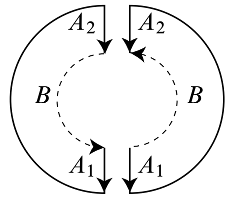
pg. 3, line 3−: Change $\bar{d}_i(x, y) < 1$ to $\bar{d}_i(x, y) \le 1$.

pg. 14: Relabel the lower left part of the central figure as

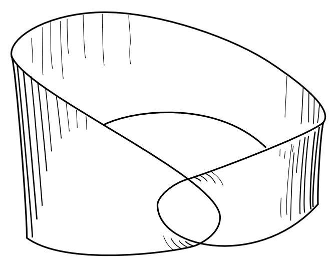


pg. 19: Replace the next-to-last paragraph with the following:

The set of points in a manifold-with-boundary that do not have a neighborhood homeomorphic to $\mathbb{R}^n$ (but only one homeomorphic to $\mathbb{H}^n$) is called the **boundary** of M and is denoted by ∂M. Equivalently, $x \in \partial M$ if and only if there is a neighborhood V of x and a homeomorphism $\phi : V \to \mathbb{H}^n$ such that $\phi(x) = 0$. If M is actually a manifold, then $\partial M = \emptyset$, and ∂M itself is always a manifold (without boundary).

Notice that if M is a subset of $\mathbb{R}^n$, then ∂M is not necessarily the same as the boundary of M in the old sense (defined for any subset of $\mathbb{R}^n$); indeed, if M is a manifold-with-boundary of dimension $< n$, then all points of M will be boundary points of M.

pg. 22: Replace the top left figure with



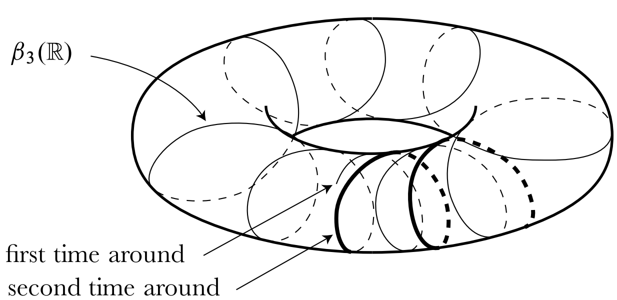
pg. 40, line 12−: $f(x, y) = x$ should be $f(x, y) = (x, 0)$.

pg. 43: Replace the last line and displayed equation with the following:

Since rank $f = k$ in a neighborhood of p, the lower rectangle in the matrix

$$\left(\frac{\partial(v^i \circ f)}{\partial x^j}\right) = \begin{pmatrix} \begin{array}{|ccc|}\hline 1 & & \\ & \ddots & \\ & & 1 \\ \hline \end{array} & 0 \\ X & \begin{array}{|ccc|}\hline D_{k+1}\psi^{k+1} & \dots & D_n\psi^{k+1} \\ \vdots & & \vdots \\ D_{k+1}\psi^m & \dots & D_n\psi^m \\ \hline \end{array} \end{pmatrix}$$

pg. 47: Replace the top figure with



pg.54. Restate Problem 5(c) as

(c) Show that the includsion $i\colon N \to M$ is C^∞. Is $\mathcal{A}'$ the only atlas with this property? (Consider $N = M = \mathbb{R}$).

pg. 57, Problem 16: For clarity, the parenthetical remark at the end may be changed to read "(the formula e^{-1/x^2} was used on page 33 just to get . . . "

pg. 59, Problem 23: Change "Let $c\colon [0, 1] \to \mathbb{R}^n$" to "Let $c\colon [0, 1] \to \mathbb{R}^n$, $n > 1$".

pp. 59–60, Problem 26: Replace part (b) with

(b) Every submanifold of $\mathbb{R}^n$ is locally of this form, after renumbering coordinates.

pg. 60, Problem 30: In part (d), change $f(Y)$ to $f(X)$.

