Innovative Applications of O.R.

Spatial Lanchester models

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ABSTRACT

Lanchester equations have been widely used to model combat for many years, nevertheless, one of their most important limitations has been their failure to model the spatial dimension of the problems. Despite the fact that some efforts have been made in order to overcome this drawback, mainly through the use of Reaction–Diffusion equations, there is not yet a consistently clear theoretical framework linking Lanchester equations with these physical systems, apart from similarity. In this paper, a spatial modeling of Lanchester equations is conceptualized on the basis of explicit movement dynamics and balance of forces, ensuring stability and theoretical consistency with the original model. This formulation allows a better understanding and interpretation of the problem, thus improving the current treatment, modeling and comprehension of warfare applications. Finally, as a numerical illustration, a new spatial model of responsive movement is developed, confirming that location influences the results of modeling attrition conflict between two opposite forces.

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1. Introduction

Lanchester equations (LEs) were introduced by F.W. Lanchester as a set of linear differential equations that describe an attrition conflict between two opposite forces concentrated on a spot, as Kimball (1950) reports. Since then, LEs have been widely used to model and theorize about combat attrition for many years. See for example Chen and Chu (1991), Kaup et al. (2005) and Hung et al. (2005) for some recent contributions. This success can be explained mainly because of the simplicity of Lanchester equations (LEs), and the fact that they are very intuitive and hence easy to apply. Additionally, at present there exist several research lines that are using LEs to analyze very distinct problems, such as: Adams and Mesterson-Gibbons (2003) in behavioral ecology, Lacasta et al. (2008) in infectology, Kimball (1957), Erickson (1997) in marketing and Hirshleifer (1991) in economics, which have maintained the interest in the Lanchester approach.

The Lanchester model makes strong simplifying assumptions that have proven to be important shortcomings in terms of real-battle outcomes forecasting. Nowadays, most modern warfare simulations are stochastic, heterogeneous and complex and in general terms give better predictions than the traditional LEs. Thus, important efforts have been made in order to generalize Lanchester original formulation and improve its performance, e.g. stochastic and heterogeneous models have been introduced, for details see Grubbs and Shuford (1973), Taylor (1974, 1983), Taylor and Brown (1983), Chen (2002), Roberts and Conolly (1992). Despite this, little attention has been paid on one of its main limitations: the fact that LEs fail to model the characteristic spatial dimension of most attrition problems.

Location matters in the evolution and state of a struggle, whether some entity is fighting a war, defining its marketing campaign or its vaccinations programs. The capacity of modeling different spatial settings in a consistent and stable manner is crucial at the time of deciding an army strategy, since it allows the modeller to take into account local battles and disaggregated allocation of resources, but at the same to keep in mind the global strategy. A publication by Protopopescu et al. (1989) was the first work that tried to model combat in a spatial setting via Lanchester equations, including one-dimensional spatial effect. This first attempt was followed by Cosner et al. (1990) that used a parabolic system with nonlinear interactions. Fields (1993) explicitly modeled the displacement of forces in a 2D model, and most recently Spradlin and Spradlin (2007) and Keane (2009) have tried to use and expand Protopopescu’s work to two dimensions.

All these efforts have concentrated in the use of partial differential equations to model combat, as Reaction–Diffusion equations systems. However, to this date and apart from their similarities, there is not a consistent and clear theoretical framework linking Lanchester equations with this physical system. As a consequence of this, interpretation of the diffusion and strategic behavior of the forces in a spatial setting could have not been properly addressed in the literature. Particularly, in these formulations there is not an explicit account for displacements, reinforcements and deaths of the engaged forces, and some stability issues are not discussed at all.
In this paper, a spatial modeling of Lanchester equations is conceptualized on the basis of an explicit balance of forces and developed in order to account not only for the time dynamics of the problem, but also for locations, movements and concentrations of the struggling forces. The resulting formulation ensures stability and theoretical consistency with the original model, allowing for a better understanding and interpretation of the spatial simulation. Besides, in order to complement the general model, the dynamics of the spatial combat is explicitly defined for some cases: troops movements, terrain modeling, responsive movement, perception, predator–prey behavior, and distance combat. It is expected, that this new taxonomy could certainly improve not only the warfare applications, but also the new research projects inspired by LEs, including stochastic behavior of both the result of the combat, as mentioned by Grubbs and Shuford (1973), Hellman (1996), Gass (1997), and the displacement of the forces, as stated by Fields (1993).

