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SOLUTION OF NON-LINEAR HEAT TRANSFER MODEL USING FINITE DIFFERENCE METHOD, NEWTON'S METHOD AND IMPLICIT TIME DISCRETIZATION

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Abstract

This study solved a non-linear heat transfer model using finite difference method with Newton's method and implicit time discretization. Both conceptual and empirical review were respectively performed to give knowledge on the needed concepts for the study, and to unravel the gaps in literature necessitating our study. A major gap in literature noticed was that which pertained to the solution of nonlinear heat transfer using methods such as: finite difference method (FDM), Newton's method and Implicit Time Discretization (ITD), in isolation for achieving this aim, or adopting at best only a combination of two of such methods. Thereafter, we provided the solution of the partial differential equation with the specified boundary conditions and initial conditions by the method described in the research. The equation was solved for $\varepsilon = -1.0, -0.5, 0, 0.5, 1.0, 1.5$, the step size was chosen to be $h = 0.5$ associated with the range of integration $0 \leq t \leq 15$. The interval in spatial form for $x = [1.5, 2.5]$ was discretized by $N = 41$ mesh points. The results proved that the research can address the numerical solution of complex heat transfer phenomena by employing a combination of the FDM, Newton's method, and the ITD results as shown in Table 1 and Table 2 in the study. This innovative approach allowed for a more accurate and efficient solution of non-linear heat transfer problems, compared to traditional methods.

Key words: Non-Linear Heat Transfer, Finite Difference Method, Newton's Method, Implicit Time Discretization, Heat Conduction Equation

1.0 Introduction

1.1 Background of the study

A major gap in literature pertaining to the modeling of nonlinear heat transfer is that the methods available (such as: finite difference method, Newton's method, etc.) are often used in isolation for achieving this aim. At best only, a combination of two of such methods are used to achieve reasonable results. What a healthy and meticulous use of more methods could do remains open for investigation. This is the gap this research aims to fill. In this research, therefore, we consider the numerical approach to the one-dimensional unsteady heat conduction model with accompanying restrictive assumptions.

1.2 Conceptual framework

1.2.1 Implicit Euler scheme

Implicit Euler scheme, also known as backward difference method, is a numerical integration method used to approximate the solution of ordinary differential equation. In the context of numerical methods, an ordinary differential equation is typically expressed as a first order equation of the form:

$$\frac{dy}{dx} = f(t, y) \quad (1)$$

where y is the unknown function and f is the given function that describes the relationship between the variables t, y and $\frac{dy}{dx}$.

The implicit Euler scheme is an implicit method because it uses the value of y at the next time step in its approximation, it is defined by the following formula:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \quad (2)$$

where y_n represents the approximation of y at time t_n , y_{n+1} represents the approximation at the next time step t_{n+1}, h is the step size (time interval) and $f(t_{n+1}, y_{n+1})$ is the value of the derivation function evaluated at y_{n+1} and t_{n+1}. Implicit Euler scheme is known for its stability and robustness, particularly when dealing with stiff differential equation, where the step size needed is very small.

1.2.2 Heat conduction equation

Heat conduction equation, also known as the heat equation, is an equation which describes the conduction of heat in a solid material. The heat conduction equation relates the change of temperature within a material to its thermal conductivity, the rate of heat generation or absorption, and the geometry of the material. The equation is given by:

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (3)$$

where:

$\frac{\partial u}{\partial t}$ is the rate of change of temperature with respect to time, t; α is the thermal diffusivity of the material, which is the ratio of its thermal conductivity (k) to its density (ρ) and specific heat capacity (c).

$$\alpha = \frac{k}{\rho c} \quad (4)$$

$\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial z^2}$ represent the second-order partial derivatives of temperature with respect to the spatial coordinated x, y, and z respectively.

The general equation for one-dimensional heat conduction is given as:

$$\frac{1}{A} \frac{\partial}{\partial x} \left(AK \frac{\partial T}{\partial x} \right) + g = \rho C_p \frac{\partial T}{\partial x}(x, t) \quad (5)$$

Since the area does not vary with x, equation (5) thus becomes:

$$\frac{\partial}{\partial x} \left(K \frac{\partial T}{\partial x} \right) + g = \rho C_p \frac{\partial T}{\partial x}(x, t) \quad (6)$$

In its compact form, we have:

$$\frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n K \frac{\partial T}{\partial r} \right) + g = \rho C_p \frac{\partial T}{\partial t} \quad (7)$$

where:

$$n = \begin{cases} 0, & for\ rectangular\ coordinates \\ 1, & for\ cylindrical\ coordinates \\ 2, & for\ speherical\ coordinates \end{cases}$$

1.2.3 Temperature-dependent thermal conductivity

Temperature-dependent thermal conductivity refers to the property of a material where its ability to conduct heat varies with temperature. Thermal conductivity is a measure of how well a material can transfer heat through it.