Also change part (f) and add part (g):

(f) If M is a connected manifold, there is a proper map $f\colon M \to \mathbb{R}$; the function f can be made C^∞ if M is a C^∞ manifold.
(g) The same is true if M has at most countably many components.

pg. 61, Problem 32: For clarity, restate part (c) as follows:

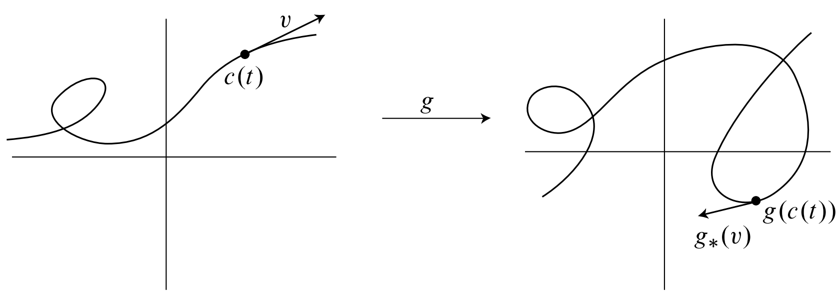
(c) This is false if $f\colon M_1 \to \mathbb{R}$ is replaced with $f\colon M_1 \to N$ for a disconnected manifold N.

pg. 62, last displayed equation should read

$$\begin{pmatrix} I_k & 0 \\ Y & I_{m-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ YA+C & YB+D \end{pmatrix}.$$

pg. 64, line 3: Replace "tangent space" with "tangent bundle".

pg. 67. Replace the first figure with the following, which makes $g_*(v)$ more clearly tangent to the curve:



pg. 70: Replace the last two lines of page 70 and the first two lines of page 71 with the following:

theorem of topology). If there were a way to map $T(M,i)$, fibre by fibre, homeomorphically onto $M \times \mathbb{R}^2$, then each v_p would correspond to $(p, \mathbf{v}(p))$ for some $\mathbf{v}(p) \in \mathbb{R}^2$, and we could continuously pick $\mathbf{w}(p) \in \mathbb{R}^2$, corresponding to a dashed vector, by using the criterion that $\mathbf{w}(p)$ should make a positive angle with $\mathbf{v}(p)$.

pg. 73, line 4: Replace M with S^1.

pg. 78: The third display should read:

$$0 = \ell(0) = \ell(fh) = f(p)\ell(h) + h(p)\ell(f) = 0 + \ell(f).$$

In the last line, $g(x)$ should be $g_i(x)$.

pg. 79, second line of the proof: $\ell(l)$ should be $\ell(1)$.

pg. 88, line 6: S^n should be S^2.

pg. 96, Problem 6: Replace this problem with the following:

6. For a bundle map $(\tilde{f}, f)$, with $f\colon B_1 \to B_2$, let K_p be the kernel of the map $\tilde{f}|{\pi_1}^{-1}(p) \to {\pi_2}^{-1}(f(p))$.

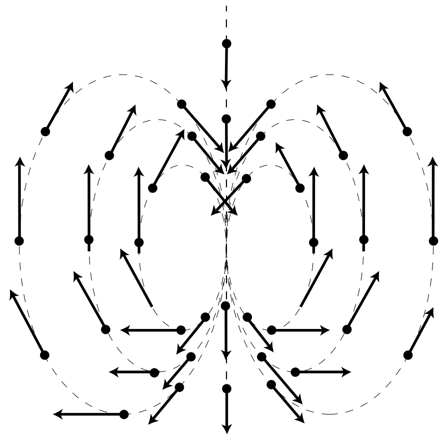
(a) If $p \mapsto \dim K_p$ is continuous, then $\ker \tilde{f}$, the union of all K_p, is a bundle over B_1.
(b) Suppose that f is one-one. For $q \in f(B_1)$, let Q_q be ${\pi_2}^{-1}(q) \,\big/\, \text{image } \tilde{f}_{f^{-1}(q)}$. Show that if $q \mapsto \dim Q_q$ is continuous, then $\operatorname{coker} \tilde{f}$, the union of all Q_q, is a bundle over $f(B_1)$.

pp. 98–99, Problem 16(e) should read:

(e) Suppose we are in the setup of Problem 2-14. Define $g\colon \partial M \times (-1,0] \to \partial N \times (-1,0]$ by $g(p,t) = (f(p),t)$. Show that TP is obtained from $TM \cup TN$ by identifying

$$v \in M_p \qquad \text{with} \qquad (\beta_*^{-1} g_* \alpha_*(v) \in N_{f(p)}.$$

and in Problem 16(g), "from M to N" should be "from ∂M to ∂N". Moreover, in Problem 18: Change the figure to



pg. 103, Problem 29(d). Add the hypothesis that M is orientable.

