(1-\alpha^H)y\frac{1}{\mu_E}\gamma\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)^2\right]}{\frac{1}{\mu_I}\frac{1}{\phi}\rho n_E^H(1-\alpha^H)y+\phi\frac{1}{\mu_E}\gamma\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)^2\right]}.$$

This is smaller than one if

$$\frac{1}{\mu_I}\left[\alpha^H\frac{\delta\rho y}{\phi}-1\right]\rho(1-\alpha^H)y\frac{1}{\mu_E}\gamma Z_{t+1} < \frac{1}{\mu_I}\frac{1}{\phi}\rho n_E^H(1-\alpha^H)y+\phi\frac{1}{\mu_E}\gamma Z_{t+1}, \qquad \text{(A.8)}$$

where

$$Z_{t+1} = 1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)^2.$$

Using the corresponding entry condition $n_E^H = \frac{1}{\mu_E}\delta\gamma\rho\left(1-\alpha^H\right)yZ_{t+1}$ to replace n_E^H in (A.8) we get

$$\frac{1}{\mu_I}\left[\alpha^H\frac{\delta\rho y}{\phi}-1\right]\rho(1-\alpha^H)y < \frac{1}{\mu_I}\frac{1}{\phi}\delta\left[\rho(1-\alpha^H)y\right]^2+\phi. \qquad \text{(A.9)}$$

We can then use (A.7) to replace ϕ in (A.9) and get

$$0 < \frac{1}{\mu_I}\frac{1}{\phi}\delta\left[\rho(1-\alpha^H)y\right]^2.$$

This condition is clearly satisfied as $\alpha^H < 1$. Thus, the high steady state equilibrium $n_{E,t}^H(\alpha^H) = n_{E,t-1}^H(\alpha^H)$ is stable. Finally recall that $dn_{E,t}^*/dn_{E,t-1} \geq 0$, with

$$\left.\frac{dn_E^*}{dn_{E-1}}\right|_{\substack{n_{E,t}=n_{E,t-1}=0 \\ \alpha_t=\alpha_{t-1}=\phi/(\delta\rho y)}}, \quad \left.\frac{dn_E^*}{dn_{E-1}}\right|_{\substack{n_{E,t}=n_{E,t-1}=n_E^H \\ \alpha_t=\alpha_{t-1}=\alpha^H}} < 1.$$

This implies that

$$\left.\frac{dn_E^*}{dn_{E-1}}\right|_{\substack{n_{E,t}=n_{E,t-1}=n_E^M \\ \alpha_t=\alpha_{t-1}=\alpha^M}} > 1.$$

Thus, the medium steady state equilibrium $n_{E,t}^M(\alpha^M) = n_{E,t-1}^M(\alpha^M)$ is unstable.

Proof of Proposition 3.

We first derive the total expected utility of all investors in the high steady state equilibrium, denoted by TU_I. Using $\widehat{\theta} = \alpha\frac{\delta\rho y}{\phi} - 1$ we get

$$\begin{aligned} TU_I &= \gamma\rho n_E \left[\int_0^{\widehat{\theta}} \left(\alpha\frac{\delta\rho y}{\phi} - \theta\right)(1-\alpha)\, y\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}}^{\mu_I} (1-\alpha)\, y\frac{1}{\mu_I}d\theta\right] \\ &= \gamma\rho n_E (1-\alpha)\, y \left[1 + \frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)^2\right], \end{aligned}$$

where $\gamma\rho n_E$ is the number of investors in each period.

Next, recall from Proof of Proposition 2 that the total expected utility of all entrepreneurs prior to entry in a given period, TU_E, is given by $TU_E = \frac{1}{2}n_E\delta\gamma\rho(1-\alpha)\, y$ (note that by definition TU_E only reflects the expected utilities in the current period, and does not account for the future expected utilities from angel investments). Using the entry condition for the high steady state equilibrium,

$$n_E = \frac{1}{\mu_E}U_E = \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)\, y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right], \tag{A.10}$$

we can write the total steady state welfare function $W = \frac{1}{1-\delta}(TU_E + TU_I)$ as

$$W = \underbrace{\frac{1}{1-\delta}\frac{1}{\mu_E}\delta(\gamma\rho y)^2 Z_{t+1}}_{\equiv X>0}(1-\alpha)^2\left[\frac{1}{2}\delta + 1 + \frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)^2\right],$$

where

$$Z_{t+1} = 1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2.$$

To show that the equilibrium equity stake α^* does not maximize W, we derive $dW/d\alpha$, and then evaluate the derivative at $\alpha = \alpha^*$. We get

$$\frac{dW}{d\alpha} = X\left[-2\left(1-\alpha\right)\left[\frac{1}{2}\delta + 1 + \frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)^2\right] + \left(1-\alpha\right)^2 \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{\delta\rho y}{\phi}\right].$$

Note that the market clearing condition for the high steady state (where $n_{E,t} = n_{E,t-1}$ and $\alpha_t = \alpha_{t-1}$) is given by

$$\phi = \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho y}{\phi} - 1\right]\rho(1-\alpha_{t-1})y,$$

which can be written as

$$(1-\alpha) = \frac{\phi}{\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\rho y}.$$

Using this expression we can evaluate $dW/d\alpha$ at $\alpha = \alpha^*$, and get after simplifying

$$\left.\frac{dW}{d\alpha}\right|_{\alpha=\alpha^*} = -2X\frac{\phi}{\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\rho y}\left[1 + \frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)^2\right] < 0.$$

Thus, $dW/d\alpha|_{\alpha=\alpha^*} < 0$, which implies that $\alpha^* > \alpha^{fb}$. Moreover, we can see from (A.10) that $dn_E^*/d\alpha < 0$. Thus, $n_E^* < n_E^{fb}$. □

Proof of Proposition 4.

With a founding subsidy S_E the market equilibrium is defined by

$$n_E = \frac{1}{\mu_E}\left[U_E + S_E\right] \quad \text{(A.11)}$$

$$\phi = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\rho\left(1-\alpha\right)y, \quad \text{(A.12)}$$

where

$$U_E = \delta\gamma\rho\left[\int_0^{\widehat{\theta}_{t+1}}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - \theta\right)\left(1-\alpha\right)y\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}_{t+1}}^{\mu_I}\left(1-\alpha\right)y\frac{1}{\mu_I}d\theta\right].$$

From (A.12) we can immediately see that $d\alpha^*(S_E)/dS_E = 0$. Using the entry condition (A.11) we can then see that $dn_E^*(S_E)/dS_E = 1/\mu_E > 0$. □

Proof of Proposition 5.

With a funding subsidy S_I the market equilibrium is defined by

$$n_E = \frac{1}{\mu_E} U_E(S_I) \tag{A.13}$$

$$\phi = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\rho\left(1-\alpha\right)y, \tag{A.14}$$

where

$$U_E(S_I) = \delta\gamma\rho\left[\int_0^{\widehat{\theta}_{t+1}+\frac{1}{\phi}\frac{1}{\gamma}S_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} + \frac{1}{\phi}\frac{1}{\gamma}S_I - \theta\right)\left(1-\alpha\right)y\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}_{t+1}+\frac{1}{\phi}\frac{1}{\gamma}S_I}^{\mu_I}\left(1-\alpha\right)y\frac{1}{\mu_I}d\theta\right].$$

Integrating the entrepreneur's expected utility $U_E(S_I)$ and using $\widehat{\theta}_{t+1} = \alpha_{t+1}\frac{\delta\rho y}{\phi} - 1$, we can write the entry condition (11) as follows:

$$n_E = \frac{1}{\mu_E}\delta\gamma\rho\left(1-\alpha\right)y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2\right]. \tag{A.15}$$

Next, using (A.14) we define

$$H \equiv \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\rho\left(1-\alpha\right)y - \phi = 0.$$

Using H we get

$$\frac{d\alpha^*(S_I)}{dS_I} = -\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\rho\left(1-\alpha\right)y}{\frac{\partial H}{\partial \alpha}}.$$

Recall the excess supply function $\Psi(\alpha)$ from our formal characterization of the dynamic equilibrium. With a funding subsidy S_I the excess supply function can be written as

$$\Psi(\alpha, S_I) = \frac{1}{\phi}\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\rho\left(1-\alpha\right)y.$$

Furthermore, for the high steady state equilibrium with $\alpha = \alpha^H$ we know that $\left.\frac{\partial\Psi(\alpha,S_I)}{\partial\alpha}\right|_{\alpha=\alpha^H} > 0$. Clearly, $H = \phi\Psi(\alpha, S_I) - \phi$. Hence, $\partial H/\partial\alpha > 0$, so that $d\alpha^*(S_I)/dS_I < 0$.

Finally, using the entry condition (A.13) we get

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{1}{\mu_E}\delta\gamma\rho y\left[-\underbrace{\frac{d\alpha^*(S_I)}{dS_I}}_{<0}\left[1+\frac{1}{2\mu_I}T_{t+1}^2\right]+(1-\alpha)\frac{1}{\mu_I}T_{t+1}\frac{1}{\phi}\frac{1}{\gamma}\right],$$

where

$$T_{t+1} = \alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I > 0.$$

Thus, $dn_E^*(S_I)/dS_I > 0$. □

Proof of Proposition 6.

Note that $n_E^*(S_I = 0) = n_E^*(S_E = 0) = n_E^*$. Thus, to show that $n_E^*(S_I) > n_E^*(S_E)$ for all $S_I = S_E$ it is sufficient to show that $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$ for all $S_I = S_E$. Recall from Proof of Proposition 4 that $dn_E^*(S_E)/dS_E = 1/\mu_E$. Moreover, using the derivations from Proof of Proposition 5 we get

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{1}{\mu_E}\delta\gamma\rho y\left[\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\rho(1-\alpha)y}{\frac{\partial H}{\partial\alpha}}\left[1+\frac{1}{2\mu_I}T_{t+1}^2\right]+(1-\alpha)\frac{1}{\mu_I}T_{t+1}\frac{1}{\phi}\frac{1}{\gamma}\right],$$

where

$$T_{t+1} = \alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I > 0,$$

and $\partial H/\partial\alpha > 0$ (see Proof of Proposition 5). Note that

$$\frac{\partial H}{\partial\alpha} = \frac{1}{\mu_I}\rho y\left[\frac{\delta\rho y}{\phi}(1-\alpha)-\left(\alpha\frac{\delta\rho y}{\phi}-1+\frac{1}{\phi}\frac{1}{\gamma}S_I\right)\right] > 0. \tag{A.16}$$

We then get after simplifying

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{1}{\mu_E}\delta\rho y(1-\alpha)\left[\frac{1+\frac{1}{2\mu_I}T_{t+1}^2+\frac{\delta\rho y}{\phi}(1-\alpha)\frac{1}{\mu_I}T_{t+1}-\frac{1}{\mu_I}TT_{t+1}}{\delta\rho y(1-\alpha)-\phi T}\right],$$

where

$$T = \alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I > 0.$$

Thus, $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$ if

$$\frac{1}{\mu_E}\delta\rho y\left(1-\alpha\right)\left[\frac{1+\frac{1}{2\mu_I}T_{t+1}^2+\frac{\delta\rho y}{\phi}\left(1-\alpha\right)\frac{1}{\mu_I}T_{t+1}-\frac{1}{\mu_I}TT_{t+1}}{\delta\rho y\left(1-\alpha\right)-\phi T}\right]>\frac{1}{\mu_E},$$

which can be rearranged to

$$\delta\rho y\left(1-\alpha\right)T_{t+1}\frac{1}{\mu_I}\left[\frac{1}{2}T_{t+1}+\underbrace{\frac{\delta\rho y}{\phi}\left(1-\alpha\right)-T}_{\equiv X}\right]>-\phi T. \tag{A.17}$$

We can immediately infer from (A.16) that $X > 0$, so that (A.17) is satisfied. Consequently, $n_E^*(S_I) > n_E^*(S_E)$ for all $S_I = S_E$.

Finally note that $\alpha^*(S_I = 0) = \alpha^*(S_E = 0) = \alpha^*$. Moreover, we know from Proposition 4 that $d\alpha^*(S_E)/dS_E = 0$, and from Proposition 5 that $d\alpha^*(S_I)/dS_I < 0$. Thus, $\alpha^*(S_I) < \alpha^*(S_E) = \alpha^*$ for all $S_I = S_E$. □

Proof of Proposition 7.

