

## PREFACE

The purpose of this book, or possibly series of books, is indicated precisely by the title *Physics for Mathematicians*. It is only necessary for me to explain what I mean by a mathematician, and what I mean by physics.

By a mathematician I mean some one who has been trained in modern mathematics and been inculcated with its general outlook. No specific mathematical knowledge is expected, but for the purposes of this book on mechanics the material in *A Comprehensive Introduction to Differential Geometry*, Volumes 1 and 2, will generally be regarded as a prerequisite, not simply because I wrote this book, but because many of the concepts of mechanics are, in fact, best expressed in terms of basic differential geometric concepts. This will always be referred to as DG, rather than Spivak [2], which is how it appears in the bibliography.

And by physics I mean . . . well, physics, what physicists mean by physics, i.e., the actual study of physical objects, even wheels, weights, ropes and pulleys (rather than the study of symplectic structures on cotangent bundles, for example). In addition to presenting the advanced physics, which mathematicians find so easy, I also want to explore the workings of elementary physics, and the mysterious maneuvers—which physicists seem to find so natural—by which one reduces a complicated physical problem to a simple mathematical question, which I have always found so hard to fathom.

As these remarks probably reveal, basically I have written this work in order to learn the subject myself, in a form that I find comprehensible. And readers familiar with some of my previous books probably realize that this has pretty much been the reason for those works also. I have been fortunate in being able to make a livelihood of sorts in this way, by indulging my desire to learn things in my own peculiar fashion while providing others with an account of the adventure. Perhaps this travelogue of an innocent abroad in a very different field will also turn out to be a book that mathematicians will enjoy (though physicists probably will not).

I am greatly indebted to many people and institutions for their help with this project. Richard Palais was, as always, an ever helpful and enthusiastic supporter of the project. Besides his help with mathematical questions, some discussions with him helped me enormously in understanding and formulating certain basic principles, though I hasten to add that he is not responsible for any heretical ideas that might appear here. Ted Shifin likewise provided unstinting help, as well as probing questions. Larry Jackal gently steered me away from some stupid mistakes and over-simplifications, and Mitch Baker heroically undertook a thorough examination of the first draft of Part I, resulting in many

corrections and improvements. John Milnor greatly contributed to my understanding of one vexing topic, and among other helpful people whom I have pestered I should mention Eisso Atzema, Robert Bryant, James Casey, Carmen Chicone, Poul Hjorth, Yildirim Hurmuzlu, Hermann Karcher, Tom Lehrer (as I couldn't plagiarize), David Nadler, Anders Persson, John Polking, and Olivier Thill.

I am grateful to the mathematics department at Rice University for affording me privileges allowing me to use the Rice University Library, and to the helpful librarians there, in particular, Erin McAfee and the science reference librarian Debra Kolah. Through the efforts of Martin Guest of Tokyo Metropolitan University and Yoshiaki Maeda of Keio University, I was able to give a series of lectures at Keio University on the material of Part I, the first time I had the opportunity to present some of this material to a live audience. A written version of the lectures was made available on the web, which providentially allowed me to be contacted by a fellow mathematician, Bruce Pourciau, who had studied many of the same questions that I puzzled through concerning Newton's work; many of his papers, listed in the Bibliography, provide additional details for the discussions of Chapters 1 and 2.



Like every self-indulgent author, I like to think that you will try to solve all the problems . . . or at least glance at them! Some problems will be used or referred to later on in the text, or sometimes in a future problem, and this crystal ball will alert you to look at them. The number in the crystal ball is the page where the problem is first used or mentioned. For example, page 41 is the first problem with a crystal ball.

I should also point out that the problems are provided mainly to help in understanding basic points, or to mention additional topics, rather than to provide proficiency in solving physics problems, and their number decreases rather rapidly after part I.

Michael Spivak  
puborperish@gmail.com

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