pg. 117: After the next to last display, $\overline{A}(X_1, \dots, X_k)(p) = A(p)(X_1(p), \dots, X_k(p))$, add:

If A is C^∞, then $\overline{A}$ is C^∞, in the sense that $\overline{A}(X_1, \dots, X_k)$ is a C^∞ function for all C^∞ vector fields $X_1, \dots, X_k$.

pg. 118: Add the following to the statement of the theorem: If $\mathcal{A}$ is C^∞, then A is also.

pg. 119: Add the following at the end of the proof:

Smoothness of A follows from the fact that the function $A_{i_1 \dots i_k}$ is $\mathcal{A}(\partial/\partial x_{i_1}, \dots, \partial/\partial x_{i_k})$.

pg. 123. In the second display, A' should be A.

pg. 127, Problem 1(d): Replace the displayed formula with

$$f^*\Bigg(\sum_{j_1,\dots,j_k} a_{j_1\dots j_k}\, dy^{j_1} \otimes \cdots \otimes dy^{j_k}\Bigg)$$

pg. 129, line 3: and the clause "if the vectors are linearly independent." at the end of the sentence.

pg. 131, Problem 9: Let F be a covariant functor from $\mathbf{V}^n$, … .

pp. 133–134. Replace these pages with

10. In classical tensor analysis there are, in addition to mixed tensor fields, other "quantities" which are defined as sets of functions which transform according to yet other rules. These new rules are of the form

$$A' = A \text{ operated on by } h\left(\frac{\partial x^\alpha}{\partial x'^\beta}\right).$$

For example, assignments of a single function a to each coordinate system x such that the function a' assigned to x' satisfies

$$a' = \left|\det\left(\frac{\partial x^i}{\partial x'^j}\right)\right| \cdot a$$

are called **scalar densities**; assignments for which

$$a' = \det\left(\frac{\partial x^i}{\partial x'^j}\right) \cdot a$$

might be called **signed scalar densities**. The Theorem in Problem 9 allows us to construct a bundle whose sections correspond to these classical entities (later we will have a more illuminating way):

(a) Let $h\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(1,\mathbb{R})$ take A into multiplication by $|\det A|$. Let F_h be the functor given by the Theorem, and consider the 1-dimensional bundle $F_h(TM)$ obtained by replacing each fibre M_p with $F_h(M_p)$. If (x, U) is a coordinate system, then

$$\alpha_x(p) = \left[\left(\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p\right), 1\right] \in F_h(M_p)$$

is non-zero, so every section on U can be expressed as $a \cdot \alpha_x$ for a unique function a. If x' is another coordinate system and $a \cdot \alpha_x = a' \cdot \alpha_{x'}$, show that

$$a' = \left|\det\left(\frac{\partial x^i}{\partial x'^j}\right)\right| \cdot a.$$

(b) If, instead, h takes A into multiplication by $\det A$, show that the corresponding equation is

$$a' = \det\left(\frac{\partial x^i}{\partial x'^j}\right)\cdot a.$$

(c) For this h, show that a non-zero element of $F_h(V)$ determines an orientation for V. Conclude that the bundle of signed scalar densities is *not* trivial if M is not orientable.
(d) We can identify $\mathcal{T}_l^{\,k}(\mathbb{R}^n)$ with $\mathbb{R}^{n^{k+l}}$ by taking

$$e^*{}_{i_1}\otimes\cdots\otimes e^*{}_{i_k}\otimes e_{j_1}\otimes\cdots\otimes e_{j_l}\mapsto (i_1,\dots,i_k,j_1,\dots,j_l)^{\text{th}}\text{ basis vector of }\mathbb{R}^{n^{k+l}}.$$

Recall that if $f\colon V\to V$, we define $\mathcal{T}_l^{\,k}(f)\colon \mathcal{T}_l^{\,k}(V)\to\mathcal{T}_l^{\,k}(V)$ by

$$\mathcal{T}_l^{\,k}(f)(T)(v_1,\dots,v_k,\lambda_1,\dots,\lambda_l)=T(f(v_1),\dots,f(v_k),\lambda_1\circ f,\dots,\lambda_l\circ f).$$