We first analyze the effects in period t. With a one-time founding subsidy $S_{E,t}$ the equilibrium is defined by the following entry and market clearing conditions:

$$n_{E,t} = \frac{1}{\mu_E}\left[\delta\gamma\rho\left(1-\alpha_t\right)y\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)^2\right]+S_{E,t}\right] \tag{A.18}$$

$$\gamma n_{E,t}\phi = \frac{1}{\mu_I}\left(\alpha_t\frac{\delta\rho y}{\phi}-1\right)\gamma\rho n_{E,t-1}(1-\alpha_{t-1})y. \tag{A.19}$$

Note that the stock of capital in period t is exogenous. We therefore define $K_t \equiv \gamma\rho n_{E,t-1}(1-\alpha_{t-1})y$. Combining the two equilibrium conditions we define

$$H \equiv \gamma\frac{1}{\mu_E}\left[\delta\gamma\rho\left(1-\alpha_t\right)yZ_{t+1}+S_{E,t}\right]\phi-\frac{1}{\mu_I}\left(\alpha_t\frac{\delta\rho y}{\phi}-1\right)K_t = 0,$$

where

$$Z_{t+1} = 1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)^2. \tag{A.20}$$

Using H we get

$$\frac{d\alpha_t^*(S_{E,t})}{dS_{E,t}} = \frac{\gamma\frac{1}{\mu_E}\phi}{\frac{1}{\mu_E}\delta\gamma^2\rho y Z_{t+1}\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}K_t} > 0.$$

Moreover, totally differentiating (A.18) and using the expression for $d\alpha_t^*(S_{E,t})/dS_{E,t}$ we get

$$\frac{dn_{E,t}^*(S_{E,t})}{dS_{E,t}} = \frac{1}{\mu_E}\left[1 - \delta\gamma\rho y Z_{t+1}\frac{d\alpha_t^*(S_{E,t})}{dS_{E,t}}\right] = \frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}K_t}{\frac{1}{\mu_E}\gamma^2 Z_{t+1}\phi + \frac{1}{\mu_I}\frac{1}{\phi}K_t} > 0.$$

Now consider the effect of a one-time funding subsidy $S_{I,t}$. The equilibrium is then defined by

$$n_{E,t} = \frac{1}{\mu_E}\delta\gamma\rho\left(1-\alpha_t\right)y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right] \quad \text{(A.21)}$$

$$\gamma n_{E,t}\phi = \frac{1}{\mu_I}\left(\alpha_t\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_{I,t}\right)K_t. \quad \text{(A.22)}$$

Combining the two equilibrium conditions we define

$$J \equiv \frac{1}{\mu_E}\delta\gamma^2\rho\left(1-\alpha_t\right)yZ_{t+1}\phi - \frac{1}{\mu_I}\left(\alpha_t\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_{I,t}\right)K_t = 0.$$

Using J we get

$$\frac{d\alpha_t^*(S_{I,t})}{dS_{I,t}} = -\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}K_t}{\frac{1}{\mu_E}\delta\gamma^2\rho y Z_{t+1}\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}K_t} < 0.$$

Furthermore, using (A.21) and the expression for $d\alpha_t^*(S_{I,t})/dS_{I,t}$,

$$\frac{dn_{E,t}^*(S_{I,t})}{dS_{I,t}} = -\frac{1}{\mu_E}\delta\gamma\rho y Z_{t+1}\frac{d\alpha_t^*(S_{I,t})}{dS_{I,t}} = \frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}K_t Z_{t+1}}{\frac{1}{\mu_E}\gamma^2 Z_{t+1}\phi + \frac{1}{\mu_I}\frac{1}{\phi}K_t} > 0.$$

Finally note that $dn_{E,t}^*(S_{I,t})/dS_{I,t} > dn_{E,t}^*(S_{E,t})/dS_{E,t}$ for all $S_{I,t} = S_{E,t}$ if

$$\frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}K_t Z_{t+1}}{\frac{1}{\mu_E}\gamma^2 Z_{t+1}\phi + \frac{1}{\mu_I}\frac{1}{\phi}K_t} > \frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}K_t}{\frac{1}{\mu_E}\gamma^2 Z_{t+1}\phi + \frac{1}{\mu_I}\frac{1}{\phi}K_t},$$

which simplifies to $Z_{t+1} > 1$. We can see from (A.20) that $Z_{t+1} > 1$, so that $dn_{E,t}^*(S_{I,t})/dS_{I,t} > dn_{E,t}^*(S_{E,t})/dS_{E,t}$ for all $S_{I,t} = S_{E,t}$. And because $n_{E,t}^*(S_{I,t} = 0) = n_{E,t}^*(S_{E,t} = 0)$, this implies that $n_{E,t}^*(S_{I,t}) > n_{E,t}^*(S_{E,t})$ for all $S_{E,t} = S_{I,t}$. Moreover, we know that $\alpha_t^*(S_{I,t} = 0) =$

$\alpha_t^*(S_{E,t}=0)$. And the fact that $d\alpha_t^*(S_{E,t})/dS_{E,t}>0$ and $d\alpha_t^*(S_{I,t})/dS_{I,t}<0$ then implies that $\alpha_t^*(S_{I,t})<\alpha_t^*(S_{E,t})$ for all $S_{E,t}=S_{I,t}$.

Next we analyze the effects of the catalyst policies in period $t+1$. The equilibrium in period $t+1$ is defined by

$$n_{E,t+1} = \frac{1}{\mu_E}\delta\gamma\rho\left(1-\alpha_{t+1}\right)y\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+2}\frac{\delta\rho y}{\phi}-1\right)^2\right] \tag{A.23}$$

$$\gamma n_{E,t+1}\phi = \frac{1}{\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)\gamma\rho n_{E,t}(S_t)(1-\alpha_t(S_t))y, \tag{A.24}$$

where $S_t\in\{S_{E,t},S_{I,t}\}$. For parsimony we define $K_{t+1}(S_t)\equiv\gamma\rho n_{E,t}(S_t)(1-\alpha_t(S_t))y$, which is the stock of capital in period $t+1$. Recall that $n_{E,t}^*(S_{I,t})>n_{E,t}^*(S_{E,t})$ and $\alpha_t^*(S_{I,t})<\alpha_t^*(S_{E,t})$ for $S_{E,t}=S_{I,t}$. Thus, $K_{t+1}(S_{I,t})>K_{t+1}(S_{E,t})$ for $S_{E,t}=S_{I,t}$. Combining the two equilibrium conditions we define

$$H\equiv\frac{1}{\mu_E}\delta\gamma^2\rho\left(1-\alpha_{t+1}\right)yZ_{t+2}\phi-\frac{1}{\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)K_{t+1}(S_t)=0,$$

where

$$Z_{t+2}=1+\frac{1}{2\mu_I}\left(\alpha_{t+2}\frac{\delta\rho y}{\phi}-1\right)^2.$$

Using H we can implicitly differentiate α_{t+1} w.r.t $K_{t+1}(S_t)$:

$$\frac{d\alpha_{t+1}^*(K_{t+1}(S_t))}{dK_{t+1}(S_t)}=-\frac{\frac{1}{\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)}{\frac{1}{\mu_E}\delta\gamma^2\rho yZ_{t+2}\phi+\frac{1}{\mu_I}\frac{\delta\rho y}{\phi}K_{t+1}(S_t)}<0.$$

This implies that $\alpha_{t+1}^*(S_{I,t})<\alpha_{t+1}^*(S_{E,t})$ for $S_{E,t}=S_{I,t}$. Thus, a one-time funding subsidy ($S_{I,t}$) also results in a higher expected wealth for entrepreneurs with good projects, compared to a one-time founding subsidy ($S_{E,t}$).

Moreover, using (A.23),

$$\frac{dn_{E,t+1}^*(K_{t+1}(S_t))}{dK_{t+1}(S_t)}=-\frac{1}{\mu_E}\delta\gamma\rho yZ_{t+1}\underbrace{\frac{d\alpha_{t+1}^*(K_{t+1}(S_t))}{dK_{t+1}(S_t)}}_{<0}>0.$$

Consequently, $n_{E,t+1}^*(S_{I,t})>n_{E,t+1}^*(S_{E,t})$ for $S_{E,t}=S_{I,t}$. Following along the lines of the last part of this proof, it is straightforward to show that $n_{E,t+i}^*(S_{I,t})>n_{E,t+i}^*(S_{E,t})$ for $i=2,3,...$

and $S_{E,t} = S_{I,t}$, until the economy reaches the high steady state equilibrium. □

Optimal Policy.

To consider optimal policies we restrict our attention to non-discriminatory policies, where the government pays the same subsidy to all entrepreneurs or investors. The proof proceeds in three steps.

Our first step is to show that a funding subsidy S_I generates a higher expected welfare level than a founding subsidy S_E, with $S_I = S_E$. The second step is to show that in our model there is also an equivalence between investment subsidies and return subsidies. That is, our funding subsidies can be equivalently structured as investment subsidies at the time of investing, or return subsidies in case of success. The final step is to show that, within the confines of our model, there are no other feasible policies that generate higher welfare levels than the funding subsidies. For parsimony we focus on the high steady state equilibrium.

(i) Welfare Comparison – Funding vs. Founding Subsidies

It is convenient to use $S_E = \eta S$ and $S_I = (1-\eta)S$, with $\eta \in [0,1]$. The total cost for the government is then given by $n_E S$. Moreover, we assume that the government chooses S such that it has a positive effect on the expected welfare. Using $\widehat{\theta} = \alpha\frac{\delta\rho y}{\phi} - 1$, we get the following expression for the total expected utility of all investors, TU_I:

$$\begin{aligned} TU_I &= \gamma\rho n_E \left[\int_0^{\widehat{\theta}+\frac{1}{\phi}\frac{1}{\gamma}(1-\eta)S}\left(\alpha\frac{\delta\rho y}{\phi} - \theta + \frac{1}{\phi}\frac{1}{\gamma}(1-\eta)S\right)(1-\alpha)\,y\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}+\frac{1}{\phi}\frac{1}{\gamma}(1-\eta)S}^{\mu_I}(1-\alpha)\,y\frac{1}{\mu_I}d\theta\right] \\ &= \gamma\rho n_E(1-\alpha)\,y\left[1 + \frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}(1-\eta)S\right)^2\right], \end{aligned}$$

where $\gamma\rho n_E$ is the number of investors in each period. Moreover, recall from Proof of Proposition 2 that the total expected utility of all entrepreneurs prior to entry, TU_E, is given by $TU_E = \frac{1}{2}n_E\delta\gamma\rho(1-\alpha)\,y$. We can then write the total steady state welfare function $W = \frac{1}{1-\delta}(TU_E + TU_I - n_E S)$ as

$$W(\cdot) = \frac{1}{1-\delta}n_E^*(\eta)\left[\gamma\rho(1-\alpha^*(\eta))\,y\left[\frac{1}{2}\delta + 1 + \frac{1}{2\mu_I}\left(\alpha^*(\eta)\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}(1-\eta)S\right)^2\right] - S\right],$$

where $n_E^*(\eta)$ and $\alpha^*(\eta)$ are defined by the following entry and market clearing conditions:

$$J \equiv \frac{1}{\mu_E}\left[\delta\gamma\rho\left(1-\alpha\right)y\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1+\frac{1}{\phi}\frac{1}{\gamma}\left(1-\eta\right)S\right)^2\right]+\eta S\right]-n_E=0 \quad \text{(A.25)}$$

$$H \equiv \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi}-1+\frac{1}{\phi}\frac{1}{\gamma}\left(1-\eta\right)S\right)\rho(1-\alpha)y-\phi=0. \quad \text{(A.26)}$$

Totally differentiating $W(n_E^*(\eta),\alpha^*(\eta),\eta)$ we get

$$\frac{dW\left(\cdot\right)}{d\eta}=\frac{\partial W}{\partial n_E^*(\eta)}\frac{dn_E^*(\eta)}{d\eta}+\frac{\partial W}{\partial\alpha^*(\eta)}\frac{d\alpha^*(\eta)}{d\eta}+\frac{\partial W}{\partial\eta}.$$

We can immediately see that

$$\frac{\partial W}{\partial n_E^*(\eta)} = \frac{1}{1-\delta}\left[\gamma\rho\left(1-\alpha^*(\eta)\right)y\left[\frac{1}{2}\delta+1+\frac{1}{2\mu_I}Z^2\right]-S\right]>0$$

$$\frac{\partial W}{\partial\eta} = -\frac{1}{1-\delta}n_E^*(\eta)\gamma\rho\left(1-\alpha^*(\eta)\right)y\frac{1}{\mu_I}Z\frac{1}{\phi}\frac{1}{\gamma}S<0,$$

where

$$Z=\alpha^*(\eta)\frac{\delta\rho y}{\phi}-1+\frac{1}{\phi}\frac{1}{\gamma}\left(1-\eta\right)S.$$

Moreover,

$$\frac{\partial W}{\partial\alpha^*(\eta)}=\frac{1}{1-\delta}n_E^*(\eta)\gamma\rho y\left[-\left[\frac{1}{2}\delta+1+\frac{1}{2\mu_I}Z^2\right]+\left(1-\alpha^*(\eta)\right)\frac{1}{\mu_I}Z\frac{\delta\rho y}{\phi}\right].$$

Note that the market clearing condition (A.26) can be written as $(1-\alpha)=\mu_I\phi/\left(\rho yZ\right)$. Using this expression we get

$$\frac{\partial W}{\partial\alpha^*(\eta)}=-\frac{1}{1-\delta}n_E^*(\eta)\gamma\rho y\left[1-\frac{1}{2}\delta+\frac{1}{2\mu_I}Z^2\right].$$

Because $\delta<1$, we find that $\partial W/\partial\alpha^*(\eta)<0$.

Next, using Cramer's rule,

$$\frac{dn_E^*(\eta)}{d\eta}=\frac{\begin{vmatrix}-\frac{\partial J}{\partial\eta} & \frac{\partial J}{\partial\alpha}\\ -\frac{\partial H}{\partial\eta} & \frac{\partial H}{\partial\alpha}\end{vmatrix}}{\begin{vmatrix}\frac{\partial J}{\partial n_E} & \frac{\partial J}{\partial\alpha}\\ \frac{\partial H}{\partial n_E} & \frac{\partial H}{\partial\alpha}\end{vmatrix}}=\frac{-\frac{\partial J}{\partial\eta}\frac{\partial H}{\partial\alpha}+\frac{\partial H}{\partial\eta}\frac{\partial J}{\partial\alpha}}{\frac{\partial J}{\partial n_E}\frac{\partial H}{\partial\alpha}-\frac{\partial H}{\partial n_E}\frac{\partial J}{\partial\alpha}},$$

where $\partial J/\partial n_E = -1$, $\partial H/\partial n_E = 0$, and

$$\begin{aligned}
\frac{\partial J}{\partial \eta} &= \frac{1}{\mu_E} S \left[1 - \delta\gamma\rho\left(1-\alpha\right) y \frac{1}{\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}\left(1-\eta\right) S\right)\frac{1}{\phi}\frac{1}{\gamma}\right] \\
\frac{\partial H}{\partial \eta} &= -\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma} S\rho(1-\alpha) y < 0 \\
\frac{\partial J}{\partial \alpha} &= -\frac{1}{\mu_E}\delta\gamma\rho y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}\left(1-\eta\right) S\right)^2\right] < 0 \\
\frac{\partial H}{\partial \alpha} &= \frac{1}{\mu_I}\rho y\left[\frac{\delta\rho y}{\phi}(1-\alpha) - \left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}\left(1-\eta\right) S\right)\right].
\end{aligned}$$