Additionally, as a numerical illustration, a new model of responsive movement is developed. In other words, the model includes the movement of the forces according to the balance of gradients of both own and enemy's troops and terrain effects. The spatio-temporal simulations confirm the fact that location influences the results of modeling attrition conflict between two opposite forces.

The rest of this paper is organized as follows. In the next section, a general formulation of the new model and its equations are presented. A continuity equation is developed accounting for displacements, generations or reinforcements and deaths of the engaged forces, that is to say: a balance of forces. In section three, the dynamics of the spatial modeling is defined. Thus, some different combat situations that could yield different movements of the forces and hence different diffusion and velocity characteristics are presented. In section four, a comparison of the model to previous works is done. In the fifth section a particular case of spatial combat is modeled: responsive movement. In section six a numerical example is developed, showing how location and concentration of forces matter in combat results. Finally, the findings are summarized, and directions for future research are discussed.

2. The general model

The first assumption needed to model the spatial Lanchester problem is to use a spatial coordinates system for the forces that will engage in combat, typically two: the Red and Blue armies. Thus, without loss of generality, the surface density, or number of soldiers per area unit, of the Blue forces will be represented by \( B(x, y, t) \) and that of the Red forces will be represented by \( R(x, y, t) \). An element of each force will have an instantaneous velocity given by \( \vec{v}_i(x, y, t) \) where \( i \) can be replaced either by \( B \) or \( R \). So the surface densities of current, or number of soldiers traveling parallel to the velocity per transversal distance unit, of the Blue and Red forces are represented by \( J_B(x, y, t) = \theta(x, y, t) \vec{v}_B(x, y, t) \).

It should be highlighted that both surface densities \( B \) and \( R \) must be non-negatively valued functions if soldiers are the elements of the forces.

Both forces will meet in a region of area \( \Delta x \Delta y \). For this purpose Fig. 1 depicts the continuity evolution of the Blue forces.

Now the temporal variation of the flow density, or the number of forces that actually cross the transversal distance per time unit, of the \( B \) forces coming into the region of area \( \Delta x \Delta y \) plus the internal generation or reinforcement \( G_B \) is \( \Delta \Phi_B \). Hence the instantaneous time variation of the Blue forces in the region should be expressed as:

\[
\Delta \Phi_B = - J_B(y + \Delta y) - J_B(y) \Delta y - J_B(x + \Delta x) - J_B(x) \Delta x + G_B \Delta x \Delta y.
\]

(1)

For the \( R \) forces, there is an analogous expression that results from changing the subscripts to \( R \).

As a result of the combat, that includes decay, spontaneous generation, regeneration or reinforcements, destruction and self-destruction, a nonlinear net result of each force is obtained. The respective density of the generic Blue or Red forces \( \theta \) resulting from the struggle, is described by:

\[
\Delta \Phi_\theta = \frac{\partial \theta}{\partial t} \Delta x \Delta y.
\]

(2)

By imposing the continuity relation and taking the limit when both \( \Delta x \) and \( \Delta y \) vanish, and by replacing (2) into (1), a general expression is obtained:

\[
G_B - \nabla \cdot \vec{J}_B = G_B - \frac{\partial J_B}{\partial x} \frac{\partial J_B}{\partial y} = \frac{\partial \theta}{\partial t}.
\]

(3)

It remains clear that \( \vec{J}_B = R - \vec{v}_B \), where \( \vec{v}_B \) is the instantaneous velocity of each part of the respective moving force. Without losing generality the same equations could be extended to a volumetric combat, adding easily the \( z \) coordinate. For the purpose of this document that discussion is left out.

The internal densities can be expressed considering the profile of the engaging forces through the Lanchester expressions:

\[
G_B = g_B(x, y, t) - \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} z_{Bj} R^j B^i \right),
\]

(4)

where the \( z \) coefficients have the same interpretation as in the original LEs, being always real valued and generally space–time dependent. There is an analogous expression for \( G_R \).