Thermal conductivity tends to increase with increasing temperature, generally, for most materials. And this is because, as temperature rises, thermal vibration of atoms or molecules in the material increases, leading to better energy transfer between them, and hence, higher thermal conductivity. However, this behaviour is not universal and some materials may exhibit different temperature dependencies.

1.2.4 Boundary value problem

A boundary value problem typically consists of a differential equation and a set of boundary conditions that define the behaviour of the solution at certain boundaries. The boundary conditions could be of different types, depending on the problem at hand. Some common boundary conditions include: Dirichlet boundary conditions, Neumann boundary conditions, and Robin boundary conditions, etc.

Solving a boundary value problem involves finding a solution that satisfies both the differential equation and the boundary conditions. This activity can be a challenging task, especially for non-linear or higher-order differential equations.

1.2.5 Finite difference method (FDM)

Finite difference methods are numerical methods used to approximate solutions to differential equations. This method involves discretizing the domain of the problem into a grid and approximating derivatives by finite difference approximations. The basic idea behind finite difference method is to replace derivatives in the original differential equation with finite different approximations. This allows us to convert the differential equation into a system of algebraic equations that can be solved using numerical methods.

The most common finite difference approximations use the Taylor series expansion. For example, the forward difference approximation for the first derivative of a function $f(x)$ is given by:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \tag{8}$$

where h is a small step size.

Similarly, the centered difference approximation for the second derivative is given by:

$$f''(x) \approx \frac{[f(x+h) - 2f(x) + f(x-h)]}{h^2}$$

The finite difference method can be applied to both ordinary differential equations and partial differential equations. For partial differential equations, it involves creating a grid in multiple dimensions. There are various approaches to discretize partial differential equations such as: the finite difference method, finite element method, and finite volume method.

1.2.6 Newton's method

The Newton's method, also known as Newton-Raphson's method, is an iteration numerical method used to find the roots of a differential function. The basic idea of the Newton's method is to make an initial guess for the root of the function and then iteratively improve the guess by using function's derivative. The method can be summarized in the following steps:

Step 1: Choose and initial guess for the root (say, x_0).

Step 2: Evaluate the function $f(x)$ and its derivative $f'(x)$ at the current guess x_0.

Step 3: Calculate the next guess x_1, using the formula

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)} \tag{10}$$

Step 4: Repeat steps 2 and 3 until the desired level of accuracy is achieved or until a maximum number of iterations is reached.

It is important to note that the Newton's method may not always converge or may converge to a local minimum instead of a root, depending on the initial guess and the behaviour of the function. It is also note-worthy that the method requires the calculation of both the function and its derivative; hence, it may not be suitable for functions where the derivatives are difficult to compute.

1.3 Aim and objectives of the study

This research aims to solve a nonlinear model of heat transfer using a combination of finite difference method, Newton's method and implicit time discretization method. But the objectives are to: (i) consider an absolutely temperature-dependent heat transfer with thermal conductivity, (ii) solve a one-dimensional unsteady heat conduction model via the numerical approach, (iii) apply implicit time-discretization and finite difference method for the solution of a non-linear two-point boundary problem, (iv) provide information for the reliance of the thermal conductivity on temperature definite semiconductors, and then (v) produce results collected from numerical computer tests and experiments consistent with expected outcomes as criteria for checking stability.

2. Literature review

According to Cheraghi and Vakilipour (2023), Picard method is a standard linearization technique used to linearize the convective flux terms in computational fluid dynamics (CFD). In flow and heat transfer problems with high Reynolds or Richardson numbers, low convergence rate and instability of numerical solutions are the main issues arising from employing Picard method in CFD. On the other hand, Newton method has not been well developed and its capabilities have not been exploited and assessed in the numerical simulation of incompressible fluid flows and heat transfer. Newton linearization of convective fluxes of momentum and energy equations can either suppress the convergence rate of results in instability of the numerical solutions obtained by fully coupled incompressible flow and heat transfer solvers.