Furthermore, using the adjusted excess supply function $\Psi(\alpha, \eta)$ one can show that $\partial H/\partial \alpha > 0$. Thus,

$$\frac{dn_E^*(\eta)}{d\eta} = \frac{\frac{\partial J}{\partial \eta}\frac{\partial H}{\partial \alpha} - \frac{\partial H}{\partial \eta}\frac{\partial J}{\partial \alpha}}{\underbrace{\frac{\partial H}{\partial \alpha}}_{>0}}.$$

Clearly, $dn_E^*(\eta)/d\eta < 0$ if the numerator is negative. Note that $\alpha = \alpha_{t+1}$ in the steady state. Thus, using the partial derivatives and simplifying we find that $dn_E^*(\eta)/d\eta < 0$ if

$$\left[1 - \delta\gamma\rho\left(1-\alpha\right) y \frac{1}{\mu_I} Z \frac{1}{\phi}\frac{1}{\gamma}\right]\left[\frac{\delta\rho y}{\phi}(1-\alpha) - Z\right] < \frac{1}{\phi}(1-\alpha)\delta\rho y\left[1 + \frac{1}{2\mu_I}Z^2\right].$$

Replacing the first $(1-\alpha)$ in this condition by using the relationship $(1-\alpha) = \mu_I\phi/\left(\rho y Z\right)$, yields

$$\left(1-\delta\right)\left[\frac{\delta\rho y}{\phi}(1-\alpha) - Z\right] < \frac{1}{\phi}(1-\alpha)\delta\rho y\left[1 + \frac{1}{2\mu_I}Z^2\right],$$

which can be rearranged to

$$0 < \frac{1}{\phi}(1-\alpha)\delta\rho y\frac{1}{2\mu_I}Z^2 + \delta\frac{\delta\rho y}{\phi}(1-\alpha) + \left(1-\delta\right) Z.$$

This condition is clearly satisfied, so that $dn_E^*(\eta)/d\eta < 0$. Likewise,

$$\frac{d\alpha^*(\eta)}{d\eta} = \frac{\begin{vmatrix} \frac{\partial J}{\partial n_E} & -\frac{\partial J}{\partial \eta} \\ \frac{\partial H}{\partial n_E} & -\frac{\partial H}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_E} & \frac{\partial J}{\partial \alpha} \\ \frac{\partial H}{\partial n_E} & \frac{\partial H}{\partial \alpha} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial n_E}\frac{\partial H}{\partial \eta} + \frac{\partial H}{\partial n_E}\frac{\partial J}{\partial \eta}}{\frac{\partial J}{\partial n_E}\frac{\partial H}{\partial \alpha} - \frac{\partial H}{\partial n_E}\frac{\partial J}{\partial \alpha}} = -\frac{\overbrace{\frac{\partial H}{\partial \eta}}^{<0}}{\underbrace{\frac{\partial H}{\partial \alpha}}_{>0}} > 0.$$

To summarize,

$$\frac{dW(\cdot)}{d\eta} = \overbrace{\frac{\partial W}{\partial n_E^*(\eta)}}^{>0}\overbrace{\frac{dn_E^*(\eta)}{d\eta}}^{<0} + \overbrace{\frac{\partial W}{\partial \alpha^*(\eta)}}^{<0}\overbrace{\frac{d\alpha^*(\eta)}{d\eta}}^{>0} + \overbrace{\frac{\partial W}{\partial \eta}}^{<0}.$$

Thus, $dW(\cdot)/d\eta < 0$, which implies that $\eta^* = 0$. Consequently, the optimal policy is a funding subsidy S_I.

(ii) Equivalence Between Investment and Return Subsidies

Consider first the benchmark model. The new market equilibrium is then defined by the following entry and market clearing conditions:

$$n_E = \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)\left(y + \frac{S_R}{\delta\gamma\rho}\right) \tag{A.27}$$

$$\gamma n_E \phi = \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho}{\phi}\left(y + \frac{S_R}{\delta\gamma\rho}\right) - 1\right]\widetilde{n}\widetilde{w}. \tag{A.28}$$

Combining these two conditions we define

$$H \equiv \frac{1}{\mu_E}\delta\gamma^2\rho(1-\alpha)\left(y + \frac{S_R}{\delta\gamma\rho}\right)\phi - \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho}{\phi}\left(y + \frac{S_R}{\delta\gamma\rho}\right) - 1\right]\widetilde{n}\widetilde{w} = 0,$$

which characterizes the equilibrium equity stake $\alpha^*(S_R)$. Implicitly differentiating $\alpha^*(S_R)$ yields

$$\frac{d\alpha^*(S_R)}{dS_R} = \frac{\frac{1}{\mu_E}\gamma(1-\alpha)\phi - \frac{1}{\mu_I}\alpha\frac{1}{\phi}\frac{1}{\gamma}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\delta\gamma^2\rho\left(y + \frac{S_R}{\delta\gamma\rho}\right)\phi + \frac{1}{\mu_I}\frac{\delta\rho}{\phi}\left(y + \frac{S_R}{\delta\gamma\rho}\right)\widetilde{n}\widetilde{w}}. \tag{A.29}$$

Using H we can derive the following expression for $\frac{1}{\mu_E}\gamma(1-\alpha)\phi$:

$$\frac{1}{\mu_E}\gamma(1-\alpha)\phi = \frac{\frac{1}{\mu_I}\left[\alpha\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right]\widetilde{n}\widetilde{w}}{\delta\gamma\rho\left(y+\frac{S_R}{\delta\gamma\rho}\right)}.$$

Using this expression we get

$$\frac{d\alpha^*(S_R)}{dS_R} = -\frac{\frac{1}{\mu_I}\widetilde{n}\widetilde{w}}{\delta\gamma\rho\left(y+\frac{S_R}{\delta\gamma\rho}\right)\left[\frac{1}{\mu_E}\delta\gamma^2\rho\left(y+\frac{S_R}{\delta\gamma\rho}\right)\phi+\frac{1}{\mu_I}\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)\widetilde{n}\widetilde{w}\right]} < 0.$$

Moreover, using the entry condition (A.27) with (A.29) we get

$$\frac{dn_E^*(S_R)}{dS_R} = \frac{1}{\mu_E}\delta\gamma\rho\left[-\frac{d\alpha^*(S_R)}{dS_R}\left(y+\frac{S_R}{\delta\gamma\rho}\right)+(1-\alpha)\frac{1}{\delta\gamma\rho}\right] = \frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\gamma^2\phi+\frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}} > 0.$$

Using the expression for $dn_E^*(S_I)/dS_I$ as derived in Proof of Proposition 2, we can immediately see that $dn_E^*(S_R)/dS_R = dn_E^*(S_I)/dS_I$. Thus, $n_E^*(S_R) = n_E^*(S_I) = n_E^*(S_E)$ for all $S_R = S_I = S_E$.

We now consider the effect of S_R in the dynamic model, focusing on the high steady state equilibrium. With a return subsidy $S_R > 0$ the expected utility of an entrepreneur in the high steady state equilibrium is given by

$$U_E(S_R) = \delta\gamma\rho\left[\int_0^{\widehat{\theta}_{t+1}}\left(\alpha_{t+1}\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-\theta\right)(1-\alpha)\left(y+\frac{S_R}{\delta\gamma\rho}\right)\frac{1}{\mu_I}d\theta+\int_{\widehat{\theta}_{t+1}}^{\mu_I}(1-\alpha)\left(y+\frac{S_R}{\delta\gamma\rho}\right)\frac{1}{\mu_I}d\theta\right],$$

where $\widehat{\theta}_{t+1} = \alpha_{t+1}\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1$. Thus,

$$U_E(S_R) = \delta\gamma\rho\,(1-\alpha)\left(y+\frac{S_R}{\delta\gamma\rho}\right)\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right)^2\right].$$

The high steady state market equilibrium is then defined by the following entry condition and market clearing condition:

$$n_E = \frac{1}{\mu_E}\delta\gamma\rho\,(1-\alpha)\left(y+\frac{S_R}{\delta\gamma\rho}\right)\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right)^2\right] \quad \text{(A.30)}$$

$$\phi = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right)\rho\,(1-\alpha)\left(y+\frac{S_R}{\delta\gamma\rho}\right), \quad \text{(A.31)}$$

where (A.31) defines $\alpha^*(S_R)$, and (A.30) then defines $n_E^*(S_R)$. Using (A.31) we define

$$H \equiv \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right)\rho\left(1-\alpha\right)\left(y+\frac{S_R}{\delta\gamma\rho}\right)-\phi=0.$$

Using H we get

$$\frac{d\alpha^*(S_R)}{dS_R}=-\frac{\frac{1}{\mu_I}\rho\left(1-\alpha\right)\left[\alpha\frac{1}{\phi}\frac{1}{\gamma}\left(y+\frac{S_R}{\delta\gamma\rho}\right)+\left(\alpha\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right)\frac{1}{\delta\gamma\rho}\right]}{\frac{\partial H}{\partial\alpha}}.$$

Furthermore, using the adjusted excess supply function $\Psi(\alpha, S_R)$ it is straightforward to show that $\partial H/\partial\alpha > 0$ in the high steady state equilibrium. Thus, $d\alpha^*(S_R)/dS_R < 0$. Moreover, totally differentiating (A.30) we find

$$\frac{dn_E^*(S_R)}{dS_R}=-\underbrace{\frac{d\alpha^*(S_R)}{dS_R}}_{<0}\frac{1}{\mu_E}\delta\gamma\rho\left(y+\frac{S_R}{\delta\gamma\rho}\right)\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right)^2\right]+\frac{\partial n_E^*(S_R)}{\partial S_R},$$

where

$$\begin{aligned}\frac{\partial n_E^*(S_R)}{\partial S_R} &= \frac{1}{\mu_E}\delta\gamma\rho\left(1-\alpha\right)\left[\frac{1}{\delta\gamma\rho}\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right)^2\right]\right.\\ &\left.+\left(y+\frac{S_R}{\delta\gamma\rho}\right)\frac{1}{\mu_I}\left(\alpha_{t+1}\frac{\delta\rho}{\phi}\left(y+\frac{S_R}{\delta\gamma\rho}\right)-1\right)\alpha_{t+1}\frac{1}{\phi}\frac{1}{\gamma}\right]>0.\end{aligned}$$

Thus, $dn_E^*(S_R)/dS_R > 0$.

Next we show that $n_E^*(S_R) = n_E^*(S_I)$ for all $S_R = S_I$. For this we consider a mix of the two subsidies, $\varpi S_R + (1-\varpi)S_I$, with $\varpi \in [0,1]$. The expected utility of an entrepreneur is then given by

$$U_E(S_R,S_I)=\delta\gamma\rho\left(1-\alpha\right)\left(y+\varpi\frac{S_R}{\delta\gamma\rho}\right)\left[\int_0^{z_{t+1}-1}\left(z_{t+1}-\theta\right)\frac{1}{\mu_I}d\theta+\int_{z_{t+1}-1}^{\mu_I}\frac{1}{\mu_I}d\theta\right],$$

where

$$z_{t+1}=\alpha_{t+1}\frac{\delta\rho}{\phi}\left(y+\varpi\frac{S_R}{\delta\gamma\rho}\right)+\frac{1}{\phi}\frac{1}{\gamma}(1-\varpi)S_I.$$

Now define

$$z\equiv\alpha\frac{\delta\rho}{\phi}\left(y+\varpi\frac{S_R}{\delta\gamma\rho}\right)+\frac{1}{\phi}\frac{1}{\gamma}(1-\varpi)S_I,$$

which can be written as

$$\alpha\delta\rho\left(y+\varpi\frac{S_R}{\delta\gamma\rho}\right)=\phi z-\frac{1}{\gamma}(1-\varpi)S_I.$$

Integrating and using this expression we get

$$U_E(S_R,S_I)=[\delta\gamma\rho y+\varpi S_R+(1-\varpi)S_I-\gamma\phi z]\left[1+\frac{1}{2\mu_I}\left(z_{t+1}-1\right)^2\right].$$

The high steady state market equilibrium is then defined by the following entry and market clearing conditions:

$$\begin{aligned} n_E &= \frac{1}{\mu_E}\left[\delta\gamma\rho y+\varpi S_R+(1-\varpi)S_I-\gamma\phi z\right]\left[1+\frac{1}{2\mu_I}\left(z_{t+1}-1\right)^2\right] && \text{(A.32)}\\ \phi &= \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho}{\phi}\left(y+\varpi\frac{S_R}{\delta\gamma\rho}\right)-1+\frac{1}{\phi}\frac{1}{\gamma}(1-\varpi)S_I\right)\rho\left(1-\alpha\right)\left(y+\varpi\frac{S_R}{\delta\gamma\rho}\right) && \text{(A.33)}\end{aligned}$$

Next we show that $dn_E^*(\varpi)/d\varpi=0$ for $S_R=S_I$. Note that in the steady state equilibrium $\alpha=\alpha_{t+1}$, so that $z=z_{t+1}$. Evaluating the total derivative at $S_R=S_I=S$ we get

$$\left.\frac{dn_E^*(\varpi)}{d\varpi}\right|_{S_R=S_I=S}=\left.\frac{\partial n_E^*(\varpi)}{\partial z}\right|_{S_R=S_I=S}\cdot\left.\frac{dz}{d\varpi}\right|_{S_R=S_I=S}+\left.\frac{\partial n_E^*(\varpi)}{\partial\varpi}\right|_{S_R=S_I=S},$$

where

$$\left.\frac{\partial n_E^*(\varpi)}{\partial\varpi}\right|_{S_R=S_I=S}=\frac{1}{\mu_E}\left[S-S\right]\left[1+\frac{1}{2\mu_I}\left(z_{t+1}-1\right)^2\right]=0.$$

Moreover,

$$\left.\frac{dz}{d\varpi}\right|_{S_R=S_I=S}=\frac{1}{\phi}\frac{1}{\gamma}\left(\delta\gamma\rho y+\varpi S\right)\left.\frac{d\alpha^*(\varpi)}{d\varpi}\right|_{S_R=S_I=S}-\left(1-\alpha\right)\frac{1}{\phi}\frac{1}{\gamma}S.$$