Given $A\in\mathrm{GL}(n,\mathbb{R})$, we can consider it as a map $A\colon\mathbb{R}^n\to\mathbb{R}^n$. Then $\mathcal{T}_l^{\,k}(A)\colon\mathcal{T}_l^{\,k}(\mathbb{R}^n)\to\mathcal{T}_l^{\,k}(\mathbb{R}^n)$ determines an element $\mathcal{T}_l^{\,k}(A)$ of $\mathrm{GL}(n^{k+l},\mathbb{R})$. Let $h\colon\ \mathrm{GL}(n,\mathbb{R})\to\mathrm{GL}(n^{k+l},\mathbb{R})$ be defined by

$$h(A)=|\det A|^w\,\mathcal{T}_l^{\,k}(A).$$

The bundle $F(TM)$ is called* the bundle of **tensor densities of type** $\binom{k}{l}$ **and weight** w. For $k=l=0$ we obtain the bundle of **scalar densities of weight** w. If $|\det A|^w$ is replaced by $\det A^w$ (w an integer), we obtain the bundle of **signed tensor densities of type** $\binom{k}{l}$ **and weight** w. Show that the transformation law for the components of sections of the bundle of tensor densities is given by

$$A'^{\beta_1\dots\beta_l}_{\alpha_1\dots\alpha_k}=\left|\det\left(\frac{\partial x^i}{\partial x'^j}\right)\right|^w\sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_l}}A^{j_1\dots j_l}_{i_1\dots i_k}\frac{\partial x^{i_1}}{\partial x'^{\alpha_1}}\cdots\frac{\partial x^{i_k}}{\partial x'^{\alpha_k}}\frac{\partial x'^{\beta_1}}{\partial x^{j_1}}\cdots\frac{\partial x'^{\beta_l}}{\partial x^{j_l}}$$

(and by same formula without the absolute value sign for the signed tensor densities).
(e) Define

$$\varepsilon_{i_1\dots i_n}=\begin{cases}+1 & \text{if } i_1,\dots,i_n \text{ is an even permutation of } 1,\dots,n\\ -1 & \text{if } i_1,\dots,i_n \text{ is an odd permutation of } 1,\dots,n\\ 0 & \text{if } i_\alpha=i_\beta \text{ for some } \alpha\neq\beta.\end{cases}$$

Show that there is a covariant signed tensor density of weight -1 with these components in every coordinate system. Also show that $\varepsilon^{i_1\dots i_n}=\varepsilon_{i_1\dots i_n}$ are the components in every coordinate system of a certain contravariant signed tensor density of weight 1. (See Problem 7-12 for a geometric interpretation of these tensor densities.)

pg. 143. The hypothesis of Theorem 3 should be changed so that it reads:

Let $x\in U$ and let α_1,α_2 be two maps on some open interval I such that $\alpha_1(I),\alpha_2(I)\subset U$,

$$\alpha_i{}'(t)=f(\alpha_i(t))\qquad i=1,2$$

and

$$\alpha_1(t_0)=\alpha_2(t_0)\qquad\text{for some } t_0\in I.$$

And the first sentence of the proof should be deleted.

pg. 162: In the multi-line equation (2), the second line, "$=2D_1(-Yf\circ\alpha_3)$" should be "$=2D_2(-Yf\circ\alpha_3)(0,0)$".

pg. 172, Problem 7: the equation at the end should read $\chi_*\circ(\partial/\partial t^1)=X\circ\chi$.

pg. 177, line 4 should begin "Using the fact that $[\widetilde{X},\widetilde{Y}]_p(f)=0$, and part (d) should begin:

(d) Let $f\colon M\to N$, and suppose that $f_{*p}=0$. For $X_p,Y_p\in M_p$ and and $g\colon N\to\mathbb{R}$ define

$$f_{**}(X_p,Y_p)(g)=\widetilde{X}_p(\widetilde{Y}(g\circ f)).$$

pg. 178, line 2 should begin "Use Theorem 2-9", and in the second line of Problem 20, "let X will be a vector field" should be "let X be a vector field".

