Chapter 4
Probability Concepts

Chapter Outline

Definition 4.1: Probability for Equally Likely Outcomes (f/N Rule)

Suppose an experiment has N possible outcomes, all equally likely. An event that can occur in f ways has probability f/N of occurring:

$$\text{Probability of an event} = \frac{f}{N}.$$

- f: Number of ways event can occur
- N: Total number of possible outcomes

Oklahoma State Officials As reported by the *World Almanac*, the top five state officials of Oklahoma are as shown in Table 4.1. Suppose that we take a simple random sample without replacement of two officials from the five officials.

Table 4.1 Five top Oklahoma state officials

Governor (G)
Lieutenant Governor (L)
Secretary of State (S)
Attorney General (A)
Treasurer (T)

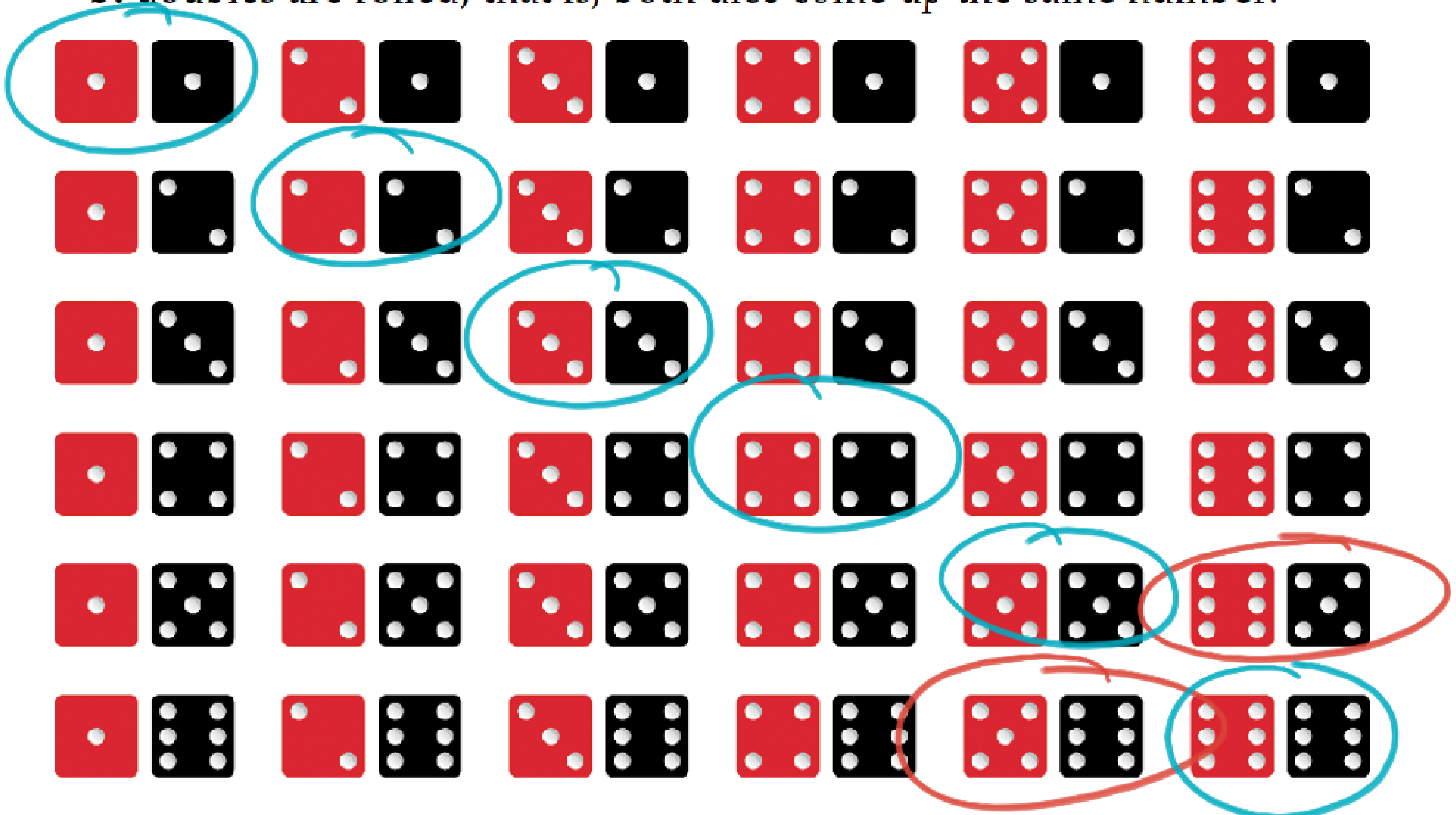
GL LS SA AT
GS LA ST
GA LT
GT

N = 10

Dice When two balanced dice are rolled, 36 equally likely outcomes are possible, as depicted in Fig. 4.1. Find the probability that

a. the sum of the dice is 11.