Using (A.33) we get

$$\begin{aligned}\left.\frac{d\alpha^*(\varpi)}{d\varpi}\right|_{S_R=S_I=S} &= -\frac{(1-\alpha)\left[\left(\alpha\frac{1}{\phi}\frac{1}{\gamma}S-\frac{1}{\phi}\frac{1}{\gamma}S\right)\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)+\left(\alpha\frac{\delta\rho}{\phi}\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)-1+\frac{1}{\phi}\frac{1}{\gamma}(1-\varpi)S\right)\frac{S}{\delta\gamma\rho}\right]}{\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)\left[\frac{\delta\rho}{\phi}\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)(1-\alpha)-\left(\alpha\frac{\delta\rho}{\phi}\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)-1+\frac{1}{\phi}\frac{1}{\gamma}(1-\varpi)S\right)\right]}\\ &= \frac{(1-\alpha)\,S\left[\frac{1}{\phi}\frac{1}{\gamma}\left(1-\alpha\right)\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)-\left(\alpha\frac{\delta\rho}{\phi}\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)-1+\frac{1}{\phi}\frac{1}{\gamma}(1-\varpi)S\right)\frac{1}{\delta\gamma\rho}\right]}{\delta\gamma\rho\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)\left[\frac{1}{\phi}\frac{1}{\gamma}\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)(1-\alpha)-\left(\alpha\frac{\delta\rho}{\phi}\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)-1+\frac{1}{\phi}\frac{1}{\gamma}(1-\varpi)S\right)\frac{1}{\delta\gamma\rho}\right]}\\ &= \frac{(1-\alpha)\,S}{\delta\gamma\rho y+\varpi S}.\end{aligned}$$

Consequently,

$$\left.\frac{dz}{d\varpi}\right|_{S_R=S_I=S}=\frac{1}{\phi}\frac{1}{\gamma}\left(\delta\gamma\rho y+\varpi S\right)\frac{\left(1-\alpha\right)S}{\delta\gamma\rho y+\varpi S}-\left(1-\alpha\right)\frac{1}{\phi}\frac{1}{\gamma}S=0,$$

so that $\left.\frac{dn_E^*(\varpi)}{d\varpi}\right|_{S_R=S_I=S}=0$. Thus, $dn_E^*(S_R)/dS_R=dn_E^*(S_I)/dS_I$ for all $S_R=S_I$, so that $n_E^*(S_R)=n_E^*(S_I)$ for all $S_R=S_I$.

Finally, let

$$E\left[w^*(\varpi)\right]\equiv\delta\gamma\rho\left(1-\alpha\right)\left(y+\varpi\frac{S_R}{\delta\gamma\rho}\right)$$

denote an entrepreneur's expected wealth in the high steady state equilibrium. Evaluating the total derivative of $E\left[w^*(\varpi)\right]$ w.r.t. ϖ at $S_R=S_I=S$ yields

$$\begin{aligned}\left.\frac{dE\left[w^*(\varpi)\right]}{d\varpi}\right|_{S_R=S_I=S} &= -\delta\gamma\rho\left.\frac{d\alpha^*(\varpi)}{d\varpi}\right|_{S_R=S_I=S}\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)+\delta\gamma\rho\left(1-\alpha\right)\frac{S}{\delta\gamma\rho}\\ &= \frac{\delta\gamma\rho y\left(1-\alpha\right)S+\left(1-\alpha\right)\varpi S^2-\delta\gamma\rho\left(1-\alpha\right)S\left(y+\varpi\frac{S}{\delta\gamma\rho}\right)}{\delta\gamma\rho y+\varpi S}=0.\end{aligned}$$

This implies that $E\left[w^*(S_R)\right]=E\left[w^*(S_I)\right]$ for all $S_R=S_I$.

(iii) Alternative Policies

To identify the set of all feasible non-discriminatory policies, we first consider what states are verifiable for the government to base a subsidy on. The first verifiable action is entry. Our founding subsidies are conditional upon entrepreneurial entry, and by definition apply only to entrepreneurs but not investors. The next verifiable action is investment. Our model already captures investment subsidies to either entrepreneurs or investors. One additional verifiable variable at the investment stage is the investment price α; we return to this shortly. Finally, the outcome of a venture is verifiable. Our model already captures return subsidies in case of success. It is easy to see that any subsidy to failure would behave equivalently in our model (but would raise concerns about moral hazard in any model extension with private effort).

The only alternative policy to consider here is therefore any subsidy that is contingent on α, as mentioned above. We claim that for any subsidy contingent on α, there exists an equivalent investment subsidy that is not contingent on α (and is therefore already accounted for in our optimality proof). Consider a generic investor subsidy $S_I(\alpha)$, where for simplicity we assume that $S_I(\alpha)$ is weakly monotonous in α (this can be relaxed too). The equilibrium of the model is again characterized by equations (11) and (12). Consider now the equilibrium level α^*, and the associated subsidy $S_I(\alpha^*)$. Next define a non-contingent subsidy $S_I^* = S_I(\alpha^*)$. It is immediate that S_I^* also satisfies the same equilibrium conditions (11) and (12) as $S_I(\alpha^*)$. Therefore it achieves the identical market outcome. It follows that any subsidy contingent on α can also be replicated with a subsidy that is not contingent on α. Consequently there cannot be any contingent subsidy that achieves a higher welfare than the non-contingent optimal funding subsidy.

Funding Subsidies and the Tax Incidence Equivalence Result.

Suppose the government either pays the monetary subsidy S_{I-E} to entrepreneurs, or the monetary subsidy S_I to investors, with $S_{I-E} = S_I$. When offering S_{I-E} to entrepreneurs, the market equilibrium is defined by the entry condition $n_E = \frac{1}{\mu_E}\left[\delta\gamma\rho(1-\alpha)y + S_{I-E}\right]$, and the market clearing condition

$$\frac{1}{\mu_E}\gamma\left[\delta\gamma\rho(1-\alpha)y + S_{I-E}\right]\phi = \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho y}{\phi} - 1\right]\widetilde{n}\widetilde{w}. \tag{A.34}$$

Implicitly differentiating (A.34) yields

$$\frac{d\alpha^*(S_{I-E})}{dS_{I-E}} = \frac{\frac{1}{\mu_E}\gamma\phi}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}} > 0.$$

Moreover, using the entry condition and the expression for $d\alpha^*(S_{I-E})/dS_{I-E}$, we get

$$\frac{dn_E^*(S_{I-E})}{dS_{I-E}} = \frac{1}{\mu_E}\left[1 - \delta\gamma\rho y\frac{d\alpha^*(S_{I-E})}{dS_{I-E}}\right] = \frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\gamma^2\phi + \frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}.$$

Finally, using the expression for $dn_E^*(S_I)/dS_I$ as derived in Proof of Proposition 2, we can immediately see that $dn_E^*(S_{I-E})/dS_{I-E} = dn_E^*(S_I)/dS_I$. Thus, $n_E^*(S_{I-E}) = n_E^*(S_I)$ for all $S_{I-E} = S_I$. Consequently it does not matter whether entrepreneurs or investors get the monetary subsidy; it always leads to the same equilibrium level of entrepreneurship n_E^*.

Leveraging Investments.

Suppose that each venture requires the investment $\omega\phi$ from smart angels; the remaining amount $(1-\omega)\phi$ is then offered by other investors. We assume that the other investors have sufficient capital, so they can always supply $\gamma n_E(1-\omega)\phi$.

Consider first the benchmark model without intergenerational dynamics, where the amount $\left(\alpha\frac{\rho y}{\phi}-1\right)\widetilde{n}\widetilde{w}$ is supplied by smart investors. The market equilibrium is then defined by the following entry condition and market clearing condition (for smart capital):

$$\begin{aligned} n_E &= U_E = \gamma\rho(1-\alpha)y \\ \gamma n_E \omega\phi &= \left(\alpha\frac{\rho y}{\phi}-1\right)\widetilde{n}\widetilde{w}, \end{aligned}$$

where the remaining capital $\gamma n_E(1-\omega)\phi$ is supplied by the other investors. Note that these equilibrium conditions are technically equivalent to (1) and (2). Thus, Propositions 1 and 2 also apply for any $\omega \in (0,1)$. Moreover, note that the total supply of capital (i.e., smart capital plus capital from other investors) is given by $\frac{1}{\omega}\left(\alpha\frac{\rho y}{\phi}-1\right)\widetilde{n}\widetilde{w}$, which, for a given α, is decreasing in ω (i.e., the smaller ω, the larger the total angel market).

Next consider the dynamic model with intergenerational linkages. When angel investments can be leveraged, the equilibrium is defined by the following entry condition and market clearing condition (for smart capital):

$$\begin{aligned} n_{E,t} &= U_{E,t} \\ \gamma n_{E,t}\omega\phi &= \left[\alpha_t\frac{\rho y}{\phi}-1\right]\gamma\rho n_{E,t-1}(1-\alpha_{t-1})y, \end{aligned}$$

where the remaining capital $\gamma n_{E,t}\left(1-\omega\right)\phi$ is supplied by the other investors, and

$$U_{E,t} = \gamma\rho\left[\int_0^{\widehat{\theta}_{t+1}}\left(\alpha_{t+1}\frac{\rho y}{\phi}-\theta\right)\left(1-\alpha_t\right)y d\theta + \int_{\widehat{\theta}_{t+1}}^1\left(1-\alpha_t\right)y d\theta\right].$$

In the high steady state the equilibrium conditions simplify to

$$\begin{aligned} n_E &= U_E \\ \phi &= \frac{1}{\omega}\left[\alpha\frac{\rho y}{\phi}-1\right]\rho(1-\alpha)y. \end{aligned}$$

Again we note that the equilibrium conditions are technically equivalent to (7) and (8) (with $n_E = n_{E,t} = n_{E,t-1}$ and $\alpha = \alpha_t = \alpha_{t-1}$ in the steady state), so that the main results from Propositions 3-6 continue to hold for any $\omega \in (0,1)$. Furthermore, note that the total capital supply $\frac{1}{\omega}\left[\alpha\frac{\rho y}{\phi}-1\right]\rho(1-\alpha)y$ is decreasing in ω (for a given α), so that a smaller ω implies a larger angel market.

Serial Angels and Entrepreneurs.

We first extend our benchmark model without intergenerational linkages by allowing for serial investors. Specifically, each investor can make another investment in the next period with probability σ_I. Using $\widehat{\theta} = \alpha \frac{\delta\rho y}{\phi} - 1$, we can write the marginal return for an investor, denoted by R, as

$$R = \int_0^{\widehat{\theta}} \left(\alpha \frac{\delta\rho y}{\phi} - \theta \right) \frac{1}{\mu_I} d\theta + \int_{\widehat{\theta}}^{\mu_I} \frac{1}{\mu_I} d\theta = 1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 \right)^2 .$$

To ensure that investors reinvest their wealth whenever possible, we assume that $\delta R > 1$. Let U_I^S denote the expected utility of a serial investor with initial wealth $\widetilde{w}$. We assume that σ_I is sufficiently small so that $\sigma_I \delta R < 1$. In the steady state U_I^S is then given by

$$U_I^S = (1 - \sigma_I)\, \widetilde{w} R \left[1 + \sigma_I \delta R + (\sigma_I \delta R)^2 + ... \right] = (1 - \sigma_I)\, \widetilde{w} \frac{R}{1 - \sigma_I \delta R}.$$

We also assume that in each period there are $\widetilde{n}$ new investors who can invest in ventures (or in the safe asset).

The expected utility of an entrepreneur is $U_E = \delta\gamma\rho(1-\alpha)y$, so that the entry condition is given by $n_E = \frac{1}{\mu_E} \delta\gamma\rho(1-\alpha)y$. Moreover, using the entry condition we get the following market clearing condition:

$$\frac{1}{\mu_E} \delta\gamma^2 \rho(1-\alpha) y\phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 \right) K,$$

where K is the total stock of capital. In the steady state we have

$$K = \widetilde{n}\widetilde{w} \left[1 + \sigma_I R + (\sigma_I R)^2 + ... \right] = \frac{1}{1 - \sigma_I R} \widetilde{n}\widetilde{w}.$$

Note that $K > \widetilde{n}\widetilde{w}$. Using the market clearing condition () we can implicitly differentiate α^* with respect to K:

$$\frac{d\alpha^*}{dK} = - \frac{\frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 \right)}{\frac{1}{\mu_E} \delta\gamma^2 \rho y\phi + \frac{1}{\mu_I} \frac{\delta\rho y}{\phi} K} < 0.$$

Consequently,

$$\frac{dn_E^*}{dK} = -\frac{1}{\mu_E}\delta\gamma\rho y \underbrace{\frac{d\alpha^*}{dK}}_{<0} > 0,$$

i.e., serial angels increase the supply of capital (because $K > \widetilde{n}\widetilde{w}$), and therefore increase the overall level of entrepreneurship.