Combining Lanchester Eqs. (4) and (3) and the constitutive relations for \( \vec{J}_B \) a new expression arises:

\[
G_B - \nabla \cdot (B \vec{v}_B) = - \nabla \cdot \vec{J}_B + g_B(x, y, t) - \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} z_{Bj} R^j B^i \right) = \frac{\partial B}{\partial t}
\]

(5)

and applying the same procedure, a twin expression results. Eq. (5) represents the general approach to spatially modeling Lanchester equations, and they can be generically written as:

\[
- \nabla \cdot (\theta \vec{v}_\theta) = \frac{\partial \theta}{\partial t} - G_\theta,
\]

(6)

where \( \theta \) can be either \( B \) or \( R \).

This equation is general and should apply to any conflict engaging two forces. The left hand side term reflects the behavior of attraction/repulsion of the forces while the term \( G_\theta \) accounts for the birth and death governing the evolution of the combat. The analysis found in the next paragraphs illustrates better this matter, specifically some particular assumptions are discussed.

Fig. 1. Divergence of the Blue forces through an element of surface.
It is useful to recall that the total number of remaining forces in the battlefield for any time is described by:

$$\theta(t) = \int_0^t \theta(x,y,t) \, dS.$$  \hfill (7)

It is important to remark that the formulation presented above has four novel and important features that are not properly addressed in the literature on spatial attrition modeling. Firstly, this model is explicitly and consistently derived from the Lanchester's original formulation and hence it is not constructed from its similarity to some physical systems. Secondly, this model is not a specific case, because it presents a general approach, allowing the inclusion of different kinds of behavior for the forces, including diffusive attitudes, attractive or repulsive, among others. Thus, arbitrary and unjustified assumptions are avoided. Thirdly, the correct and explicit balance of forces is modeled, guaranteeing that no soldiers or forces can arbitrarily appear or disappear. In fact it allows to account for the forces at any time. Usually Fick's law, as indicated by Dekker (1959), Smith (2004) can be applied with the balance of forces in a continuous setting if stability and consistency have to be achieved. Finally, a 2D formulation is derived, which can be very helpful for didactic and visual purposes.

3. Defining the spatial dynamics of attrition modeling

In order to bound the solution to the general problem already described, the dynamics of the spatial modeling must be defined.

Thus, it is possible to identify some combat situations that could rise from the attitudes of the forces. We discuss each situation in turn, starting with the more basic.

3.1. Troop directed movement without other effects

Since most movements on the battlefield are directed towards a specific location or target, a basic movement of the forces will be to follow a given path. It is assumed that the forces are not responsive to any other information or force but they will just follow their orders unless they fight, fact that can change the average velocity due to any other information or force but they will just follow their or-...
In this case the problem is intrinsically nonlinear in $B$ and $R$. This equation can be trivially modified for the dynamics of the Red forces.

3.3.2. Direct relation between the density of current and a weighted difference of the gradients of the two forces

The group of forces located at some position will move towards or away from the other forces according to the perceived strength of the opponent.

$$J_B = -f_B - p_B (h_B \nabla B - h_B \nabla R) + B f_{BB}.$$  \hspace{1cm} (14)

where $p_B$ and $f_B$ are proportionality constants and $f_B$ and $f_R$ are friction terms, as described early.

By replacing (14) into (3) and proceeding in the same way for the other forces, the densities of current no longer appear in the equations, leaving the problem with a resemblance to the classic Poisson equation in terms of the $B$ and $R$ force densities:

$$p_B \left( h_B \nabla^2 B - h_B \nabla^2 R \right) - \nabla B = \frac{\partial B}{\partial t} - G_B$$  \hspace{1cm} (15)

and also an analog equation obtained by swapping the $B$ and $R$ subscripts.

Due to the right side of this equation, as Eq. (4) expresses, in general this is still a nonlinear problem.

3.4. Nonlocal movements and attrition

Two effects arise from nonlocal struggle, movement and destruction. These consequences are analyzed here.