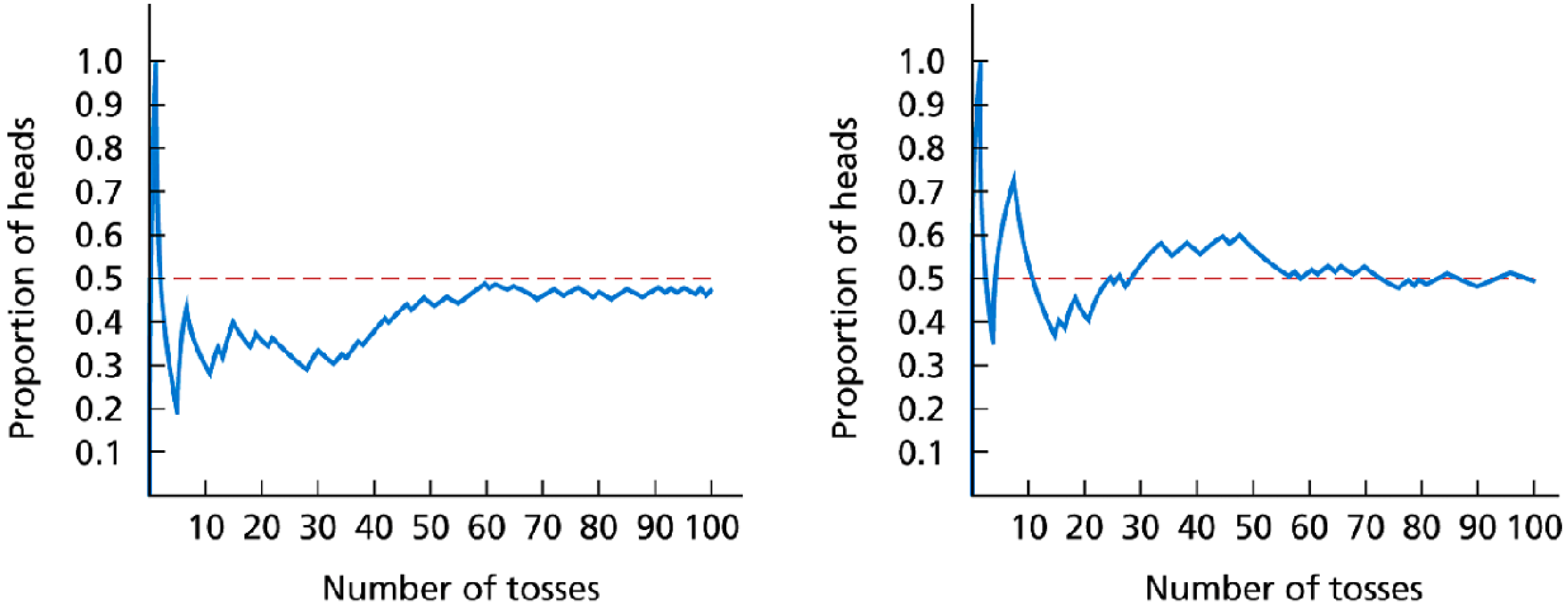
b. doubles are rolled; that is, both dice come up the same number.



a) $\frac{2}{36}$

b) $\frac{6}{36}$

Figure 4.2



Two computer simulations of tossing a balanced coin 100 times

Key Fact 4.1: Basic Properties of Probabilities

Property 1: The probability of an event is always between 0 and 1, inclusive.