Now consider the effect of a founding subsidy S_E. The market equilibrium is then defined by the following entry and market clearing conditions:

$$\begin{aligned} n_E &= \frac{1}{\mu_E}\left[\delta\gamma\rho(1-\alpha)y + S_E\right] \\ \frac{1}{\mu_E}\gamma\left[\delta\gamma\rho(1-\alpha)y + S_E\right]\phi &= \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{1}{1-\sigma_I R}\widetilde{n}\widetilde{w}. \end{aligned}$$

Using the market clearing condition (A.35) we get

$$\frac{d\alpha^*(S_E)}{dS_E} = \frac{\frac{1}{\mu_E}\gamma\phi}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\widetilde{n}\widetilde{w}\left[\frac{\delta\rho y}{\phi}\frac{1}{1-\sigma_I R} + \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{1}{[1-\sigma_I R]^2}\sigma_I\frac{\partial R}{\partial\alpha}\right]},$$

where

$$\frac{\partial R}{\partial\alpha} = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{\delta\rho y}{\phi} > 0.$$

Moreover, using the entry condition (A.35),

$$\frac{dn_E^*(S_E)}{dS_E} = \frac{1}{\mu_E}\left[1 - \delta\gamma\rho y\frac{d\alpha^*(S_E)}{dS_E}\right] = \frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\widetilde{n}\widetilde{w}\left[\frac{\delta\rho y}{\phi}\frac{1}{1-\sigma_I R} + \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{1}{[1-\sigma_I R]^2}\sigma_I\frac{\partial R}{\partial\alpha}\right]}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\widetilde{n}\widetilde{w}\left[\frac{\delta\rho y}{\phi}\frac{1}{1-\sigma_I R} + \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{1}{[1-\sigma_I R]^2}\sigma_I\frac{\partial R}{\partial\alpha}\right]}.$$

Next, consider the effect of a funding subsidy S_I. The market equilibrium is then defined by

$$\begin{aligned} n_E &= \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y \\ \frac{1}{\mu_E}\gamma\delta\gamma\rho(1-\alpha)y\phi &= \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{1-\sigma_I R(S_I)}\widetilde{n}\widetilde{w}, \end{aligned}$$

where

$$R(S_I) = 1 + \frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2.$$

Using the market clearing condition (A.35) we get

$$\frac{d\alpha^*(S_I)}{dS_I} = -\frac{\frac{1}{\mu_I}\widetilde{n}\widetilde{w}\left[\frac{1}{\phi}\frac{1}{\gamma}\frac{1}{1-\sigma_I R(S_I)} + \left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{[1-\sigma_I R(S_I)]^2}\sigma_I\frac{\partial R(S_I)}{\partial S_I}\right]}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\widetilde{n}\widetilde{w}\left[\frac{\delta\rho y}{\phi}\frac{1}{1-\sigma_I R(S_I)} + \left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{[1-\sigma_I R(S_I)]^2}\sigma_I\frac{\partial R(S_I)}{\partial\alpha}\right]}.$$

Consequently, using the entry condition (A.35),

$$\frac{dn_E^*(S_I)}{dS_I} = -\frac{1}{\mu_E}\delta\gamma\rho y\frac{d\alpha^*(S_I)}{dS_I} = \frac{\frac{1}{\mu_E}\delta\gamma\rho y\frac{1}{\mu_I}\widetilde{n}\widetilde{w}\left[\frac{1}{\phi}\frac{1}{\gamma}\frac{1}{1-\sigma_I R(S_I)} + \left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{[1-\sigma_I R(S_I)]^2}\sigma_I\frac{\partial R(S_I)}{\partial S_I}\right]}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\widetilde{n}\widetilde{w}\left[\frac{\delta\rho y}{\phi}\frac{1}{1-\sigma_I R(S_I)} + \left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{[1-\sigma_I R(S_I)]^2}\sigma_I\frac{\partial R(S_I)}{\partial\alpha}\right]}.$$

Next we show that $dn_E^*(S_E)/dS_E = dn_E^*(S_I)/dS_I$ for $S_E = S_I \to 0$. Note that we have $\alpha(S_E) = \alpha(S_I) = \alpha$ and $R(S_I) = R$ for $S_E = S_I \to 0$. We can then immediately see that $dn_E^*(S_E)/dS_E = dn_E^*(S_I)/dS_I$ for $S_E = S_I \to 0$, is equivalent to

$$\begin{aligned} \frac{\delta\rho y}{\phi}\frac{1}{1-\sigma_I R} + \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{1}{[1-\sigma_I R]^2}\sigma_I\frac{\partial R}{\partial\alpha} &= \delta\gamma\rho y\left[\frac{1}{\phi}\frac{1}{\gamma}\frac{1}{1-\sigma_I R} + \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{1}{[1-\sigma_I R]^2}\sigma_I\left.\frac{\partial R(S_I)}{\partial S_I}\right|_{S_I=0}\right] \\ \Leftrightarrow \qquad \frac{\partial R}{\partial\alpha} &= \delta\gamma\rho y\left.\frac{\partial R(S_I)}{\partial S_I}\right|_{S_I=0}. \end{aligned}$$

Note that

$$\frac{\partial R}{\partial\alpha} = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{\delta\rho y}{\phi} \qquad \left.\frac{\partial R(S_I)}{\partial S_I}\right|_{S_I=0} = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{1}{\phi}\frac{1}{\gamma}.$$

Thus, we can write the above condition as

$$\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{\delta\rho y}{\phi} = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\frac{\delta\rho y}{\phi},$$

which is clearly satisfied. Thus, $dn_E^*(S_E)/dS_E = dn_E^*(S_I)/dS_I$, and therefore $n_E^*(S_E) = n_E^*(S_I)$, for all $S_E = S_I \to 0$, i.e., both policies lead to the same level of entrepreneurship.

Finally we extend our dynamic model with intergenerational linkages by allowing for serial angels and serial entrepreneurs. We then show that our main insight, namely that funding subsidies generate more entrepreneurial activities than founding subsidies, remains intact. For simplicity we focus on the high steady state equilibrium.

Suppose again that each investor can make another investment in the next period with probability σ_I; the marginal return is still given by R as defined above. In the steady state the expected utility of a 'first-time' serial investor (i.e., an investor who just succeeded as entrepreneur) is now given by

$$U_I^S = (1-\sigma_I)(1-\alpha)yR\left[1+\sigma_I\delta R+(\sigma_I\delta R)^2+...\right] = (1-\sigma_I)(1-\alpha)y\frac{R}{1-\sigma_I\delta R},$$

where $\sigma_I\delta R < 1$.

Furthermore, suppose that each entrepreneur can start another venture in the next period with probability σ_E.[2] For tractability we assume that formerly successful entrepreneurs can start a new venture and make angel investments at the same time. In this case we also assume that the entire wealth is invested in other start-ups or in the safe asset, so that ventures started by wealthy serial entrepreneurs are financed by different angels.[3] The expected utility of a serial entrepreneur, denoted by U_E^S, is then given by

$$U_E^S = \delta\gamma\rho U_{I,t+1}^S + \sigma_E\delta U_E^S, \tag{A.35}$$

where

$$U_{I,t+1}^S = (1-\sigma_I)(1-\alpha)y\frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}}, \qquad R_{t+1} = 1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)^2.$$

Solving (A.35) for U_E^S we get $U_E^S = \frac{1}{1-\sigma_E\delta}\delta\gamma\rho U_{I,t+1}^S$, so that the entry condition for entrepreneurs can be written as

$$n_E = \frac{1}{\mu_E}U_E^S = \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}}.$$

Moreover, the market clearing condition is given by

$$\gamma N_E^S\phi = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi}-1\right)K,$$

[2] One could set $\sigma_E = 0$ to only allow for serial investors in our dynamic model with intergenerational linkages. However, as it will become clear from the derivations below, our main result (namely, that a funding subsidy leads to more entrepreneurial entry) also holds for $\sigma_E = 0$.

[3] Note that this model is equivalent to a model where the entrepreneur can invest in her own company. This is because an entrepreneur can always use her wealth to buy back the equity stake from the investor, which provides her with the same expected return as investing her wealth in other ventures.

where N_E^S is the total number of entrepreneurs (with good or bad projects) in the market, and K is the total stock of capital. In the steady state we have

$$\begin{aligned} N_E^S &= n_E^S + \sigma_E n_{E-1}^S + \sigma_E^2 n_{E-2}^S + ... = n_E^S \left[1 + \sigma_E + \sigma_E^2 + ...\right] = \frac{1}{1-\sigma_E} n_E^S \\ K &= \rho\gamma n_{E-1}^S (1-\alpha) y + \sigma_I \rho\gamma n_{E-2}^S (1-\alpha) y R + \sigma_I^2 \rho\gamma n_{E-3}^S (1-\alpha) y R^2 + ... \\ &= \rho\gamma n_E^S (1-\alpha) y \left[1 + \sigma_I R + (\sigma_I R)^2 + ...\right] = \frac{1}{1-\sigma_I R} \rho\gamma n_E^S (1-\alpha) y. \end{aligned}$$

Using the definition of R we can then write the market clearing condition as follows:

$$\left(1 - \sigma_I - \sigma_I \frac{1}{2\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1\right)^2\right) \phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1\right) (1-\sigma_E) \rho (1-\alpha) y.$$

Now consider the effect of a founding subsidy S_E. Using the entry condition and market clearing condition we define

$$\begin{aligned} J &\equiv \frac{1}{\mu_E} \left[\delta\gamma\rho(1-\alpha) y \frac{1-\sigma_I}{1-\sigma_E \delta} \frac{R_{t+1}}{1-\sigma_I \delta R_{t+1}} + S_E\right] - n_E = 0 \\ H &\equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1\right) (1-\sigma_E) \rho (1-\alpha) y - \left(1 - \sigma_I - \sigma_I \frac{1}{2\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1\right)^2\right) \phi = 0. \end{aligned}$$

We get

$$\frac{dn_E^*(S_E)}{dS_E} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial S_E} & \frac{\partial J}{\partial \alpha} \\ -\frac{\partial H}{\partial S_E} & \frac{\partial H}{\partial \alpha} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_E} & \frac{\partial J}{\partial \alpha} \\ \frac{\partial H}{\partial n_E} & \frac{\partial H}{\partial \alpha} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial S_E}\frac{\partial H}{\partial \alpha} + \frac{\partial H}{\partial S_E}\frac{\partial J}{\partial \alpha}}{\frac{\partial J}{\partial n_E}\frac{\partial H}{\partial \alpha} - \frac{\partial H}{\partial n_E}\frac{\partial J}{\partial \alpha}},$$

where $\partial J/\partial S_E = 1/\mu_E, \partial H/\partial S_E = 0, \partial J/\partial n_E = -1$, and $\partial H/\partial n_E = 0$. Thus, $dn_E^*(S_E)/dS_E = \partial J/\partial S_E = 1/\mu_E > 0$.

Next, consider the effect of a funding subsidy S_I. The marginal return function for an investor, R, then becomes

$$R = \int_0^{\widehat{\theta} + \frac{1}{\phi}\frac{1}{\gamma}S_I} \left(\alpha \frac{\delta\rho y}{\phi} + \frac{1}{\phi}\frac{1}{\gamma} S_I - \theta\right) \frac{1}{\mu_I} d\theta + \int_{\widehat{\theta} + \frac{1}{\phi}\frac{1}{\gamma}S_I}^{\mu_I} \frac{1}{\mu_I} d\theta = 1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma} S_I\right)^2.$$

With $S_I > 0$ the high steady state market equilibrium is then defined by the following entry condition and market clearing condition:

$$\begin{aligned} J &\equiv \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}} - n_E = 0 \\ H &\equiv \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)(1-\sigma_E)\,\rho\,(1-\alpha)\,y \\ &\quad - \left(1-\sigma_I-\sigma_I\frac{1}{2\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2\right)\phi = 0, \end{aligned}$$

where

$$R_{t+1} = 1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2.$$

Using the equilibrium conditions we get

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial S_I} & \frac{\partial J}{\partial \alpha} \\ -\frac{\partial H}{\partial S_I} & \frac{\partial H}{\partial \alpha} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_E} & \frac{\partial J}{\partial \alpha} \\ \frac{\partial H}{\partial n_E} & \frac{\partial H}{\partial \alpha} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial S_I}\frac{\partial H}{\partial \alpha} + \frac{\partial H}{\partial S_I}\frac{\partial J}{\partial \alpha}}{\frac{\partial J}{\partial n_E}\frac{\partial H}{\partial \alpha} - \frac{\partial H}{\partial n_E}\frac{\partial J}{\partial \alpha}},$$

where $\partial J/\partial n_E = -1$, $\partial H/\partial n_E = 0$, and

$$\begin{aligned} \frac{\partial J}{\partial S_I} &= \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)}{\left[1-\sigma_I\delta R_{t+1}\right]^2} \\ \frac{\partial H}{\partial S_I} &= \frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}(1-\sigma_E)\,\rho\,(1-\alpha)\,y + \sigma_I\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{\gamma} \\ \frac{\partial J}{\partial \alpha} &= -\frac{1}{\mu_E}\delta\gamma\rho y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}} \\ \frac{\partial H}{\partial \alpha} &= \frac{1}{\mu_I}(1-\sigma_E)\,\rho y\left[\frac{\delta\rho y}{\phi}(1-\alpha) - \left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\right] + \sigma_I\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\delta\rho y. \end{aligned}$$

Moreover, using the adjusted excess supply function $\Psi(\alpha, S_I, \sigma_E, \sigma_I)$ it is straightforward to show that $\partial H/\partial\alpha > 0$ for the high steady state equilibrium. Thus,

$$\begin{aligned}\frac{dn_E^*(S_I)}{dS_I} &= \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}T_{t+1}}{[1-\sigma_I\delta R_{t+1}]^2} \\ &\quad + \frac{\left[\frac{1}{\mu_I}\frac{1}{\phi}(1-\sigma_E)\rho(1-\alpha)y+\sigma_I\frac{1}{\mu_I}T\right]\frac{1}{\mu_E}\delta\rho y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}}}{\frac{\partial H}{\partial\alpha}},\end{aligned}$$

where

$$T = \alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I \qquad T_{t+1} = \alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I.$$

Note that $n_E^*(S_I = 0) = n_E^*(S_E = 0) = n_E^*$. Thus, we have $n_E^*(S_I) > n_E^*(S_E)$ for all $S_I = S_E > 0$ if $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$, which is equivalent to

$$\begin{aligned}&\delta\gamma\rho(1-\alpha)y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}T_{t+1}}{[1-\sigma_I\delta R_{t+1}]^2}\frac{\partial H}{\partial\alpha} \\ &+\left[\frac{1}{\mu_I}\frac{1}{\phi}(1-\sigma_E)\rho(1-\alpha)y+\sigma_I\frac{1}{\mu_I}T\right]\delta\rho y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}} > \frac{\partial H}{\partial\alpha}.\end{aligned}$$

Using $\partial H/\partial\alpha$ and simplifying we can write this condition as

$$\begin{aligned}&\delta\gamma\rho(1-\alpha)y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}T_{t+1}}{[1-\sigma_I\delta R_{t+1}]^2}\left[(1-\sigma_E)\left[\frac{\delta\rho y}{\phi}(1-\alpha)-T\right]+\sigma_I T\delta\right] \\ &+\left[\frac{1}{\phi}(1-\sigma_E)\rho(1-\alpha)y+\sigma_I T\right]\delta\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}} \\ &> (1-\sigma_E)\left[\frac{\delta\rho y}{\phi}(1-\alpha)-T\right]+\sigma_I T\delta.\end{aligned}$$

Rearranging yields

$$\begin{aligned}&\delta\gamma\rho(1-\alpha)y\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}T_{t+1}}{[1-\sigma_I\delta R_{t+1}]^2}\overbrace{\left[(1-\sigma_E)\left[\frac{\delta\rho y}{\phi}(1-\alpha)-T\right]+\sigma_I T\delta\right]}^{\equiv X_1} \\ &+\delta\left[\frac{1}{\phi}(1-\sigma_E)\rho(1-\alpha)y+\sigma_I T\right]\underbrace{\left[\frac{1-\sigma_I}{1-\sigma_E\delta}\frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}}-1\right]}_{\equiv X_2} > -(1-\sigma_E)T. \quad \text{(A.36)}\end{aligned}$$

Recall that $\partial H/\partial\alpha > 0$ in the high steady state equilibrium, which implies that $X_1 > 0$. Moreover, we can write X_2 as

$$X_2 = \underbrace{\frac{1-\sigma_I}{1-\sigma_I\delta R_{t+1}}}_{\equiv L_1}\underbrace{\frac{R_{t+1}}{1-\sigma_E\delta}}_{\equiv L_2} - 1.$$

Note that $L_1 > 1$ because $\delta R_{t+1} > 1$. Moreover, we can immediately see that $R_{t+1} > 1$; and because $\sigma_E\delta < 1$ we have $L_2 > 1$. Consequently, $X_2 > 0$, so that condition (A.36) is satisfied. Thus, $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$, and therefore $dn_E^*(S_I) > n_E^*(S_E)$ for all $S_I = S_E > 0$. $\square$

Growth Options.