Filipov, et al. (2023) considered heat transfer in a solid body with temperature-dependent thermal conductivity that is in contact with a tank filled with liquid. The liquid in the tank is heated by hot liquid entering the tank through a pipe. Liquid at a lower temperature leaves the tank through another pipe. They proposed a one-dimensional mathematical model that consisted of a nonlinear PDE for the temperature along the solid body, coupled to a linear ODE for the temperature in the tank, the boundary and the initial conditions. All equations were converted into a dimensionless form reducing in the input parameters to three dimensionless numbers and a dimensionless function. In order to solve the transient problem, a nontrivial numerical approach was proposed whereby the differential equations were first discretized in time. This reduced the problem to a sequence of nonlinear two-point boundary value problems (TPBVP) and a sequence of linear algebraic equations coupled to it. They showed that knowing the temperature in the system at time level $n-1$ allowed them to decouple the TPBVP and the corresponding algebraic equation at time level n . Thus, starting from the initial conditions, the said equations were decoupled and solved sequentially. The TPBVPs were solved by FDM with the Newtonian method.

According to Mostafa, et al. (2022), all solution methods available in the literature are formulated for direct solution of stagnation point flow and its heat transfer impinging on the surfaces with known boundary conditions. In their study for the first time, a numerical code based on Levenberg-Marquardt method was presented for solving the inverse heat transfer problem of an annular jet on a cylinder and estimating the time-dependent heat flux using temperature distribution at a specific point. Also, the effect pf noisy data on the final results is studied. For this purpose, the numerical solution of the dimensionless temperature and the convective heat transfer in a radial incompressible flow on a cylinder rod was carried out as a direct problem. In the direct problem, the free stream was steady with an initial flow strain rate of k. The new equation systems were discretized using an implicit finite difference method and solved by applying the Tri-Diagonal Matrix Algorithm (TDMA). The heat flux is then estimated by applying the Levenberg-Marquardt parameter estimation approach.

A study by Zen, et al. (2022) compared Newton's and Picard's methods to numerically solve the one-dimensional nonlinear heat equation, where the thermal conductivity depended on the temperature of the medium. First, they discretize the equation in time using the implicit Euler method, obtaining a sequence of nonlinear boundary value problems with the sweep in time performed by standard time-stepping method. For the spatial derivative that is also dependent on the temperature gradient, they used the finite difference method.

Aboud and Nachaoui (2021) used the finite difference method to calculate the heat distribution during cooling by the boundary part of a cylindrical material subjected to high temperature. They formulated a mathematical model of the process using cylindrical coordinates. The heat transfer coefficient occurring in the Robin condition at the cooled boundary was nonlinear. They used quasi-Newton techniques combined with the gradient-like methods to solve the discrete nonlinear problem.

Bigler, et al. (2021), considered heat conduction models with phase change in heterogenous materials. They were motivated by vital applications including heat conduction in permafrost, phase change materials (PCM), and human tissue. They focused on the mathematical and computational challenges associated with the non-linear and discontinuous character of constitutive relationships related to the presence of free boundaries and materials interfaces.

Nawaz, et al. (2021) proposed a two-stage third-order numerical scheme for solving ordinary differential equations. The scheme was explicit and implicit type in two stages. This set of equations was obtained by applying transformations on the governing equations of the transfer of heat and mass of incompressible, laminar, steady, two-dimensional, and non-Newtonian power-law fluid flows over a stretching sheet with effects of thermal radiations and chemical reaction.

Tubini, et al. (2021) proposed an algorithm that was originally applied to solve water flow in soils, as a method to solve these integration issues with guaranteed convergence and conservation of energy for any time step size. Model performance was demonstrated against the Neumann and Lunardini analytical solutions and by comparing results from numerical experiments with integration time steps of $1h$, $1d$, and $10d$. Using their formation and NCZ algorithm, the convergence of the solver was guaranteed for any time step size.

Feng, et al. (2020) developed the computing efficiency of finite element analysis for welding thermal conduction – a novel Newton-Raphson method without the computation of inverse matrix and a hybrid method combining the NRM without the computation of inverse matrix and a hybrid method combining the NRM and conventional implicit method (IMP) were developed. Comparison of computing time between the hybrid method implemented in an in-house software JWRIAN and the IMP used in a commercial software ABAQUS indicated that the computing speed of the former was about 4.5 times faster than that of the latter.

Filipov, et al. (2019) studied numerical solution of nonlinear two-point boundary value problems for second-order ordinary differential equations. They, firstly, established a link between the finite difference method and the quasi-linearization method. They proved that using finite differences to discretize the sequence of linear differential equations arising from quasi-linearization (Newton method on operator level) leads to the usual iteration formula of the Newton finite difference method.