3.4.1. Struggle-driven movement: Predator–prey behavior

It is expected that an army will move towards or away the enemies depending on its own balance of forces. In simple terms, fighting units are expected to direct their fire at a single specific opposite unit that they consider they can destroy. The notion of accessibility, whether they can win or not, obviously will depend upon their expectations of superiority. Thus, a tentative form of "intelligence" under a perfect information assumption that can evaluate a unit superiority against a determined target can be expressed as follows:

$$\int_{t_0}^{t_{r+1}} \int_A \left( B_{R B} - R_{R B} \right) \, dA \, dt \geq 0.$$  \hspace{1cm} (16)

Fig. 2 shows the incremental analysis needed to formulate a displacement policy for the $B$ forces.

If Eq. (16) holds true, then $B$ is going to move towards $R$, if not, $B$ is going to escape in the opposite direction or evaluate a different target. On the other hand, the velocity of $p_B$, once the direction is set, is going to depend upon the distance of the two forces.

Assuming that an element of the Blue forces moves in the orientation where it perceives extreme weakness or robustness, a spatial function should define the attitude of that element according to the relative distance while moving on the line that joins the positions of the antagonistic elements of the forces. As perceptions are involved, the effect of integration over the $S_{R B}$ domain can drive the $B$ forces through a twisting path, away from the one obtained when information is perfect. Red forces can experience the same winding in their movements.

$$\vec{v_B} = \int \int_{S_{R B}} h_B B R \, dS_{R B}.$$  \hspace{1cm} (17)

The great importance of the introduction of this kind of velocity is that it exists even if no elements of the force are present at some spot, as it happens with any potential function. Also, it should be highlighted that this interaction would induce a force alignment for the combat or the escape, quite long before.

3.4.2. Nonlocal attrition

If attrition takes place at a significant distance, for example due to the use of artillery, the killing rate over each unit of the $B(\vec{R}_{B})$ force would be needed. If it is assumed that a nonlocal portion of the $R$ forces inflict losses from that remote location, then by using the notation given in Eq. (3), the relative contribution to the decay of each elementary individual of the $B$ forces can be calculated from the effect of the remote portion of the enemy forces $R(\vec{R}_{B})$.

That value is:

$$d\left( \frac{G_B}{B} \right)_{\text{remote}} = d\left( \frac{1}{B} \frac{dB}{dt} \right)_{\text{remote}}.$$  \hspace{1cm} (18)

In this model $R$ forces fire according to their ammunition stock $a_R(\vec{R}_{B})$, distance to the $B$ forces $||\vec{R}_{B} - \vec{R}_{R}||$, relative targeting on the $B$ forces $z_R(\vec{R}_{B})$ (no information about the ammunition distribution of the enemy) and perceived dis-balance of the struggling forces $\delta(\vec{R}_{B}, \vec{R}_{R}) = h_B B(\vec{R}_{B}) - h_B B(\vec{R}_{R})$. The relative targeting on the $B$ forces is assumed to be conditioned by the perception observed by the $R$ forces over the $B$ forces, that can also be a function of the spatial coordinates.

$$z_R(\vec{R}_{B}) = h_B B(\vec{R}_{B}) \int_{S_{R B}} h_B B(\vec{R}_{B}) \, dS_{R B}.$$  \hspace{1cm} (19)

And the relative rate of firing of the $R$ forces located at $\vec{R}_{B}$ should be $\eta_R$, which can be expressed as a function of the four already mentioned variables.

Hence, the absolute contribution to the spatial Lanchester equations is:

$$G_B|_{\text{remote}} = \frac{dB}{dt} |_{\text{remote}},$$  \hspace{1cm} (20)

or

$$G_R|_{\text{remote}} = -B(\vec{R}_{B}) \int_{S_{R B}} R(\vec{R}_{R}) \eta_R(\vec{R}_{R}) ||\vec{R}_{R} - \vec{R}_{B}||, z_R(\vec{R}_{B}), a_R(\vec{R}_{B}), \delta(\vec{R}_{R}, \vec{R}_{R}) \, dS_{R B}.$$  \hspace{1cm} (21)