Property 2: The probability of an event that cannot occur is 0. (An event that cannot occur is called an **impossible event.**)

Property 3: The probability of an event that must occur is 1. (An event that must occur is called a **certain event.**)

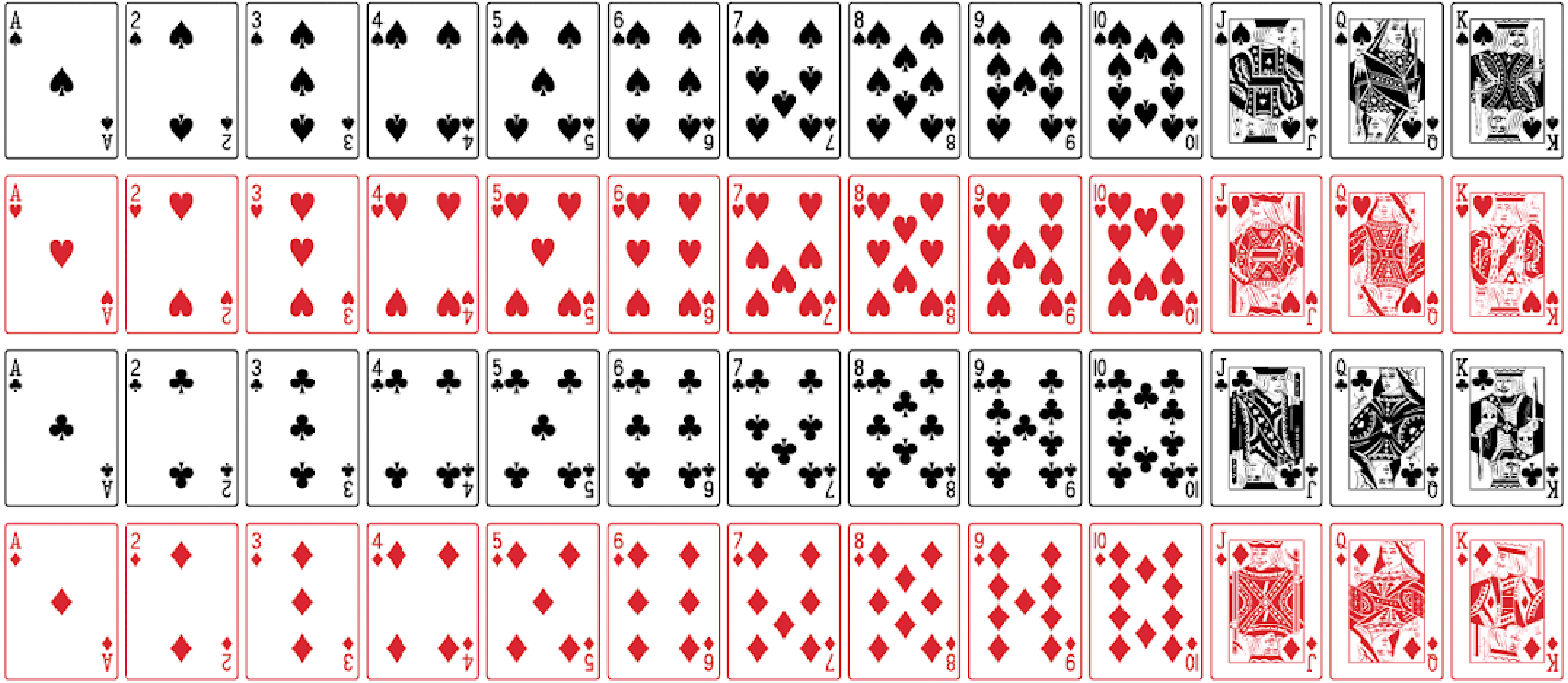
Definition 4.2: Sample Space and Event

Sample space: The collection of all possible outcomes for an experiment.

Event: A collection of outcomes for the experiment, that is, any subset of the sample space. An event **occurs** if and only if the outcome of the experiment is a member of the event.

Playing Cards A deck of playing cards contains 52 cards, as displayed in Fig. 4.3. When we perform the experiment of randomly selecting one card from the deck, we will get one of these 52 cards. The collection of all 52 cards—the possible outcomes—is called the **sample space** for this experiment.

Figure 4.3



Definition 4.3: Relationships Among Events

(not E): The event "E does not occur"

(A & B): The event "both A and B occur" $(A \cap B)$

(A or B): The event "either A or B or both occur"

$(A \cup B)$

not
(A or B)
A
B
A an B
A or B

Definition 4.4: Mutually Exclusive Events

Two or more events are **mutually exclusive events** if no two of them have outcomes in common.

Playing Cards For the experiment of randomly selecting one card from a deck of 52, let

A = event the card selected is the king of hearts,

B = event the card selected is a king,

C = event the card selected is a heart, and

D = event the card selected is a face card.