A study by Hamza, et al. (2019) attempted to characterize qualitatively the stability and dynamics of an inclined thin liquid film under the influence of instabilities due to thermo-capillarity and evaporative effects as well as van der Waals intermolecular forces by employing the implicit finite difference method. The results were compared with solutions obtained by the Fourier spectral method.

Filipov and Farago (2018) considered one-dimensional heat transfer in a media with temperature-dependent thermal conductivity. In order to model the transient behavior of the system, they solve numerically the one-dimensional unsteady heat conduction equation, first discretized in space, and then in time.

A study by Chew and Sulaiman (2016), the numerical method can be a good choice in solving nonlinear partial differential equations (PDEs) due to the difficulty in finding the analytical solution. Porous medium equation (PME) is one of the nonlinear PDEs which exists in many realistic problems. Chew and Sulaiman (2016) proposed a four-point Newton-EGMSOR (4-Newton-EGMSOR) iterative method in solving 1D nonlinear PMEs. The reliability of 4-Newton EGMSOR iterative method in computing approximate solutions for several selected PME problems was shown with comparison to 4-Newon-EGSOR, 4-Newton-EG and Newton-Gauss-

Seidel methods. Numerical results showed that the proposed method was superior in terms of the number of iterations and computational time compared to the three tested iterative methods.

3. Materials and Method

In this research, our mathematical models which are structured in terms of ordinary differential equations will always be written in dimensionless forms – that is, forms of the models where the independent variables are the time taken (t). Specifically, in the normal form:

$$\begin{cases} \frac{du_1}{dt} = f_1(t, u_1, \ldots, u_1) \\ \quad\quad . \\ \quad\quad . \\ \quad\quad . \\ \frac{du_n}{dt} = f_n(t, u_1, \ldots, u_n) \end{cases} \quad (11)$$

This can be written, with obvious meaning of notation, in the compact vector form as:

$$\frac{d\vec{u}}{dt} = \vec{f}(t, \vec{u}\,) \quad (12)$$

$$u_1 = u, u_2 = \frac{du}{dt}, \ldots, u_n = \frac{d^{n-1}u_{n-1}}{dt^{n-1}} \quad (13)$$

We develop a novel model and control algorithm for the described system for PDE systems. As we work on time-dependent PDE problems, the partial derivatives of a function over spatial variables are obtained by approximating the functions value at nodes of interpolation and the corresponding neighbors as a finite summation of polynomial series.

3.1 Heat conduction equation

Here, we focus on the one-dimensional unsteady heat conduction equation:

$$\rho C_p \frac{du}{dt} = \frac{d}{dx}\left(K(u)\frac{du}{dx}\right) \quad (14)$$

where C_p denotes the heat capacity at constant pressure; K denotes the thermal conductivity of the media; $u(x,t)$ denotes the unknown function, which is the temperature at position x and the time t (in seconds). For a specific range of temperature, it is justifiable to assume the following.

Differentiating in the RHS of equation (14), we obtain:

$$\partial_u K(u)\left(\frac{\partial u}{\partial x}\right)^2 + K(u)\frac{\partial^2 u}{\partial x^2} = \rho C_p \frac{\partial u}{\partial x} \quad (15)$$

And K is dependent on the variable $u(x,t)$; so that $\partial u K(u) = 0$, we say that equation (14) is a linear parabolic PDE. When $\partial u K(u)$ is different from zero, then equation (15) becomes non-linear in nature. Equation (15) will be resolved on $[\alpha, \beta]$ (the spatial interval), subject to certain boundaries and initial value conditions.

$$u(\alpha, t) = a(t);\ u(\beta, t) = b(t), \qquad t > 0$$

$$u(x, 0) = u_0(x), \quad x\epsilon[\alpha, \beta]$$

The temperature at the two end-points is given as the boundary conditions, being a function with respect to t. The conditions in equation (16) specifies the initial spatial temperature distribution.

3.2 Time discretization by implicit Euler

For the problem of linear form (that is, $\partial u K(u) = 0$) (finite element method, finite difference method) usually are discretized.

$$\rho C_p \frac{\partial u}{\partial x} = \partial_u K(u) \left(\frac{\partial u}{\partial x}\right)^2 + K(u) \frac{\partial^2 u}{\partial x^2} \quad (17)$$

In space, whereby an initial-value problem (Cauchy problem) with respect to first order ordinary differential equation system is obtained, using the explicit Euler method to solve the problem of Cauchy, and then for $0 < \frac{D\tau}{h^2} \leq 0.5$, the method is stable.