4. Comparison to other formulations

In an early work, Protopopescu et al. (1989) model the spatial behavior of the forces, using only one spatial dimension:

$$\frac{\partial B}{\partial t} = \vec{v}_{BB} \cdot \nabla B + \nabla \cdot \left( D_B \nabla B \right) + I_B.$$  \hspace{1cm} (22)
This model can be interpreted with the forces simultaneously starting to fight and trying to stay around or chasing their neighboring adversaries. The authors explicitly left the explanation of the parameters for future work. By recurring to the model shown in Eq. (6), it is easy to reverse-engineer the formulations of the authors already mentioned:

$$
\hat{v}_B = -\hat{v}_0 - \frac{1}{B} (\hat{f}_0 + D_l \nabla B).
$$

The actual velocity that emerges from that formulation is:

$$
\vec{v}_B = -C_B \vec{B} - A_v(K_a \ast B) - A_B(B(K_R \ast B) - \frac{1}{B} (\vec{f}_0 + D_B(B) \nabla (B)),
$$

where $A_v$ and $A_B$ are constants, and $K_a$ and $K_R$ must be vectorial operators, not described by the authors.

It is perhaps due to the problems of explicit definitions pointed out above in the original Protopopescu's seminal work, that only a few papers have followed up this line of research. Indeed, with limited theoretical development and shallow interpretation, it could be difficult to generate useful spatial applications. Thus, most of the work on spatial attrition models previously discussed could benefit from the explicit and consistent general framework of analysis introduced here, allowing a more clear understanding and interpretation of the spatial modeling. It is expected that from this clarification, further work on spatial LE will be developed and not only for warfare applications.

As a conclusion of the analysis already presented, this new spatial specification of LEs can be seen as a general formulation, bringing a taxonomy of models and explicit definitions for spatial Lanchester Models.

5. A new model of responsive local movement

Even though any of the formulations presented above could be analytically and numerically implemented, this section develops an original responsive movement dynamics, which highlights the combat intelligence rather than the troop movement. In any case, it is important to remark that these two dynamics (troop directed and responsive/intelligent movement) are not competitive and can be seen as complementary. For purposes of explanation, the analytical and numerical example here developed is thought to show combat dynamics in the battlefield, where emphasis is given to the combat intelligence.

Specifically, a responsive movement of the soldiers will be modeled. Thus, the densities of currents of the forces will move towards or away from the enemies depending on the balance of their gradients. In other words, each force will evaluate dynamically its strength against the opposite forces, and it will move accordingly.

Note that if friction is neglected and $\rho_{BH}/B = \rho_{BH}/h_{BH}$, then both forces will have the same instantaneous direction at the same point of the surface, but with different signs. This condition is equivalent to having the determinant of the associated linear system equal to zero.

5.1. An application on the time side

The situation to be analyzed assumes no spontaneous generation (no reinforcements) and only linear dependencies, whereas the terms from Eq. (4),

$$
\begin{align*}
\alpha_{B10} &= E_a(= k_{R}) \quad \alpha_{B01} = M_{B}, \\
\alpha_{B10} &= E_a(= k_{R}) \quad \alpha_{B01} = M_{B}
\end{align*}
$$

are used in order to compare with the example of Bach et al. (1962), and thus results in:

$$
[\Theta] \begin{bmatrix} B \\ R \end{bmatrix} - \frac{\partial}{\partial t} \begin{bmatrix} B \\ R \end{bmatrix} = 0,
$$

where the $[\Theta]$ matrix operator is:

$$
[\Theta] = \begin{bmatrix} p_A h_a \nabla^2 B - M_B - (\vec{v}_{BH} \cdot \nabla) & -p_B h_B \nabla^2 R - E_B \\
-p_A h_a \nabla^2 R - E_B & p_B h_B \nabla^2 B - M_R - (\vec{v}_{BH} \cdot \nabla)
\end{bmatrix}.
$$
5.2. Numerical solution for a 2D uniform grid

As $B$ and $R$ must be non-negatively valued for each point of the space–time domain, the solution to this problem can not be treated as in the Dirichlet (zero-order boundary conditions) or Neumann (first-order boundary conditions) problem, because the non-negativity can be regarded as a time-dependent boundary condition. Bearing in mind this statement, a numerical approach suits this problem better, where the formulation given by (31) drives to a time-stepping formulation for the spatial profile of the $B$ and $R$ scalar fields. The procedure should provide a way to adjust the time step so as to limit the maximum deviation of negative values and then to reset acceptable deviations to zero. The time stepping approach can be faced with a stable Crank–Nicolson (C–N) method, leaving the formula to adjust the time stepping and to the center lines. All the surfaces are plotted using $UC$ and cell side units.