Playing Cards For the experiment of randomly selecting one card from a deck of 52, let

C = event the card selected is a heart,

D = event the card selected is a face card,

E = event the card selected is an ace,

F = event the card selected is an 8, and

G = event the card selected is a 10 or a jack.

Definition 4.5: Probability Notation

If E is an event, then $\boldsymbol{P(E)}$ represents the probability that event E occurs. It is read "the probability of E."

3 or 5

$\frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

Formula 4.1: The Special Addition Rule

If event A and event B are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B).$$

odd 5

$\frac{3}{6}$ $\frac{1}{6} = \frac{4}{6}$

More generally, if events A, B, C, . . . are mutually exclusive, then

$$P(A \text{ or } B \text{ or } C \text{ or } \cdots) = P(A) + P(B) + P(C) + \cdots .$$

Size (acres)	Relative frequency	Event
Under 10	0.106	*A*
10–49	0.281	*B*
50–179	0.300	*C*
180–499	0.167	*D*
500–999	0.068	*E*
1000–1999	0.042	*F*
2000 & over	0.036	*G*

Formula 4.2: The Complementation Rule

For any event E,

$$P(E) = 1 - P\left(\text{not } E\right).$$

Size of Farms We saw that the first two columns of Table 4.5 provide a relative- frequency distribution for the size of U.S. farms. Find the probability that a randomly selected farm has

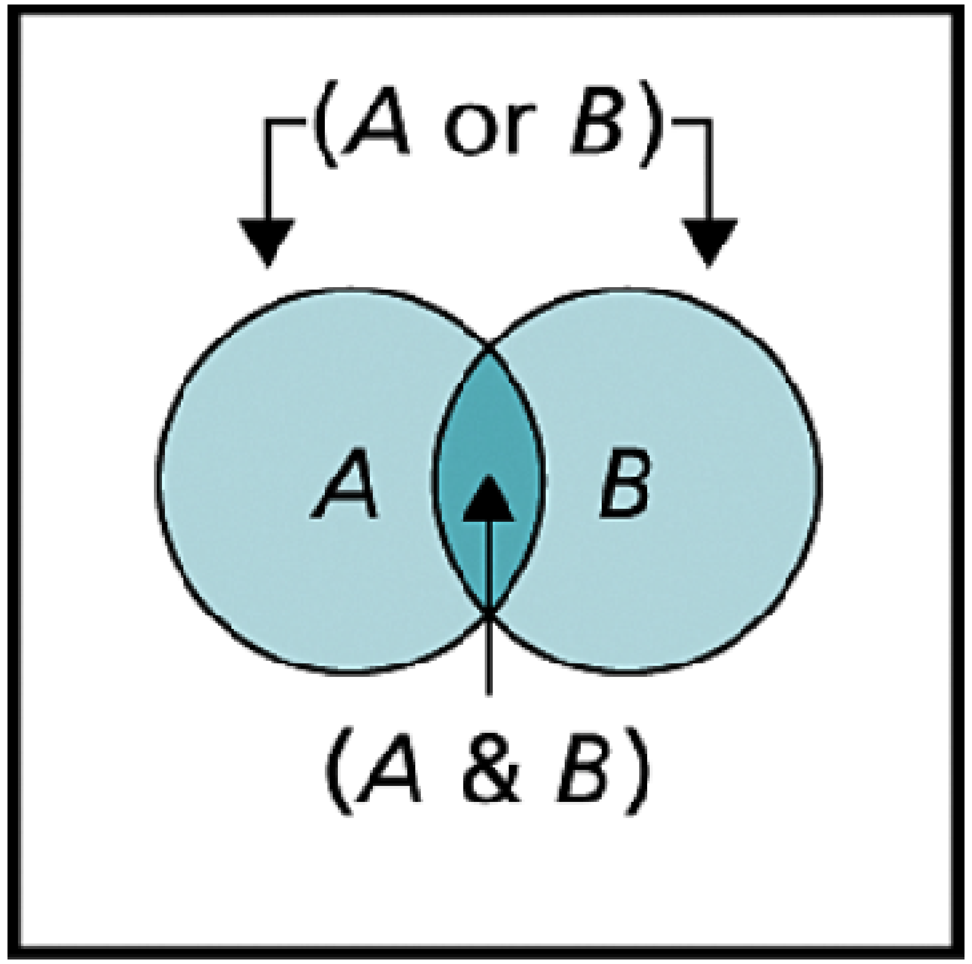
a. less than 2000 acres.

b. 50 acres or more.