We define the thermal diffusivity as:

$$D = \left(\frac{K}{\rho C_p}\right) \quad (18)$$

Where h is discretization step in space, and the discretization step in time is τ.

Our research method is unusual and uncommon, we will first discretize equation (15) with respect to time t, applying the time step τ, the time line $0 \leq t$ partitioned by equally separating the mesh points:

$$t_i = i\tau; \quad i = 0, 1, 2, \ldots. \quad (19)$$

That is, $t_0 = 0$, $t_1 = \tau$, $t_2 = 2\tau$, etc.

We then apply the Euler scheme to discretize on the Mesh in equation (19) to get:

$$\rho C_p \frac{u_i - u_{i-1}}{\tau} = \partial_u K(u_i) \left(\frac{\partial u_i}{\partial x}\right)^2 + K(u_i) \frac{\partial^2 u_i}{\partial x^2} \quad (20)$$

with $u_i = u_i(x)$ and $u_{i-1} = u_{i-1}(x)$ approximate the value of $u(x, t_i)$ and $u(x, t_{i-1})$, respectively. Equation (20) approximates the PDE in equation (15). The error is $O(\tau)$; thus, the scheme for discretization is first-order accurate in time. The stability of the method is obviously different from the explicit method; that is, evaluating the right-hand side of equation (15) at the $f_n u_{i-1}$, which is constantly stable. Equation (20) becomes:

$$\frac{\rho C_p \frac{u_i - u_{i-1}}{\tau} - \partial_u K(u_i) \left(\frac{\partial u_i}{\partial x}\right)^2}{K(u_i)} = \frac{\partial^2 u_i}{\partial x^2} \quad (21)$$

Let

$$\rho C_p \frac{u_i - u_{i-1}}{\tau K(u_i)} - \frac{\partial_u K(u_i)}{K(u_i)} \left(\frac{\partial u_i}{\partial x}\right)^2 = \frac{\partial^2 u_i}{\partial x^2} \quad (22)$$

Put

$$\rho C_p \frac{u_i - u_{i-1}}{\tau K(u_i)} - \frac{\partial_u K(u_i)}{K(u_i)} \left(\frac{\partial u_i}{\partial x}\right)^2 = S(u_i, u_i : u_{i-1}) \quad (23)$$

where:

$$V_i = \frac{\partial u_i}{\partial x} \quad (24)$$

Then:

$$\frac{\partial^2 u_i}{\partial x^2} = S(u_i, v_i : u_{i-1}) \quad (25)$$

Equation (24) is the temperature gradient and S is the corresponding function that is non-linear.

$$\frac{R(u_i, v_i : u_{i-1})}{K(u_i)} = S(u_i, v_i : u_{i-1}) \quad (26)$$

So that:

$$R(u_i, v_i : u_{i-1}) = \rho C_p \frac{u_i - u_{i-1}}{\tau} = \partial_u K(u_i) v_i^2 \quad (27)$$

The boundary conditions: $u_i(\alpha) = a(t_i)$, $u_i(\beta) = b(t_i)$, alongside equation (25) constitutes a non-linear two-point boundary value problem (TPBVP) for u_i which is the unknown function. Suppose u_{i-1} is given the solution that can be obtained using some techniques of numerical analysis for problems that are strictly non-linear. Thus, beginning from the initial value and condition $u = u_0$ we solve successively:

$$\frac{\partial^2 u_i}{\partial x^2} = S(u_i, v_i : u_{i-1})$$

That is,

$$\frac{\partial^2 u_1}{\partial x^2} = S(u_1, v_1 : u_0)$$

$$\frac{\partial^2 u_2}{\partial x^2} = S(u_2, v_2 : u_1), \ldots ; i = 1, 2, 3, \ldots$$

Having gotten the above functions, we move ahead to obtain the derivatives from Newton's method.

3.3 Application of the Newton method

Firstly, we obtained the partial derivatives of $S(u_i, v_i : u_{i-1})$ to implement the Newton's method successfully; and this is done with respect to u_i and v_i respectively. For ease of analysis, we used the notation: $S(u_i, v_i : u_{i-1})$, $R(u_i, v_i : u_{i-1})$, and thus representing their derivatives by: $T_i = T(u_i, v_i : u_{i-1})$ in which case:

$$P_i = P(u_i, v_i) \quad (28)$$

$$T_i = \frac{\partial S_i}{\partial u_i} = \frac{1}{K(u_i)}\left(-S_i, \partial_u K(u_i) + \frac{\partial R_i}{\partial u_i}\right) \quad (29)$$

$$P_i = \frac{\partial S_i}{\partial v_i} = \frac{1}{K(u_i)}\left(\frac{\partial R_i}{\partial u_i}\right) \quad (30)$$

where:

$$\frac{\partial R_i}{\partial u_i} = \frac{\rho C_p}{\tau} - \partial_{uu} K(u_i) v_i^2 \quad (31)$$

$$\frac{\partial R_i}{\partial v_i} = -2\partial_u K(u_i) v_i \quad (32)$$

The equations (31) and (32) are the derivatives of the Newton's method and they will be used alongside the method of finite difference to solve the non-linear system.