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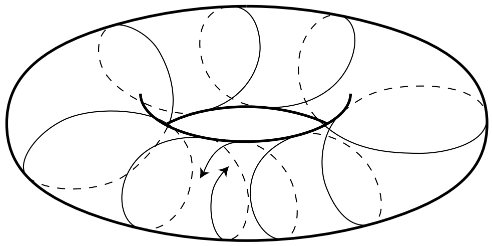
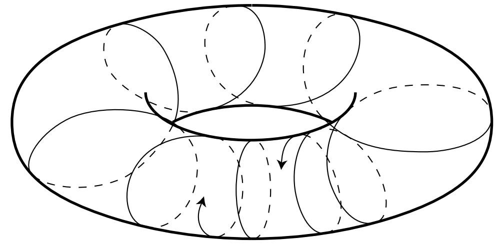
* There is actually considerable variation in terminology. For example, sometimes the term "densities" is applied only to those weight 1, while others are called "relative tensors"; and the terminology "signed densities" is not standard.

pg. 180: The bottom two figures should be replaced with



and



pg. 190, line 7: $f \circ g$ should be $g \circ f$.

pg. 198. In Problem 5, we must also assume that each $\Delta_i \oplus \Delta_j$ is integrable.

pg. 209, line 7: change "even scalar densities" to "signed scalar densities".

pg. 226. In the comutative diagram, the lower right entry should be "l-forms on N".

pg. 228, Problem 7: ω should be assumed non-zero.

pg. 229, Problem 8(c): Replace with

(c) If $\omega = \sum_{i<j} a_{ij}\psi_i \wedge \psi_j$, and A is the upper triangular matrix with $A_{ij} = a_{ij}$ for $i < j$, then the rank of A is the rank of ω. If A is the skew-symmetric matrix with $A_{ij} = a_{ij}$ for $i < j$, then the rank of A is twice the rank of ω.

pg. 231 and all of **pg. 232** up to Problem 13 should read

This shows that sections of $\Omega^n(T^*M)$ are the geometric objects corresponding to the signed scalar densities of weight -1 in Problem 4-10.
(b) Let $\mathcal{T}_l^{k[m]}(V)$ denote the vector space of all multilinear functions

$$\underbrace{V \times \cdots \times V}_{k \text{ times}} \times \underbrace{V^* \times \cdots \times V^*}_{l \text{ times}} \to \Omega^m(V).$$

Show that sections of $\mathcal{T}_l^{k[n]}(TM)$ correspond to signed tensor densities of type $\binom{k}{l}$ and weight 1. (Notice that if $v_1, \ldots, v_n$ is a basis for V, then elements of $\Omega^n(V)$ can be represented by real numbers [times the element $v^*{}_1 \wedge \cdots \wedge v^*{}_n$].)
(c) If $\mathcal{T}^k_{l[m]}(V)$ is defined similarly, except that $\Omega^m(V)$ is replaced by $\Omega^m(V^*)$, show that sections of $\mathcal{T}^k_{l[n]}(TM)$ correspond to signed tensor densities of type $\binom{k}{l}$ and weight -1.
(d) Show that the contravariant signed tensor density of type $\binom{0}{n}$ and weight 1 defined in Problem 4-10, with components $\varepsilon^{i_1 \ldots i_n}$, corresponds to the map

$$\underbrace{V^* \times \cdots \times V^*}_{n \text{ times}} \to \Omega^n(V)$$

given by $(\phi_1, \ldots, \phi_n) \mapsto \phi_1 \wedge \cdots \wedge \phi_n$. Interpret the covariant signed tensor density with components $\varepsilon_{i_1 \ldots i_n}$ similarly.
(e) Suppose $\Omega^{n;w}(V)$ denotes all functions $\eta\colon V \times \cdots \times V \to \mathbb{R}$ which are of the form

$$\eta(v_1, \ldots, v_n) = [\omega(v_1, \ldots, v_n)]^w \qquad w \text{ an integer}$$

for some $\omega \in \Omega^n(V)$. Let $\mathcal{T}_l^{k[n;w]}(V)$ be defined like $\mathcal{T}_l^{k[n]}$, except that $\Omega^n(V)$ is replaced by $\Omega^{n;w}(V)$. Show that sections of $\mathcal{T}_l^{k[n;w]}(TM)$ correspond to signed tensor densities of type $\binom{k}{l}$ and weight w. Similarly for $\mathcal{T}^k_{l[n;w]}$.
(f) For those who know about tensor products $V \otimes W$ and exterior algebras $\Lambda^k(V)$, these results can all be restated. We can identify $\mathcal{T}_l^k(V)$ with

$$\bigotimes^k V^* \otimes \bigotimes^l V = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l \text{ times}}.$$