Formula 4.3: The General Addition Rule

If A and B are any two events, then

$$P(A \text{ or } B) = P(A) + P(B) - P\left(A \,\&\, B\right).$$



Playing Cards Consider again the experiment of selecting one card at random from a deck of 52 playing cards. Find the probability that the card selected is either a spade or a face card

a. without using the general addition rule.

b. by using the general addition rule.

A → spade

B → face card

$$P(A) = \frac{13}{52}$$

$$+ \; P(B) = \frac{12}{52}$$

$$- \; P(A \text{ and } B) = \frac{3}{52}$$

$$P(A \text{ or } B) = 22/52$$

Characteristics of People Arrested Data on people who have been arrested are published by the Federal Bureau of Investigation in *Uniform Crime Reports*. Records for one year show that 73.9% of the people arrested were male, 12.0% were under 18 years of age, and 8.5% were males under 18 years of age. If a person arrested that year is selected at random, what is the probability that that person is either male or under 18?

A → male $P(A) = 73.9\%$

B → under 18 $P(B) = 12.0\%$

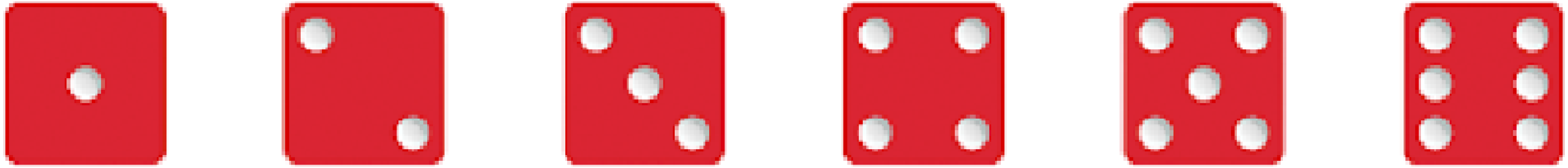
$P(A \text{ and } B) = 8.5\%$

$P(A \text{ or } B) = 77.4\%$

Rolling a Die When a balanced die is rolled once, six equally likely outcomes are possible, a

displayed in Fig. 4.22.

Figure 4.22



Sample space for rolling a die once

Let $P(F|O) = \frac{1}{3}$ $P(F) = \frac{1}{6}$

F = event a 5 is rolled, and $P(O) = \frac{3}{6}$

O = event the die comes up odd.

Definition 4.6: Conditional Probability

The probability that event B occurs given that event A occurs is called a **conditional probability**. It is denoted $\boldsymbol{P(B \mid A)}$ which is read "the probability of B given A." We call A the **given event.**

Rank

		Full professor R_1	Associate professor R_2	Assistant professor R_3	Instructor R_4	**Total**
Age (yr)	Under 30 A_1	2	3	57	6	68
	30–39 A_2	52	170	163	17	402
	40–49 A_3	156	125	61	6	348
	50–59 A_4	145	68	36	4	253
	60 & over A_5	75	15	3	0	93
	Total	430	381	320	33	1164

$$P(R_1) = \frac{430}{1164}$$

$$P(A_2 | R_1) = \frac{52}{430}$$

$$P(A_4 \text{ or } A_5 | R_4) \frac{4}{33}$$

a. Determine the (unconditional) probability that the selected faculty member is in his or her 50s.

b. Determine the (conditional) probability that the selected faculty member is in his or her 50s given that an assistant professor is selected.

Formula 4.4: The Conditional Probability Rule

If A and B are any two events with $P(A) > 0$, then

$$P(B \mid A) = \frac{P(A \,\&\, B)}{P(A)}.$$

Marital Status and Gender From *America's Families and Living Arrangements*, a publication of the U.S. Census Bureau, we obtained a joint probability distribution for the marital status of U.S. adults by gender, as shown in Table 4.9. We used "Single" to mean "Never married."