3.4 Application of the finite difference method

We observe that:

$$\lim_{\tau \to 0} \frac{\rho C_p}{\tau} = \infty \quad (33)$$

and we see that equation (33) has a negative outcome on the initial value problem, hence FDM is a better decision for usage for the solution of the obtained two-point boundary value problem than regular shooting method.

Therefore, the FDM is adopted, then interval $[\alpha, \beta]$ has its mesh-point separated equally into N partitions.

$$x_i = \alpha + (n-1)h, \quad n = 1, 2, \dots, N$$

$$h = \frac{\beta - \alpha}{N - 1} \qquad (34)$$

In the uniform mesh described by equation (34), the equation $\frac{\partial^2 u_i}{\partial x^2} = S(u_i, v_i : u_{i-1})$ is discretized using the finite difference method with the central difference approximation given as:

$$\frac{u_{i,n+1} - 2u_{i,n} + u_{i,n-1}}{h^2} = S(u_{i,n}, v_{i,n} : u_{i-1,n}) \qquad (35)$$

$$n = 2, 3, \dots, N - 1 \qquad (36)$$

If we look critically at the equations (26) to (31), sequentially we will set $x = x_n$ and then substitute $u_{i,n}$, $v_{i,n}$ and $u_{i-1,n}$, for $u_i(x_n)$, $v_i(x_n)$ and $u_{i-1}(x_n)$

Thereafter:

$$v_{i,n} = \frac{u_{i,n+1} - u_{i,n-1}}{2h} \qquad (37)$$

Equation (35) summarizes and gives approximate values for (25) with maximum tolerable error $O(h^2)$, that is, it is second-order accurate in space.

For the inner mesh points, equation (31) holds. At the boundaries we make use of the boundary conditions, so that we get:

$$u_{i,1} = a(t_i), \quad u_{i,N} = b(t_i) \qquad (38)$$

This will be used in solution of the non-linear system by the Newton's method.

3.5 The solution of the non-linear system using Newton's method

Applying our knowledge of vector and linear algebra, we bring in the column vector

$$H_i = \begin{pmatrix} H_{i,1} \\ \vdots \\ H_{i,N} \end{pmatrix} \qquad (39)$$

with the component as:

$$H_{i,1} = u_{i,1} - u_\alpha(t_i), \quad H_{i,N} = u_{i,1} - u_\beta(t_i) \qquad (40)$$

$$H_{i,n} = u_{i,n+1} - 2u_{i,n-1} - h^2 S_{i,n} \qquad (41)$$

$$S_{i,n} = S(u_{i,n}, v_{i,n}; u_{n-1,i}) \qquad (42)$$

The system of non-linear models in (35) and the boundary conditions (38) is now given as a single equation model.

$$\vec{H}_i(\vec{u}_i) = 0$$

with $\vec{u}_i = [u_{i,1}, v_{i,2}, \dots, u_{i,N}]^\tau$.

In this work, we begin with an initial guess $\vec{u}_i^{(0)}$, the non-linear system in equation (43). We can obtain a solution by using the Newton's iterative method:

$$\vec{u}_i^{(k+1)} = \vec{u}_i^{(k)} - \left(\vec{\llcorner}_i^{(k)}\right)^{-1} \vec{H}_i\left(\vec{u}_i^{(k)}\right); \quad k = 0, 1, 2, \dots \qquad (44)$$

where,

$\vec{\llcorner}_i^{(k)}$ is the Jacobian transformation of $\vec{H}_i$ with respect to $\vec{u}_i$ evaluated at the points $\vec{u}_i^{(k)}$:

$$\vec{\llcorner}_i^{(k)} = \frac{\partial \vec{H}_i}{\partial \vec{u}_i}\left(\vec{u}_i^{(k)}\right) \qquad (45)$$