Since $\Omega^m(V) \approx \Lambda^m(V^*) \approx [\Lambda^m(V)]^*$, we can identify

$$\begin{aligned} \mathcal{T}_l^{k[m]}(V) &\quad \text{with} \quad \bigotimes^k V^* \otimes \bigotimes^l V \otimes \Lambda^m(V) \\ \mathcal{T}^k_{l[m]}(V) &\quad \text{with} \quad \bigotimes^k V^* \otimes \bigotimes^l V \otimes \Lambda^m(V^*). \end{aligned}$$

Consider, more generally,

$$\begin{aligned} \mathcal{T}_l^{k[m;w]}(V) &= \bigotimes^k V^* \otimes \bigotimes^l V \otimes \bigotimes^w \Lambda^m(V) \\ \mathcal{T}^k_{l[m;w]}(V) &= \bigotimes^k V^* \otimes \bigotimes^l V \otimes \bigotimes^w \Lambda^m(V^*), \end{aligned}$$

for an integer $w > 0$. Noting that $\Lambda^n(V) \otimes \cdots \otimes \Lambda^n(V)$ is always 1-dimensional, show that sections of $\mathcal{T}_l^{k[n;w]}(TM)$ and $\mathcal{T}^k_{l[n;w]}(TM)$ correspond to signed tensor densities of type $\binom{k}{l}$ and weight w and $-w$, respectively.

pg. 233: The reference "pg. V.375" refers to pg. 375 of Volume V.

pg. 236, Problem 24(a): U should be assumed connected, and in the last line of the paragraph $\leq 1/k$ should be replaced by $\geq 1/k$.

pg. 237. In Problem 26, replace parts (b) and (c) with:

(b) Determine the i^{th} component of $v_1 \times \cdots \times v_{n-1}$ in terms of the $(n-1) \times (n-1)$ submatrices of the matrix

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

In particular, for $\mathbb{R}^3$, show that

$$v \times w = (v^2w^3 - v^3w^2,\ v^3w^1 - v^1w^3,\ v^1w^2 - v^2w^1).$$

pg. 259, line 7: replace "odd scalar densities" with "scalar densities".

pg. 278: The first displayed equation after the proof of Corollary 15 should be

$$X(p) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{n+1}, x_n);$$

pg. 288, Problem 11 should read

11. Following Problem 7-12, define geometric objects corresponding to tensor densities of of type $\binom{k}{l}$ and weight w (w any real number).

pg. 292. In Problem 20, the condition $U_i \cap U_j \neq \emptyset$ should be $U_i \cap U_{i+1} \neq \emptyset$.

pg. 372. The displayed equation should read

$$(A^{-1})_{ji} = (-1)^{i+j} \det A^{ij} \big/ \det A,$$

and in the last line $\mathbb{R}^2$ should be $\mathbb{C} - \{0\}$.

pg. 373, line 2: It is known that ...

pg. 374, line 2: $A \subset \mathrm{O}(n)$ should be $A \in \mathrm{O}(n)$, and the fourth display should read

$$\begin{aligned} A\tau_a(B\tau_b)^{-1} &= A\tau_a{\tau_b}^{-1}B^{-1} = A\tau_{a-b}B^{-1} \\ &= AB^{-1}\tau_{B(a-b)}, \end{aligned}$$

pg. 377, line 7: $x^{kl} \circ L_A \colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ should read $x^{kl} \circ L_A \colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$.

pg. 406. Change Problem 6 to:

6. For the multiplication $\mu \colon G \times G \to G$ of a Lie group G, we have

$$\mu_* \colon T(G \times G) \to TG,$$

as well as the standard equivalence $\simeq \colon TG \times TG \to T(G \times G)$ of Problem 3-26. Show that the composition $\tilde{\mu} = \mu_* \circ \simeq \colon TG \times TG \to TG$ is a Lie group structure on TG where, under the identification of TG with $G \times \mathfrak{g}$, the map $\tilde{\mu} \colon (G \times \mathfrak{g}) \times (G \times \mathfrak{g}) \to (G \times \mathfrak{g})$ is

$$\big((g_1, X_1),\ (g_2, X_2)\big) \longmapsto (g_1g_2,\ R_{g_2*}X_1 + L_{g_1*}X_2).$$

(Choose curves α_1, α_2 with $\alpha_i(0) = g_i$ and ${\alpha_i}'(0) = X_i$, and look at $(\alpha_1 \cdot \alpha_2)'(0)$.) Compare Problem 19(b).

pg. 408, Problem 16(a): Show that if $-1 \notin U$, then there are elements $a \in U$ which have square roots outside U in addition to their square roots in U.