Table 4.9 Joint probability distribution of marital status and gender

		Marital status				
		Single M_1	Married M_2	Widowed M_3	Divorced M_4	$P(S_i)$
Gender	Male S_1	0.147	0.281	0.013	0.044	0.485
	Female S_2	0.121	0.284	0.050	0.060	0.515
	$P(M_j)$	0.268	0.565	0.063	0.104	1.000

$P(M_4 | S_1)$

$\frac{0.044}{0.485} = 0.091$

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	$\boldsymbol{P(M_j)}$	0.268	0.565	0.063	0.104	1.000

$P(S_2 \mid M_3)$

$\frac{0.050}{0.063} = .794$

Formula 4.5: The General Multiplication Rule

If A and B are any two events, then

$$P(A \,\&\, B) = P(A) \cdot P(B \mid A).$$

$$P(S) \cdot P(D \mid S)$$

$$P(D \text{ and } S) = .187 \times .53$$

$$= 0.099$$

$$= 9.9\%$$

U.S. Congress The U.S. Congress, Joint Committee on Printing, provides information on the composition of the Congress in the *Congressional Directory*. For the 113th Congress, 18.7% of the members are senators and 53% of the senators are Democrats. What is the probability that a randomly selected member of the 113th Congress is a Democratic senator?

D → Democrats

S → Senators → $P(S) = 18.7\%$

$P(D|S) = 53\%$

$P(D \text{ and } S) =$

Gender of Students In Professor Weiss's introductory statistics class, the number of males and females are as shown in the frequency distribution presented in Table 4.10. Two students are selected at random from the class. The first student selected is not returned to the class for possible reselection; that is, the sampling is without replacement. Find the probability that the first student selected is female and the second is male.

Table 4.10 Frequency distribution of males and females in Professor Weiss's introductory statistics class

Gender	Frequency
Male	17
Female	23
	40

$P(F_1) = \frac{23}{40}$

$P(M_2 | F_1) = \frac{17}{39}$

$P(F_1 \text{ and } M_2) = .251$

Definition 4.7: Independent Events

Event B is said to be **independent** of event A if $P(B \mid A) = P(B)$.

Playing Cards Consider again the experiment of randomly selecting one card from a deck of 52 playing cards. Let

$$F = \text{event a face card is selected},$$

$$K = \text{event a king is selected, and}$$

$$H = \text{event a heart is selected.}$$

a. Determine whether event K is independent of event F.

b. Determine whether event K is independent of event H.

Formula 4.6: The Special Multiplication Rule (for Two Independent Events)

If A and B are independent events, then

$$P(A \,\&\, B) = P(A) \cdot P(B),$$

and conversely, if $P(A \,\&\, B) = P(A) \cdot P(B)$, then A and B are independent events.

or → addition

and → multiplication

Roulette An American roulette wheel contains 38 numbers, of which 18 are red, 18 are black, and 2 are green. When the roulette wheel is spun, the ball is equally likely to land on any of the 38 numbers. In three plays at a roulette wheel, what is the probability that the ball will land on green the first time and on black the second and third times?

$P(A) \rightarrow \frac{2}{38}$

$P(B) \rightarrow \frac{18}{38}$

$P(C) \rightarrow \frac{18}{38}$

$0.012 \rightarrow 1.2\%$

License Plates The license plates of a state consist of three letters followed by three digits.

a. How many different license plates are possible?

b. How many possibilities are there for license plates on which no letter or digit is repeated?

a) 26 26 26 10 10 10

17,576,000

b) 26 25 24 10 9 8

11,232,000

Definition 4.8: Factorials

The product of the first k positive integers (counting numbers) is called $\boldsymbol{k}$ **factorial** and is denoted $\boldsymbol{k!}$. In symbols,

$$k! = k\,(k-1)\cdots 2\cdot 1.$$

We also define $0! = 1$.

Formula 4.10: The Permutations Rule

The number of possible permutations of r objects from a collection of m objects is given by the formula

$$_mP_r = \frac{m!}{(m-r)!}$$

Exacta Wagering In an exacta wager at the race track, a bettor picks the two horses that he or she thinks will finish first and second in a specified order. For a race with 12 entrants, determine the number of possible exacta wagers.

$$_nP_r = {}_{12}P_2 = 132$$

Formula 4.12: The Combinations Rule

The number of possible combinations of r objects from a collection of m objects is given by the formula

$$_{m}C_{r} = \frac{m!}{r!(m-r)!}.$$

$_{69}C_4 = 864{,}501$

CD-Club Introductory Offer To recruit new members, a compact-disc (CD) club advertises a special introductory offer: A new member agrees to buy 1 CD at regular club prices and receives free any 4 CDs of his or her choice from a collection of 69 CDs. How many possibilities does a new member have for the selection of the 4 free CDs?