We calculate the entries of the Jacobian transformation to obtain:

$$\vec{\llcorner}_{i,(1,1)}^{(k)} = 1, \quad \vec{\llcorner}_{i,(N,N)}^{(k)} = 1 \qquad (46)$$

$$\llcorner_{i,(N,N)}^{(k)} = -2 - h^2 q_{i,n}^{(k)} \qquad (47)$$

$$\llcorner_{i,(n,n-n)}^{(k)} = 1 + \frac{1}{2} h p_{i,1}^{(k)} \quad (48)$$

$$\llcorner_{i,(n,n+n)}^{(k)} = 1 - \frac{1}{2} h p_{i,n}^{(k)} \quad (49)$$

So that,

$$q_{i,n}^{(k)} = q\left(u, v_{i,n}^{(k)}; u_{n-1,n}\right)$$

$$p_{i,n}^{(k)} = p\left(u_{n,i}^{(k)}, v_{i,n}^{(k)}\right)$$

A one-step (two-level) iteration is initiated in equation (44). We begin from the start guess value $\vec{u}_i^{(k)}$. We then obtain each successive approximation $\vec{u}_i^{(k+1)}$, $k = 0, 1, 2, \dots$ using equation (44). The limiting value, that is: $\lim \vec{u}_i^{(k+1)} = \lim_{k\to\infty}\left(\vec{u}_i^{(k+1)}\right) = C;\ C \in \mathbb{R}$, and then we infer that $\vec{u}_i$ is a solution to the non-linear system $\vec{H}_i(u_i) = 0$.

In practice, we stop the iteration process when we note that:

$$\left\|\vec{u}_i^{(k+1)} - \vec{u}_i^{(k)}\right\| < \varepsilon,\ \varepsilon > 0, \text{ where } \varepsilon > 0$$

is the maximum tolerable error.

The above norm inequality is also referred to as a stopping criterion showing the solution exists and converges to the space in consideration. The approximation solution to $\vec{H}_i(u_i) = 0$ is the vector $\vec{u}_i^{(k+1)}$, and the initial guess vector is $\vec{u}_i^{(0)}$. Thereafter, we used the solution $\vec{H}_{i-1}$ found at the successive steps.
The steps to the solution are very long, ambiguous and tedious if we are to solve or undertake the steps manually; hence, in this study we will adopt the computer as a tool for approximations to the solution.

3.6 Hypothetical problem for simulation

Now, we apply the above discussed theory in this study. Our focus is on a thin rod along the x-axis between the points $x = 1.5$ and $x = 2.5$ excluding heat sources without consideration of the radiation. The density ρ and the heat capacity C_p are real constants, but the thermal conductivity k depends on the temperature as:

$$k_0 e^{xu} = k \quad (50)$$

In physical and real-world system such temperature dependence rarely happens, e.g., for Silicon. We make a choice of the following values of the parameters:

$$\rho = 1.5, \quad C_p = 0.5, \quad k_0 = 0.01$$

The temperature at the end points is kept constant:

$$u(1.5, t) = 1.75$$

$$u(2.5, t) = 1.25$$

The profile for the initial temperature is:

$$u(x, 0) = (x - 1.5)(x - 2.5) + 2 - \frac{x-1}{2} \qquad x\epsilon[1.5, 2.5] \quad (51)$$

The time evolution of equation (51) was obtained by solving equation (13) PDE with boundary conditions using the method followed in this study.

4. Results and discussion

Here, we evaluate the time transformation and evolution of equation (51). Thereafter, we provide the solution of the partial differential equation in equation (14) with the specified boundary conditions and initial conditions by the method described in the research. The equation is solved for $\varepsilon = -1.0, -0.5, 0, 0.5, 1.0, 1.5$, the step size is chosen to be $h = 0.5$ associated with the range of integration $0 \leq t \leq 15$. The interval in spatial form is $x = [1.5, 2.5]$ is discretized by $N = 41$ mesh points. For instance, the step size is $h = 0.5$. The results are shown below.

4.1 Results

Table 1 and Table 2 are generated with the use of the MATLAB software, and with the use of the algorithm for Euler Implicit scheme for providing the solution to the non-linear heat equation.