Suppose each entrepreneur has a growth option with probability ξ, generating the exit value y_2. With probability $1-\xi$, an entrepreneur does not have a growth option, and the exit value is given by y_1, with $y_1 < \delta y_2$. We assume that y_2 is large enough, so that entrepreneurs would always prefer to take advantage of the growth option. The expected utility of an entrepreneur is then given by

$$\begin{aligned} U_{E,t} &= \delta\gamma\rho(1-\xi)\left[\int_0^{\widehat{\theta}_{t+1}}\left(\alpha_{t+1}\frac{\delta\rho\widetilde{y}}{\phi}-\theta\right)(1-\alpha_t)\,y_1\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}_{t+1}}^{\mu_I}(1-\alpha_t)\,y_1\frac{1}{\mu_I}d\theta\right] \\ &\quad +\delta\gamma\rho\xi\delta\left[\int_0^{\widehat{\theta}_{t+2}}\left(\alpha_{t+2}\frac{\delta\rho\widetilde{y}}{\phi}-\theta\right)(1-\alpha_t)\,y_2\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}_{t+2}}^{\mu_I}(1-\alpha_t)\,y_2\frac{1}{\mu_I}d\theta\right], \end{aligned}$$

where

$$\widehat{\theta}_{t+1} = \alpha_{t+1}\frac{\delta\rho\widetilde{y}}{\phi} - 1 \qquad \widehat{\theta}_{t+2} = \alpha_{t+2}\frac{\delta\rho\widetilde{y}}{\phi} - 1 \qquad \widetilde{y} = (1-\xi)y_1 + \xi\delta y_2.$$

Note that $\alpha_{t+1} = \alpha_{t+2}$ in the steady state. Following along the lines of our previous derivations of $U_{E,t}$, we get

$$U_{E,t} = \delta\gamma\rho\,(1-\alpha_t)\,\widetilde{y}\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho\widetilde{y}}{\phi} - 1\right)^2\right].$$

Thus, the entry condition is given by

$$n_{E,t} = \frac{1}{\mu_E}\delta\gamma\rho\,(1-\alpha_t)\,\widetilde{y}\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho\widetilde{y}}{\phi} - 1\right)^2\right].$$

Note that the capital demand is given by $\gamma n_{E,t}\phi$. Moreover, note that in period t capital is provided by former entrepreneurs without growth options (in $t-1$), and by former entrepreneurs with growth options (in $t-2$). Thus, the market clearing condition can be written as

$$\gamma n_{E,t}\phi = \frac{1}{\mu_I}\left[\alpha_t \frac{\delta\rho\widetilde{y}}{\phi} - 1\right]\gamma\rho\left(1-\xi\right)n_{E,t-1}(1-\alpha_{t-1})y_1 + \frac{1}{\mu_I}\left[\alpha_t \frac{\delta\rho\widetilde{y}}{\phi} - 1\right]\gamma\rho\xi n_{E,t-2}(1-\alpha_{t-2})y_2.$$

In the steady state we have $n_E = n_{E,t-1} = n_{E,t-2}$ and $\alpha = \alpha_t = \alpha_{t-1} = \alpha_{t-2}$. Thus, the high steady state market equilibrium is defined by the following (simplified) entry and market clearing conditions:

$$\begin{aligned} n_E &= \frac{1}{\mu_E}\delta\gamma\rho\left(1-\alpha\right)\widetilde{y}\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho\widetilde{y}}{\phi} - 1\right)^2\right] \\ \phi &= \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho\widetilde{y}}{\phi} - 1\right]\rho(1-\alpha)\left[\left(1-\xi\right)y_1 + \xi y_2\right], \end{aligned}$$

where $\widetilde{y} = (1-\xi)y_1 + \xi\delta y_2$. Note that the structure of the equilibrium conditions is the same as for our model without growth options. Thus, both models are equivalent, implying that all of our main results carry over to the model with growth options.

Next, define

$$H \equiv \frac{1}{\mu_I}\left[\alpha\frac{\delta\rho}{\phi}\left[y_1 + \xi\left(\delta y_2 - y_1\right)\right] - 1\right]\rho(1-\alpha)\left[y_1 + \xi\left(y_2 - y_1\right)\right] - \phi = 0.$$

Thus,

$$\frac{d\alpha^*}{d\xi} = -\frac{\frac{1}{\mu_I}\rho(1-\alpha)\left[\alpha\frac{\delta\rho}{\phi}\left(\delta y_2 - y_1\right)\left[y_1 + \xi\left(y_2 - y_1\right)\right] + \overbrace{\left[\alpha\frac{\delta\rho}{\phi}\left[y_1 + \xi\left(\delta y_2 - y_1\right)\right] - 1\right]}^{>0}\left(y_2 - y_1\right)\right]}{\frac{\partial H}{\partial \alpha}}.$$

Recall that $\partial H/\partial\alpha > 0$ in the high steady state equilibrium. Thus, $d\alpha^*/d\xi < 0$. Furthermore, we can immediately see that n_E^* is decreasing in α, and increasing in ξ; thus, $dn_E^*/d\xi > 0$.

Finally note that the number of angels in period t is given by $n_{A,t} = \gamma\rho(1-\xi) n_{E,t-1} + \gamma\rho\xi n_{E,t-2}$. Thus, the number of angels in the high steady state equilibrium (where $n_{E,t-1} = n_{E,t-2}$) is $n_A^* = \gamma\rho n_E^*$. We then find that

$$\frac{dn_A^*}{d\xi} = \gamma\rho \underbrace{\frac{dn_E^*}{d\xi}}_{>0} > 0.$$

Small Open Ecosystem – Proof.

Throughout this proof we focus on the case with $\lambda < \widehat{\theta} = \frac{\delta\rho y}{\phi} - 1$, so there is always some foreign investment for sufficiently high α.

The entry condition and the market clearing condition are then given by

$$J \equiv \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha) y \left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right] - n_E = 0 \tag{A.37}$$

$$H \equiv \frac{1}{\mu_I}\left[\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\gamma\rho(1-\alpha)y + \left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda\right)\frac{1}{n_E}\widetilde{n}\widetilde{w}\right] - \gamma\phi = 0. \tag{A.38}$$

Applying Cramer's rule we get

$$\frac{d\alpha^*}{d\widetilde{n}\widetilde{w}} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial(\widetilde{n}\widetilde{w})} & \frac{\partial J}{\partial n_E} \\ -\frac{\partial H}{\partial(\widetilde{n}\widetilde{w})} & \frac{\partial H}{\partial n_E} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial \alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial(\widetilde{n}\widetilde{w})}\frac{\partial H}{\partial n_E} + \frac{\partial H}{\partial(\widetilde{n}\widetilde{w})}\frac{\partial J}{\partial n_E}}{\frac{\partial J}{\partial \alpha}\frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial \alpha}\frac{\partial J}{\partial n_E}},$$

where $\partial J/\partial\lambda = 0$, $\partial J/\partial(\widetilde{n}\widetilde{w}) = 0$, $\partial J/\partial n_E = -1$, and

$$\begin{aligned}
\frac{\partial H}{\partial n_E} &= -\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda\right)\frac{1}{[n_E]^2}\widetilde{n}\widetilde{w} < 0 \\
\frac{\partial J}{\partial \alpha} &= -\frac{1}{\mu_E}\delta\gamma\rho y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right] < 0 \\
\frac{\partial H}{\partial(\widetilde{n}\widetilde{w})} &= \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda\right)\frac{1}{n_E} > 0.
\end{aligned}$$

Using the adjusted excess supply function $\Psi(\alpha)$ one can show that $\partial H/\partial\alpha > 0$ for the high and low steady state equilibrium. Thus, $d\alpha^*/d(\widetilde{n}\widetilde{w}) < 0$ Likewise,

$$\frac{dn_E^*}{d(\widetilde{n}\widetilde{w})} = \frac{\begin{vmatrix} \frac{\partial J}{\partial\alpha} & -\frac{\partial J}{\partial(\widetilde{n}\widetilde{w})} \\ \frac{\partial H}{\partial\alpha} & -\frac{\partial H}{\partial(\widetilde{n}\widetilde{w})} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial\alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial\alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial\alpha}\frac{\partial H}{\partial(\widetilde{n}\widetilde{w})} + \frac{\partial H}{\partial\alpha}\frac{\partial J}{\partial(\widetilde{n}\widetilde{w})}}{\frac{\partial J}{\partial\alpha}\frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial\alpha}\frac{\partial J}{\partial n_E}} = \frac{-\overbrace{\frac{\partial J}{\partial\alpha}}^{<0}\overbrace{\frac{\partial H}{\partial(\widetilde{n}\widetilde{w})}}^{>0}}{\underbrace{\frac{\partial J}{\partial\alpha}}_{<0}\underbrace{\frac{\partial H}{\partial n_E}}_{<0} + \underbrace{\frac{\partial H}{\partial\alpha}}_{>0}} > 0.$$

Next we show that the low steady state equilibrium disappears for sufficiently high $\widetilde{n}\widetilde{w}$. For this we use the adjusted excess supply function $\Psi(\alpha)$, with

$$\Psi(\alpha) = \frac{I(\alpha)}{E(\alpha)} = \frac{1}{\mu_I}\frac{1}{\phi}\left[\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\rho(1-\alpha)y + \frac{1}{\gamma}\left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda\right)\frac{1}{n_E(\alpha)}\widetilde{n}\widetilde{w}\right] = 1,$$

which, in conjunction with the entry condition (A.37), describes all ownership stakes α that are associated with all possible steady-state equilibria. We can immediately see that for $\alpha \leq \phi/(\delta\rho y)$ there will be no capital supply, i.e., $I(\alpha \leq \phi/(\delta\rho y)) = 0$, whereas the entry condition (A.37) implies that $n_E(\alpha \leq \phi/(\delta\rho y)) > 0$. Consequently, $\Psi(\alpha \leq \phi/(\delta\rho y)) = 0 < 1$. And when $\alpha = 1$ we can infer from (A.37) that $n_E(\alpha = 1) = 0$. Thus, $\lim_{\alpha\to 1}\Psi(\alpha) = \infty > 1$. Moreover,

$$\begin{aligned} \frac{d\Psi(\alpha)}{d\alpha} = {} & \frac{1}{\mu_I}\frac{1}{\phi}\left[\rho y\frac{\delta\rho y}{\phi}(1-\alpha) + \frac{1}{\gamma}\frac{\delta\rho y}{\phi}\frac{1}{n_E(\alpha)}\widetilde{n}\widetilde{w} + \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\left[\frac{1}{\gamma}\frac{1}{[n_E(\alpha)]^2}\left(-\frac{dn_E(\alpha)}{d\alpha}\right)\widetilde{n}\widetilde{w} - \rho y\right]\right. \\ & \left. -\lambda\frac{1}{\gamma}\frac{1}{[n_E(\alpha)]^2}\left(-\frac{dn_E(\alpha)}{d\alpha}\right)\widetilde{n}\widetilde{w}\right]. \end{aligned}$$

From the entry condition (A.37) we can see that $dn_E(\alpha)/d\alpha$. Thus, $\Psi(\alpha)$ is monotone and increasing in α when $\widetilde{n}\widetilde{w} \to \infty$. This in turn implies that then there exists only one α, denoted by α', satisfying the above excess supply function $\Psi(\alpha)$. And because $\left.\frac{d\Psi(\alpha)}{d\alpha}\right|_{\alpha=\alpha'} > 0$, the unique steady state equilibrium $n_E^*(\alpha')$ is stable.