The arbitrary units taken, can account for a plausible situation where the physical units are given together with the equivalences in Table 1:

From Figs. 3 and 4 it is clear that the final distribution of the Blue Forces depends on the shift of the centroid of the forces. The effect also happens with the Red Forces, condition that can be seen in Figs. 5 and 6. As expected, Figs. 3 and 5 are symmetric in relation to the diagonals, and to the center lines. All the surfaces are plotted using $UC$ and cell side units.

The arbitrary units taken, can account for a plausible situation where the physical units are given together with the equivalences in Table 1:

Table 1

<table>
<thead>
<tr>
<th>Concept</th>
<th>Arbitrary unit</th>
<th>Physical unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>$UT$</td>
<td>$531.91$ [minutes]</td>
</tr>
<tr>
<td>Distance</td>
<td>$UL$</td>
<td>$10^3$ metres</td>
</tr>
<tr>
<td>Cell side</td>
<td>$10^{-1}$ $UL$</td>
<td>$100$ metres</td>
</tr>
<tr>
<td>Time step</td>
<td>$10^{-4}$ $UT$</td>
<td>$3.1915$ seconds</td>
</tr>
<tr>
<td>Soldiers</td>
<td>$US$</td>
<td>$10^4$ soldier</td>
</tr>
<tr>
<td>Concentration of Forces</td>
<td>$UC = US$</td>
<td>$10^{-6}$ soldier metre$^{-2}$</td>
</tr>
<tr>
<td>(B or R)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Velocity ($v$) or Speed ($v$)</td>
<td>$UL$ or $UL^{-1}$</td>
<td>$188$ metres$^{-1}$ minutes$^{-1}$</td>
</tr>
<tr>
<td>Density of current ($j$)</td>
<td>$US$ or $US^{-1}$</td>
<td>$1.88 \times 10^{-3}$ soldier metres$^{-1}$ minutes$^{-1}$</td>
</tr>
</tbody>
</table>

It must be remarked that this model accounts for the change in combat duration regarding the classical Lanchester equations, situation that is explained by the diffusive behavior of this kind of forces.

It can be pointed out that even though the second combat lasts longer, the number of casualties of the triumphant rebels (Red force) is significantly reduced. When physical units are used, the total amount of rebels is initially 23,757 soldiers while the sieging army has 7919 soldiers. Evolution times are compared for 95%
annihilation of one of the forces: the results obtained show some unexpected behavior, where $t_{95} = 318.72$ minutes but for off-center combat $t_{95} = 341.06$ minutes. A plausible explanation can be found in the unbalanced values of $E_X, E_B, M_X$ and $M_B$ that resemble the example analyzed by Bach et al. (1962). Interestingly, the overall results change from the original LEs, and more information

Fig. 3. Evolution of Blue forces for centered combat. $B(x,y)$ vs. spatial coordinates.

Fig. 4. Evolution of Blue forces for shifting. $B(x,y)$ vs. spatial coordinates.
is obtained about the final location of the forces. Despite the Blue army having well trained troops, in both analyzed cases, the attempt of the Red forces to destroy that sieging army ends up in an successful rebellion, where it seems to be a better strategy for them to have the forces in contact but not sharing the same shape of the Blue forces distribution. In fact, the result of these combats

Fig. 5. Evolution of Red forces for centered combat. $R(x,y)$ vs. spatial coordinates.

Fig. 6. Evolution of Red forces for shifting. $R(x,y)$ vs. spatial coordinates.
show for the centered combat that the Red forces finish with 4749 soldiers while for the off-centered combat the rebels end up with 8124 soldiers. (In both cases the Blue forces are defeated when they reach the size of 396 soldiers). An interesting situation can arise if there is a threat of a small reinforcement for the sieging army. If that is the case, maybe it would be wise to take the decision of rebelling for a shorter combat rather than a strategy of minimizing losses.