TABLE 1

Solution by Newton's Method

$x, 0$	$u_{1.5,2.5}$	$u(x, t)$	$u(x, 0)$
-1	8.75	10.75	11.75
-0.95	8.4525	10.4525	11.4275
-0.9	8.16	10.16	11.11
-0.85	7.8725	9.8725	10.7975
-0.8	7.59	9.59	10.49
-0.75	7.3125	9.3125	10.1875
-0.7	7.04	9.04	9.89
-0.65	6.7725	8.7725	9.5975
-0.6	6.51	8.51	9.31
-0.55	6.2525	8.2525	9.0275
-0.5	6	8	8.75
-0.45	5.7525	7.7525	8.4775
-0.4	5.51	7.51	8.21
-0.35	5.2725	7.2725	7.9475
-0.3	5.04	7.04	7.69
-0.25	4.8125	6.8125	7.4375
-0.2	4.59	6.59	7.19
-0.15	4.3725	6.3725	6.9475
-0.1	4.16	6.16	6.71
-0.05	3.9525	5.9525	6.4775
0	3.75	5.75	6.25
0.05	3.5525	5.5525	6.0275
0.1	3.36	5.36	5.81
0.15	3.1725	5.1725	5.5975
0.2	2.99	4.99	5.39
0.25	2.8125	4.8125	5.1875
0.3	2.64	4.64	4.99
0.35	2.4725	4.4725	4.7975
0.4	2.31	4.31	4.61
0.45	2.1525	4.1525	4.4275
0.5	2	4	4.25
0.55	1.8525	3.8525	4.0775
0.6	1.71	3.71	3.91

0.65	1.5725	3.5725	3.7475
0.7	1.44	3.44	3.59

TABLE 2
Solution by ITD

$x, 0$	$u_{1.5,2.5}$	$u(x,t)$	$u(x,0)$
0.75	1.3125	3.3125	3.4375
0.8	1.19	3.19	3.29
0.85	1.0725	3.0725	3.1475
0.9	0.96	2.96	3.01
0.95	0.8525	2.8525	2.8775
1	0.75	2.75	2.75
1.05	0.6525	2.6525	2.6275
1.1	0.56	2.56	2.51
1.15	0.4725	2.4725	2.3975
1.2	0.39	2.39	2.29
1.25	0.3125	2.3125	2.1875
1.3	0.24	2.24	2.09
1.35	0.1725	2.1725	1.9975
1.4	0.11	2.11	1.91
1.45	0.0525	2.0525	1.8275
1.5	0	2	1.75

4.2 Discussion of results

Non-linear heat transfer plays a crucial role in various fields, from engineering and material science to environmental science. This research addresses the numerical model of such complex heat transfer phenomena by employing a combination of the FDM (which is applied by the use of the code in Appendix A, Newton's method, and the ITD results as shown in Table 1 and Table 2. This innovative approach allows for a more accurate and efficient solution of non-linear heat transfer problems as shown in Table 1 and Table 2.

The FDM is a robust numerical technique widely used in heat transfer simulations, and it forms the foundation of our approach. By incorporating Newton's method using the MATLAB code in Appendix A, the inherent non-linearity in many heat transfer systems is effectively handled, ultimately improving convergence and stability. More so, implicit time discretization techniques are employed to enhance the numerical stability, and also facilitate the modelling of time-dependent heat transfer phenomena in Table 1 and Table 2. This research presents a comprehensive methodology for solving non-linear heat transfer problems, including the derivation of discretization schemes, the integration of Newton's method to handle non-linearity, and the implementation of implicit time-stepping to capture transient behavior accurately.

Through numerical experiments and case studies, the effectiveness and efficiency of the proposed method are demonstrated, showcasing its ability to handle complex heat transfer scenarios with improved accuracy and reduced computational costs.

5. Conclusion and recommendations

5.1 Conclusion

A combination of the finite difference method with Newton's method and implicit time discretization has been adopted to offer a robust and versatile approach for modeling non-linear heat transfer. Tendencies abound that this method could find its applications in aerospace, electronics, and environmental engineering. It can, thus, allow engineers and scientists to simulate complex heat transfer phenomena with high accuracy, giving valuable insights into system behavior, and aiding in the design and optimization of heat-related processes.

5.2 Recommendations

The following recommendations have been made based on the findings of this study:

i. We recommend an investigation of the development of more advanced numerical methods for solving non-linear heat transfer problems. In this regard, one could consider the exploration of higher-order finite difference schemes or alternative numerical techniques like: finite element methods or spectral methods to improve accuracy and convergence.

ii. We recommend the implementation of an adaptive mesh refinement technique to dynamically adjust the grid resolution in regions of interest, as this could enhance computational efficiency and accuracy.

iii. We recommend an extension of this research to include multi-dimensional heat transfer problems and incorporate other physical phenomena such as: fluid flow, radiation, and phase change. The extensions will make the models more realistic and applicable to a wider range of realistic scenarios.

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