Next, we analyze the effects of a founding subsidy S_E. The entry and the market clearing conditions are then given by

$$J \equiv \frac{1}{\mu_E}\left[\delta\gamma\rho(1-\alpha)y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right] + S_E\right] - n_E = 0$$

$$H \equiv \frac{1}{\mu_I}\left[\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\gamma\rho(1-\alpha)y + \left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda\right)\frac{1}{n_E}\widetilde{n}\widetilde{w}\right] - \gamma\phi = 0.$$

Applying Cramer's rule we get

$$\frac{d\alpha^*}{dS_E} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial S_E} & \frac{\partial J}{\partial n_E} \\ -\frac{\partial H}{\partial S_E} & \frac{\partial H}{\partial n_E} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial \alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial S_E}\frac{\partial H}{\partial n_E} + \frac{\partial H}{\partial S_E}\frac{\partial J}{\partial n_E}}{\frac{\partial J}{\partial \alpha}\frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial \alpha}\frac{\partial J}{\partial n_E}},$$

here $\partial J/\partial S_E = 1/\mu_E$, $\partial H/\partial S_E = 0$, $\partial J/\partial n_E = -1$, and

$$\begin{aligned} \frac{\partial H}{\partial n_E} &= -\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda\right)\frac{1}{[n_E]^2}\widetilde{n}\widetilde{w} < 0 \\ \frac{\partial J}{\partial \alpha} &= -\frac{1}{\mu_E}\delta\gamma\rho y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right] < 0. \end{aligned}$$

Moreover, using the adjusted excess supply function $\Psi(\alpha, S_E)$ one can show that $\partial H/\partial\alpha > 0$ at any stable steady state equilibrium. Consequently, $d\alpha^*/dS_E > 0$. Likewise,

$$\begin{aligned} \frac{dn_E^*(S_E)}{dS_E} &= \frac{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & -\frac{\partial J}{\partial S_E} \\ \frac{\partial H}{\partial \alpha} & -\frac{\partial H}{\partial S_E} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial \alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial \alpha}\frac{\partial H}{\partial S_E} + \frac{\partial H}{\partial \alpha}\frac{\partial J}{\partial S_E}}{\frac{\partial J}{\partial \alpha}\frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial \alpha}\frac{\partial J}{\partial n_E}} \\ &= \frac{\overbrace{\frac{\partial H}{\partial \alpha}}^{>0}\frac{1}{\mu_E}}{\frac{1}{\mu_E}\delta\gamma\rho y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right]\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda\right)\frac{1}{[n_E]^2}\widetilde{n}\widetilde{w} + \underbrace{\frac{\partial H}{\partial \alpha}}_{>0}} > 0. \end{aligned}$$

Consider now the effects of a funding subsidy S_I. For this it is convenient to write the entry and the market clearing conditions as follows:

$$\begin{aligned} J &\equiv \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha)y\left[1 + \varpi\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2\right] - n_E = 0 \\ H &\equiv \frac{1}{\mu_I}\left[\varpi\left(\alpha\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\gamma\rho(1-\alpha)y + \left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{n_E}\widetilde{n}\widetilde{w}\right] - \gamma\phi = 0, \end{aligned}$$

with $\varpi \in \{0,1\}$. For $\varpi = 0$ and $\widetilde{n} > 0$ (only foreign angel investors) we can combine J and H so that the equilibrium ownership stake $\alpha^*(S_I, \varpi = 0, \widetilde{n} > 0)$ is defined by

$$\frac{1}{\mu_E}\delta\gamma^2\rho\left(1-\alpha\right)y\phi = \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\widetilde{n}\widetilde{w}.$$

Implicitly differentiating this equilibrium condition we get

$$\frac{d\alpha^*(S_I, \varpi = 0, \widetilde{n} > 0)}{dS_I} = -\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}} < 0.$$

Moreover, we know from Proposition 5 that $d\alpha^*(S_I, \varpi = 1, \widetilde{n} = 0)/dS_I < 0$. This implies that for $\varpi = 1$ and $\widetilde{n} > 0$ we have $d\alpha^*(S_I)/dS_I < 0$. Furthermore,

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & -\frac{\partial J}{\partial S_I} \\ \frac{\partial H}{\partial \alpha} & -\frac{\partial H}{\partial S_I} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial \alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial \alpha}\frac{\partial H}{\partial S_I} + \frac{\partial H}{\partial \alpha}\frac{\partial J}{\partial S_I}}{\frac{\partial J}{\partial \alpha}\frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial \alpha}\frac{\partial J}{\partial n_E}},$$

where $\partial J/\partial n_E = -1$, and

$$\begin{aligned}
\frac{\partial J}{\partial S_I} &= \frac{1}{\mu_E}\delta\gamma\rho\left(1-\alpha\right)y\frac{1}{\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{\phi}\frac{1}{\gamma} > 0 \\
\frac{\partial H}{\partial S_I} &= \frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\left[\gamma\rho(1-\alpha)y + \frac{1}{n_E}\widetilde{n}\widetilde{w}\right] > 0 \\
\frac{\partial H}{\partial n_E} &= -\frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1 - \lambda + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{\left[n_E\right]^2}\widetilde{n}\widetilde{w} < 0 \\
\frac{\partial J}{\partial \alpha} &= -\frac{1}{\mu_E}\delta\gamma\rho y\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2\right] < 0.
\end{aligned}$$

Moreover, using the adjusted excess supply function $\Psi(\alpha, S_I)$ one can show again that $\partial H/\partial\alpha > 0$ at any stable steady state equilibrium. Thus,

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{\frac{1}{\mu_E}\delta\gamma\rho y\left[1+\frac{1}{2\mu_I}\left(\widehat{\theta}_{t+1}+\frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2\right]\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\left[\gamma\rho(1-\alpha)y+\frac{1}{n_E}\widetilde{n}\widetilde{w}\right]}{\frac{1}{\mu_E}\delta\gamma\rho y\left[1+\frac{1}{2\mu_I}\left(\widehat{\theta}_{t+1}+\frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2\right]\frac{1}{\mu_I}\left(\widehat{\theta}-\lambda+\frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{[n_E]^2}\widetilde{n}\widetilde{w}+\underbrace{\frac{\partial H}{\partial\alpha}}_{>0}}$$

$$+\frac{\overbrace{\frac{\partial H}{\partial\alpha}}^{>0}\frac{1}{\mu_E}\delta\gamma\rho\left(1-\alpha\right)y\frac{1}{\mu_I}\left(\widehat{\theta}_{t+1}+\frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{\phi}\frac{1}{\gamma}}{\frac{1}{\mu_E}\delta\gamma\rho y\left[1+\frac{1}{2\mu_I}\left(\widehat{\theta}_{t+1}+\frac{1}{\phi}\frac{1}{\gamma}S_I\right)^2\right]\frac{1}{\mu_I}\left(\widehat{\theta}-\lambda+\frac{1}{\phi}\frac{1}{\gamma}S_I\right)\frac{1}{[n_E]^2}\widetilde{n}\widetilde{w}+\underbrace{\frac{\partial H}{\partial\alpha}}_{>0}} > 0,$$

where

$$\widehat{\theta} = \alpha\frac{\delta\rho y}{\phi}-1 \qquad \widehat{\theta}_{t+1} = \alpha_{t+1}\frac{\delta\rho y}{\phi}-1.$$

For the policy comparison we evaluate the two derivatives, $dn_E^*(S_E)/dS_E$ and $dn_E^*(S_I)/dS_I$, at $S_E = S_I = 0$. Note that $\alpha^*(S_E) = \alpha^*(S_I)$ for $S_E = S_I \to 0$, and

$$\left.\frac{\partial H(S_E)}{\partial\alpha}\right|_{S_E=S_I=0} = \left.\frac{\partial H(S_I)}{\partial\alpha}\right|_{S_E=S_I=0} = \frac{1}{\mu_I}\rho y\left[\frac{\delta\rho y}{\phi}\gamma(1-\alpha)-\gamma\widehat{\theta}+\frac{\delta}{\phi}\frac{1}{n_E}\widetilde{n}\widetilde{w}\right]. \quad \text{(A.39)}$$

Thus, for $S_E = S_I \to 0$, we can immediately see that $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$ if

$$\begin{aligned}&\delta\gamma\rho y\left[1+\frac{1}{2\mu_I}\widehat{\theta}_{t+1}^2\right]\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\left[\gamma\rho(1-\alpha)y+\frac{1}{n_E}\widetilde{n}\widetilde{w}\right]\\&+\left.\frac{\partial H(S_I)}{\partial\alpha}\right|_{S_E=S_I=0}\delta\gamma\rho\left(1-\alpha\right)y\frac{1}{\mu_I}\widehat{\theta}_{t+1}\frac{1}{\phi}\frac{1}{\gamma} > \left.\frac{\partial H(S_E)}{\partial\alpha}\right|_{S_E=S_I=0}.\end{aligned} \quad \text{(A.40)}$$

Using (A.39) we can see that a sufficient condition for (A.40) to hold is

$$\delta\gamma\rho y\left[1+\frac{1}{2\mu_I}\widehat{\theta}_{t+1}^2\right]\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\left[\gamma\rho(1-\alpha)y+\frac{1}{n_E}\widetilde{n}\widetilde{w}\right] > \frac{1}{\mu_I}\rho y\left[\frac{\delta\rho y}{\phi}\gamma(1-\alpha)-\gamma\widehat{\theta}+\frac{\delta}{\phi}\frac{1}{n_E}\widetilde{n}\widetilde{w}\right].$$

Note that this sufficient condition can be simplified to

$$\frac{1}{2\mu_I}\widehat{\theta}_{t+1}^2\left[\frac{\delta\rho y}{\phi}\gamma(1-\alpha)+\frac{\delta}{\phi}\frac{1}{n_E}\widetilde{n}\widetilde{w}\right] > -\gamma\widehat{\theta},$$

which is clearly satisfied. Thus, $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$ for $S_E = S_I \to 0$, so that $n_E^*(S_I) > n_E^*(S_E)$ for $S_E = S_I \to 0$.

Monetary Entry Subsidy.

Suppose entrepreneurs receive a monetary subsidy $\widetilde{S}_E$. This would allow entrepreneurs with good ideas to buy back the equity share β from their investors. The equity stake β is such that investors are indifferent between accepting $\widetilde{S}_E$ and giving up the equity β, and keeping the equity stake α, i.e., $(\alpha - \beta)\,\delta\rho y + \widetilde{S}_E = \alpha\delta\rho y$. Thus, $\beta = \widetilde{S}_E \frac{1}{\delta\rho y}$. Entrepreneurs with bad ideas, on the other hand, simply consume $\widetilde{S}_E$.

Consider first the benchmark model without intergenerational dynamics. The expected utility of an entrepreneur is then given by

$$U_E = \delta\gamma\rho(1-\alpha+\beta)y + (1-\gamma)\widetilde{S}_E = \delta\gamma\rho\,(1-\alpha)\,y + \widetilde{S}_E,$$

so that the entry condition can be written as $n_E = \frac{1}{\mu_E}\left[\delta\gamma\rho\,(1-\alpha)\,y + \widetilde{S}_E\right]$. Moreover, the market clearing condition can be written as

$$\begin{aligned} \gamma n_E \phi &= \frac{1}{\mu_I}\left[\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\widetilde{w} + \widetilde{S}_E - \beta\delta\rho y\right]\widetilde{n} \\ \Leftrightarrow \quad \frac{1}{\mu_E}\gamma\left[\delta\gamma\rho\,(1-\alpha)\,y + \widetilde{S}_E\right]\phi &= \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\widetilde{w}\widetilde{n}. \end{aligned}$$

Note that the new entry and market clearing conditions are formally equivalent to (A.3) and (A.4) (with $S_E = \widetilde{S}_E$). Consequently, the expected wealth of an entrepreneur is higher under a funding subsidy than a monetary founding subsidy, i.e., $E[w_E(\widetilde{S}_E)] < E[w_E(S_I)]$ for all $\widetilde{S}_E = S_I$.

Next consider the dynamic model with intergenerational linkages. Entrepreneurs have now $\widetilde{S}_E$, in addition to $(1-\alpha)\,y$ (in case of success), available for investments. Using $\widehat{\theta}_{t+1} = \alpha_{t+1}\frac{\delta\rho y}{\phi} - 1$ we can write the new expected utility for an entrepreneurs as follows:

$$\begin{aligned} U_E &= \int_0^{\widehat{\theta}_{t+1}} \left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - \theta\right)\left[\delta\gamma\rho\,(1-\alpha)\,y + \widetilde{S}_E\right]\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}_{t+1}}^{\mu_I}\left[\delta\gamma\rho\,(1-\alpha)\,y + \widetilde{S}_E\right]\frac{1}{\mu_I}d\theta \\ &= \left[\delta\gamma\rho\,(1-\alpha)\,y + \widetilde{S}_E\right]\left[1 + \frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi} - 1\right)^2\right]. \end{aligned}$$

Thus, the new entry and market clearing conditions for the high steady state are given by

$$\begin{aligned} n_E &= \frac{1}{\mu_E}U_E = \frac{1}{\mu_E}\left[\delta\gamma\rho\left(1-\alpha\right)y + \widetilde{S}_E\right]\left[1+\frac{1}{2\mu_I}\left(\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right)^2\right] \\ \phi &= \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi}-1\right)\left[\rho\left(1-\alpha\right)y+\frac{1}{\gamma}\widetilde{S}_E\right]. \end{aligned}$$

We define

$$H(\widetilde{S}_E) \equiv \frac{1}{\mu_I}\left(\alpha\frac{\delta\rho y}{\phi}-1\right)\left[\rho\left(1-\alpha\right)y+\frac{1}{\gamma}\widetilde{S}_E\right]-\phi = 0.$$

Using H we get

$$\frac{d\alpha^*(\widetilde{S}_E)}{d\widetilde{S}_E} = -\frac{\frac{1}{\mu_I}\left(\alpha^*(\widetilde{S}_E)\frac{\delta\rho y}{\phi}-1\right)\frac{1}{\gamma}}{\frac{\partial H(\widetilde{S}_E)}{\partial\alpha}},$$

where

$$\frac{\partial H(\widetilde{S}_E)}{\partial\alpha} = \frac{1}{\mu_I}\left[\frac{\delta\rho y}{\phi}\left[\rho\left(1-\alpha\right)y+\frac{1}{\gamma}\widetilde{S}_E\right]-\left(\alpha\frac{\delta\rho y}{\phi}-1\right)\rho y\right] > 0.$$

The expected wealth of an entrepreneur with a monetary founding subsidy $\widetilde{S}_E$ is $E[w(\widetilde{S}_E)] = \delta\gamma\rho\left(1-\alpha^*(\widetilde{S}_E)\right)y+\widetilde{S}_E$. Thus,

$$\frac{dE[w(\widetilde{S}_E)]}{d\widetilde{S}_E} = 1-\delta\gamma\rho y\frac{d\alpha^*(\widetilde{S}_E)}{d\widetilde{S}_E} = 1+\frac{\frac{1}{\mu_I}\delta\rho y\left(\alpha^*(\widetilde{S}_E)\frac{\delta\rho y}{\phi}-1\right)}{\frac{\partial H(\widetilde{S}_E)}{\partial\alpha}}.$$