From the last two figures, Figs. 7 and 8, both shown using UT and US units, it is evident that the damage inflicted on the Blue forces is less significant if the rebels of the Red forces place their force distribution with a similar distribution to that exhibited by the Blue forces, even though the final result is plain defeat for the Blue forces. Classical Lanchester equations cannot predict the forces distribution once the combat has finished, Figs. 5 and 6, show this feature of the spatial modeling. Also an examination of the space–time evolution could be helpful in planning for eventual reinforcement of troops.

Another simulation that makes a reversal in the situation takes a change in amplitude (21,993 soldiers) and a slight relative spreading of the sieging army \( \langle \rho \rangle = 20 \), yielding 250 Red soldiers against 11,881 Blue soldiers in 920.2 minutes, with all the remaining rebels quartered, and still fighting, in the center of the battlefield.

More research could be done on trying to obtain, if possible, a distribution of the Blue forces that could lead to a reversal in the result of this combat for the same given distribution of the Blue forces.

7. Conclusions and outlook

A clear link between partial differential equations (Reaction–Diffusion equations) and Lanchester formulations is established, including the attitudes and perceptions of each force towards its opponents. The general model proposed here closes a gap not addressed in previous formulations, giving sense and interpretation to those previous models presented on the bases of analogies with other fields of knowledge.

Once the explicit movement dynamics and balance of forces are incorporated to the traditional LEs, this new Lanchester Spatial Model can be seen as a general formulation, where all the few attempts done in this direction in the literature are particular cases which could have a consistent comparison and review.
Furthermore, an original model of responsive movement is developed. As shown in the comparison to other formulations, before this work only one attempt to model velocity as a function of the opposite forces has been done. Here, troops are considered to move towards or away from the enemies depending on the balance of forces. This new feature allows to model local attrition in a more realistic way.

In this new formulation, it is possible to confirm that location influences the results of modeling attrition conflict between two opposite forces. The spatial distribution of the forces, their concentration and movement (diffusion) capabilities affect the overall results of the traditional LEs. The combat between an occupation army and a local army presented here can also be extended to a one-to-one conflict between special forces that infiltrate an enemy camp.

Specifically, it is shown that spatial concentration of forces will affect the time of annihilation and army’s losses. Thus, the optimum location of forces needed to minimize cost or maximize damage is not intuitive and hence requires further investigation, most likely in the field of dynamic optimization (optimal control). However, armies’ size, effectiveness and availability of supplies are still crucial to model the battle.

The model can also be extended, with suitable computational resources, quite straightforwardly to more massive competing forces in the struggle. In the same way, resource partitioning in a military conflict could be easily incorporated. For instance, the work done by Sheeba and Ghose (2005) can be enhanced using our formulation.

An important conclusion of the model is that a new square law must be established when space is taken into account, since the disaggregated analysis of a battle can not be modeled on a spot, unless there exist both spatial homogeneity among the forces, and negligible movements of the struggling forces constrained to a small domain. If these two conditions are not met, the spatial Lanchester approach becomes a better tool for combat modeling while the square law does not survive as such. The same conclusion applies to the linear law.

Clearly, the results of the LEs are diverted by these new modeling possibilities in a number of directions, giving the modeler a whole new spectrum of variables and parameters to simulate in a more realistic fashion the current problems modeled with LEs. Additionally, new valuable information is generated by the model, since the final location and distribution of the armies can be determined, and not just the time of annihilation.

Given the general approach employed here, it is easy to adapt this model to allow different force movements and strategic behaviors to an even wider range of problems and applications, including 3D situations. Furthermore, the space–time modeling of antagonistic forces could improve a number of applications such as: crime prevention, marketing strategies, epidemiology, population evolution, pollution interaction, economic modeling, among many others.

Finally, this publication presents hints for further work, enhancing the search for solving attrition-like problems that benefit from the spatial dimension, as the introduction of remotely driven attraction/repulsion behavior of the forces, as shown in Subsection 3.4. This general model can have stochastic analysis, either by specifically modeling the velocity field or by a characterization of the term that accounts for the struggle.

References