Problem 16 (b) should read: "For any Lie group G, show that ... ".

pp. 408–410. For consistency with standard usage, Aut should be replaced with Aut, and then replace End with End. Moreover, Problem 19 should be replaced with the following (note that H is connected in part (h)).

19. For $a \in G$, consider the map $b \mapsto aba^{-1} = L_a{R_a}^{-1}(b)$. The map

$$(L_a{R_a}^{-1})_* \colon \mathfrak{g} \to \mathfrak{g}$$

is denoted by $\mathrm{Ad}(a)$; usually $\mathrm{Ad}(a)(X)$ is denoted simply by $\mathrm{Ad}(a)X$.
(a) $\mathrm{Ad}(ab) = \mathrm{Ad}(a) \circ \mathrm{Ad}(b)$. Thus we have a homomorphism $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$, where $\mathrm{Aut}(\mathfrak{g})$, the automorphism group of $\mathfrak{g}$, is the set of all non-singular linear transformations of the vector space $\mathfrak{g}$ onto itself (thus, isomorphic to $\mathrm{GL}(n,\mathbb{R})$ if $\mathfrak{g}$ has dimension n). The map Ad is called the **adjoint representation**.

(b) Show that the Lie group structure on TG defined in Problem 6(b) is isomorphic to the semidirect product of G and $\mathfrak{g}$ via the map Ad.
(c) Show that

$$\exp(\mathrm{Ad}(a)X) = a(\exp X)a^{-1}.$$

Hint: This follows immediately from one of our propositions.
(d) For $A \in \mathrm{GL}(n, \mathbb{R})$ and $M \in \mathfrak{gl}(n, \mathbb{R})$ show that

$$\mathrm{Ad}(A)M = AMA^{-1}.$$

(It suffices to show this for M in a neighborhood of 0.)
(e) Show that

$$\mathrm{Ad}(\exp tX)Y = Y + t[X, Y] + O(t^2).$$

(f) Since $\mathrm{Ad}\colon G \to \mathfrak{g}$, we have the map

$$\mathrm{Ad}_{*e}\colon \mathfrak{g}\ (= G_e) \ \to \begin{array}{l}\text{tangent space of } \mathrm{Aut}(\mathfrak{g}) \text{ at the} \\ \text{identity map 1 of } \mathfrak{g} \text{ to itself.}\end{array}$$

This tangent space is isomorphic to $\mathrm{End}(\mathfrak{g})$, where $\mathrm{End}(\mathfrak{g})$ is the vector space of all linear transformations of $\mathfrak{g}$ into itself: If c is a curve in $\mathrm{Aut}(\mathfrak{g})$ with $c(0) = 1$, then to regard $c'(0)$ as an element of $\mathrm{Aut}(\mathfrak{g})$, we let it operate on $Y \in \mathfrak{g}$ by

$$c'(0)(Y) = \left.\frac{d}{dt}\right|_{t=0} c(Y).$$

(Compare with the case $\mathfrak{g} = \mathbb{R}^n$, $\mathrm{Aut}(\mathfrak{g}) = \mathrm{GL}(n, \mathbb{R})$, $\mathrm{End}(\mathfrak{g}) = n \times n$ matrices.) Use (e) to show that

$$\mathrm{Ad}_{*e}(X)(Y) = [X, Y].$$

(A proof may also be given using the fact that $[\widetilde{X}, \widetilde{Y}] = L_{\widetilde{X}}\widetilde{Y}$.) The map $Y \mapsto [X, Y]$ is denoted by $\mathrm{ad}\, X \in \mathrm{End}(\mathfrak{g})$.
(g) Conclude that

$$\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}\, X) = 1 + \mathrm{ad}\, X + \frac{(\mathrm{ad}\, X)^2}{2!} + \cdots.$$

(h) Let G be a connected Lie group and $H \subset G$ a connected Lie subgroup. Show that H is a normal subgroup of G if and only if $\mathfrak{h} = \mathcal{L}(H)$ is an **ideal** of $\mathfrak{g} = \mathcal{L}(G)$, that is, if and only if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$.

pg. 411. The display in Problem 21, part (c) should read:

$$(-1)^{km}[\omega \wedge [\eta \wedge \lambda]] + (-1)^{kl}[\eta \wedge [\lambda \wedge \omega]] + (-1)^{lm}[\lambda \wedge [\omega \wedge \eta]] = 0.$$