With a funding subsidy S_I the expected wealth of an entrepreneur is given by $E[w(S_I)] = \delta\gamma\rho\left(1-\alpha^*(S_I)\right)y$. Recall from Proof of Proposition 5 that

$$\frac{d\alpha^*(S_I)}{dS_I} = -\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\rho\left(1-\alpha^*(S_I)\right)y}{\frac{\partial H(S_I)}{\partial\alpha}},$$

where

$$\frac{\partial H(S_I)}{\partial\alpha} = \frac{1}{\mu_I}\left[\frac{\delta\rho y}{\phi}\rho\left(1-\alpha\right)y-\left(\alpha\frac{\delta\rho y}{\phi}-1+\frac{1}{\phi}\frac{1}{\gamma}S_I\right)\rho y\right] > 0.$$

Thus,

$$\frac{dE[w(S_I)]}{dS_I} = -\delta\gamma\rho y\frac{d\alpha^*(S_I)}{dS_I} = \frac{\frac{1}{\mu_I}\frac{1}{\phi}\delta\rho y\rho\left(1-\alpha^*(S_I)\right)y}{\frac{\partial H(S_I)}{\partial\alpha}}.$$

Next we show that $dE[w(S_I)]/dS_I > dE[w(\widetilde{S}_E)]/d\widetilde{S}_E$ for the limit case $S_I, \widetilde{S}_E \to 0$. Note that $\alpha(S_I) = \alpha(\widetilde{S}_E) = \alpha$ for $S_I, \widetilde{S}_E \to 0$, and

$$\left.\frac{\partial H(S_I)}{\partial \alpha}\right|_{S_I=0} = \left.\frac{\partial H(\widetilde{S}_E)}{\partial \alpha}\right|_{\widetilde{S}_E=0} = \frac{1}{\mu_I}\rho y \left[\frac{\delta\rho y}{\phi}(1-\alpha) - \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\right] > 0.$$

Thus, $dE[w(S_I)]/dS_I > dE[w(\widetilde{S}_E)]/d\widetilde{S}_E$ is equivalent to

$$\begin{aligned} \delta\rho y\frac{1}{\mu_I}\left[\frac{\rho y}{\phi}(1-\alpha) - \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)\right] &> \frac{\partial H}{\partial \alpha} \\ \Leftrightarrow \quad \frac{\delta\rho y}{\phi}(1-\alpha) - \delta\left(\alpha\frac{\delta\rho y}{\phi} - 1\right) &> \frac{\delta\rho y}{\phi}(1-\alpha) - \left(\alpha\frac{\delta\rho y}{\phi} - 1\right) \\ \Leftrightarrow \quad 1 &> \delta \end{aligned}$$

which is clearly satisfied. Thus, $dE[w(S_I)]/dS_I > dE[w(\widetilde{S}_E)]/d\widetilde{S}_E$ for $S_I, \widetilde{S}_E \to 0$, and consequently, $E[w(S_I)] > E[w(\widetilde{S}_E)]$.

Boundary Condition – Serial Angels with Non-monetary Subsidies.

Consider our benchmark model without intergenerational dynamics. Moreover, assume that these external angels can invest in two periods, t and $t+1$ (serial angels). We also treat the proportional entry cost of angels θk, as well as the financing subsidy S_I, as non-monetary. For example, we could interpret θk as an angel's private cost of effort when investing the amount k in new ventures. Without subsidies, it is then easy to see that the critical investment cost $\widehat{\theta}$ is still defined by $\widehat{\theta} = \alpha \frac{\delta \rho y}{\phi} - 1$.

With a founding subsidy $S_{E,t}$ the market equilibrium in t is defined by

$$n_{E,t} = \frac{1}{\mu_E} \left[\delta \gamma \rho (1-\alpha_t) y + S_{E,t} \right] \tag{A.41}$$

$$\frac{1}{\mu_E} \gamma \left[\delta \gamma \rho (1-\alpha_t) y + S_{E,t} \right] \phi = \frac{1}{\mu_I} \left[\alpha_t \frac{\delta \rho y}{\phi} - 1 \right] \widetilde{n} \widetilde{w}. \tag{A.42}$$

Using the market clearing condition (A.42) we get

$$\frac{d\alpha_t^*(S_{E,t})}{dS_{E,t}} = \frac{\frac{1}{\mu_E} \gamma \phi}{\frac{1}{\mu_E} \delta \gamma^2 \rho y \phi + \frac{1}{\mu_I} \frac{\delta \rho y}{\phi} \widetilde{n} \widetilde{w}} > 0.$$

Moreover, using the entry condition (A.41),

$$\frac{dn_{E,t}^*(S_{E,t})}{dS_{E,t}} = \frac{1}{\mu_E} \left[1 - \delta \gamma \rho y \frac{d\alpha_t^*(S_{E,t})}{dS_{E,t}} \right] = \frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} \widetilde{n} \widetilde{w}}{\frac{1}{\mu_E} \gamma^2 \phi + \frac{1}{\mu_I} \frac{1}{\phi} \widetilde{n} \widetilde{w}} > 0.$$

The expected wealth of an angel in $t+1$, denoted $\overline{w}_{t+1}(S_{E,t})$ is then given by

$$\overline{w}_{t+1}(S_{E,t}) = \frac{1}{\delta} \widetilde{w} \left[\int_0^{\widehat{\theta}} \alpha_t(S_{E,t}) \frac{\delta \rho y}{\phi} \frac{1}{\mu_I} d\theta + \int_{\widehat{\theta}}^{\mu_I} \frac{1}{\mu_I} d\theta \right] = \frac{1}{\delta} \widetilde{w} \left[1 + \frac{1}{\mu_I} \left(\alpha_t(S_{E,t}) \frac{\delta \rho y}{\phi} - 1 \right)^2 \right].$$

Thus,

$$\frac{d\overline{w}_{t+1}(S_{E,t})}{dS_{E,t}} = 2 \frac{1}{\delta} \widetilde{w} \frac{1}{\mu_I} \left(\alpha_t(S_{E,t}) \frac{\delta \rho y}{\phi} - 1 \right) \frac{d\alpha_t^*(S_{E,t})}{dS_{E,t}} \frac{\delta \rho y}{\phi} = \frac{2 \frac{1}{\delta} \frac{1}{\mu_E} \frac{1}{\mu_I} \gamma \widetilde{w} \left(\alpha_t(S_{E,t}) \frac{\delta \rho y}{\phi} - 1 \right)}{\frac{1}{\mu_E} \gamma^2 \phi + \frac{1}{\mu_I} \frac{1}{\phi} \widetilde{n} \widetilde{w}} > 0.$$

Next, consider a non-monetary funding subsidy $S_{I,t}$. The market equilibrium is then defined by

$$
\begin{aligned}
n_{E,t} &= \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha_t)y && \text{(A.43)} \\
\frac{1}{\mu_E}\delta\gamma^2\rho(1-\alpha_t)y\phi &= \frac{1}{\mu_I}\left[\alpha_t\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_{I,t}\right]\widetilde{n}\widetilde{w}. && \text{(A.44)}
\end{aligned}
$$

Using the market clearing condition (A.44),

$$
\frac{d\alpha_t^*(S_{I,t})}{dS_{I,t}} = -\frac{\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\gamma}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\widetilde{w}} < 0.
$$

And using the entry condition (A.43),

$$
\frac{dn_{E,t}^*(S_{I,t})}{dS_{I,t}} = -\frac{1}{\mu_E}\delta\gamma\rho y\frac{d\alpha_t^*(S_{I,t})}{dS_{I,t}} = \frac{\frac{1}{\mu_E}\frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}{\frac{1}{\mu_E}\gamma^2\phi + \frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}} > 0.
$$

Furthermore, the expected wealth of an angel in $t+1$, $\overline{w}_{t+1}(S_{I,t})$, is given by

$$
\begin{aligned}
\overline{w}_{t+1}(S_{I,t}) &= \frac{1}{\delta}\widetilde{w}\left[\int_0^{\widehat{\theta}+\frac{1}{\phi}\frac{1}{\gamma}S_{I,t}}\alpha_t(S_{I,t})\frac{\delta\rho y}{\phi}\frac{1}{\mu_I}d\theta + \int_{\widehat{\theta}+\frac{1}{\phi}\frac{1}{\gamma}S_{I,t}}^{\mu_I}\frac{1}{\mu_I}d\theta\right] \\
&= \frac{1}{\delta}\widetilde{w}\left[1 + \frac{1}{\mu_I}\left(\alpha_t(S_{I,t})\frac{\delta\rho y}{\phi} - 1\right)\left(\alpha_t(S_{I,t})\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_{I,t}\right)\right].
\end{aligned}
$$

Consequently, using the expression for $d\alpha_t^*(S_{I,t})/dS_{I,t}$,

$$
\begin{aligned}
\frac{d\overline{w}_{t+1}(S_{I,t})}{dS_{I,t}} &= \frac{1}{\delta}\widetilde{w}\frac{1}{\mu_I}\left[\frac{d\alpha_t^*(S_{I,t})}{dS_{I,t}}\frac{\delta\rho y}{\phi}\left(\alpha_t(S_{I,t})\frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi}\frac{1}{\gamma}S_{I,t}\right) + \left(\alpha_t(S_{I,t})\frac{\delta\rho y}{\phi} - 1\right)\frac{d\alpha_t^*(S_{I,t})}{dS_{I,t}}\frac{\delta\rho y}{\phi}\right] \\
&\quad + \frac{1}{\delta}\widetilde{w}\frac{1}{\mu_I}\left(\alpha_t(S_{I,t})\frac{\delta\rho y}{\phi} - 1\right)\frac{1}{\phi}\frac{1}{\gamma} \\
&= \frac{\frac{1}{\delta}\frac{1}{\mu_E}\frac{1}{\mu_I}\gamma\widetilde{w}\left(\alpha_t(S_{I,t})\frac{\delta\rho y}{\phi} - 1\right) - \frac{1}{\delta}\frac{1}{\mu_I}\frac{1}{\mu_I}\frac{1}{\phi}\frac{1}{\phi}\frac{1}{\gamma}\widetilde{n}\widetilde{w}^2\left[\left(\alpha_t(S_{I,t})\frac{\delta\rho y}{\phi} - 1\right) + \frac{1}{\phi}\frac{1}{\gamma}S_{I,t}\right]}{\frac{1}{\mu_E}\gamma^2\phi + \frac{1}{\mu_I}\frac{1}{\phi}\widetilde{n}\widetilde{w}}.
\end{aligned}
$$

Note that $\frac{dn_{E,t}^*(S_{E,t})}{dS_{E,t}} = \frac{dn_{E,t}^*(S_{I,t})}{dS_{E,t}}$. Thus, $n_{E,t}^*(S_{E,t}) = n_{E,t}^*(S_{I,t})$ for all $S_{E,t} = S_{I,t}$. Moreover, because $\frac{d\alpha_t^*(S_{E,t})}{dS_{E,t}} > 0$ and $\frac{d\alpha_t^*(S_{I,t})}{dS_{E,t}} < 0$, we have $\alpha_t^*(S_{E,t}) > \alpha_t^*(S_{I,t})$ for all $S_{E,t} = S_{I,t}$. It is then easy to see that $\frac{d\overline{w}_{t+1}(S_{E,t})}{dS_{E,t}} > \frac{d\overline{w}_{t+1}(S_{I,t})}{dS_{I,t}}$, so that $\overline{w}_{t+1}(S_{E,t}) > \overline{w}_{t+1}(S_{I,t})$ for all $S_{E,t} = S_{I,t}$.

Next consider period $t+1$. For parsimony we assume that the government does not provide a subsidy in $t+1$. However, it is straightforward to show that allowing for a subsidy in $t+1$ does not change the results below.

The market equilibrium in $t+1$ is defined by

$$n_{E,t+1} = \frac{1}{\mu_E}\delta\gamma\rho(1-\alpha_{t+1})y \tag{A.45}$$

$$\frac{1}{\mu_E}\delta\gamma^2\rho(1-\alpha_{t+1})y\phi = \frac{1}{\mu_I}\left[\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right]\widetilde{n}\overline{w}_{t+1}(S_t), \tag{A.46}$$

where $S_t \in \{S_{E,t}, S_{I,t}\}$. Using the market clearing condition (A.46), we get

$$\frac{d\alpha^*_{t+1}(S_t)}{dS_t} = -\frac{\frac{1}{\mu_I}\left[\alpha_{t+1}\frac{\delta\rho y}{\phi}-1\right]\widetilde{n}\frac{d\overline{w}_{t+1}(S_t)}{dS_t}}{\frac{1}{\mu_E}\delta\gamma^2\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\widetilde{n}\overline{w}_{t+1}(S_t)}.$$

Recall that $\frac{d\overline{w}_{t+1}(S_{E,t})}{dS_{E,t}} > \frac{d\overline{w}_{t+1}(S_{I,t})}{dS_{I,t}}$. For $S_{E,t} = S_{I,t} \to 0$ we then find that $\left(-\frac{d\alpha^*_{t+1}(S_{E,t})}{dS_{E,t}}\right) > \left(-\frac{d\alpha^*_{t+1}(S_{I,t})}{dS_{E,t}}\right)$. Furthermore, using the entry condition (A.45),

$$\frac{dn^*_{E,t+1}}{dS_t} = \frac{1}{\mu_E}\delta\gamma\rho y\left(-\frac{d\alpha^*_{t+1}}{dS_t}\right).$$

The fact that $\left(-\frac{d\alpha^*_{t+1}(S_{E,t})}{dS_{E,t}}\right) > \left(-\frac{d\alpha^*_{t+1}(S_{I,t})}{dS_{E,t}}\right)$ then implies that $n^*_{E,t+1}(S_{E,t}) > n^*_{E,t+1}(S_{I,t})$ for $S_{E,t} = S_{I,t} \to 0$.